Limits and Convergence

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1 Basic Definitions

Definition 1.1. Let S be a set and $P(x)$ be a proposition depending on x for $x \in S$

1. We say $\exists x \in S \mid P(x)$ if there exists some x in S such that $P(x)$ is true.
2. We say $\forall x \in S, P(x)$ if $P(x)$ is true for all x in S.

$\exists$ is called the **existential quantifier** and $\forall$ is called the **universal quantifier**.

Definition 1.2. A **sequence** in $\mathbb{R}$ is a function $a : \mathbb{N} \to \mathbb{R}$. However, we more often think of sequences in $\mathbb{R}$ as a family of numbers in $\mathbb{R}$ indexed by $\mathbb{N}$. In particular, we denote a by $\{a_n\}_{n=1}^{\infty}$, where $a_n = a(n)$ for each $n \in \mathbb{N}$

Definition 1.3. We say that L is the **limit** of the sequence $\{a_n\}_{n=1}^{\infty}$ as n tends to ∞, or equivalently,

$$\lim_{n\to\infty} a_n = L,$$

if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\mid a_n - L \mid < \varepsilon$ whenever $n \geq N$.

Definition 1.4 (Limit of a Function)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function, and $c \in \mathbb{R}$. We say that L is the **limit** of $f(x)$ as x tends to c, or equivalently,

$$\lim_{x\to c} f(x) = L,$$

if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

Definition 1.5 (Convergence, $\mathbb{R}$)**.** A sequence $\{a_n\}$ **converges** to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ for which $n > N$ implies $|a_n - L| < \varepsilon$.

2 Exercises

1. Which of the following statements are true
   a) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \mid xy = 1$
   b) $\exists x \in \mathbb{Z} \mid \forall y \in \mathbb{Z}, xy = 1$
   c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \mid xy = 0$
   d) $\exists x \in \mathbb{Z} \mid \forall y \in \mathbb{Z}, xy = 0$
2. Prove that $\lim_{n\to 2} \dfrac{n-1}{n+1} = \dfrac{1}{3}$.
3. Prove that $\lim_{x\to 5} 3x^2 - 1 = 74$
4. What is $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$?
5. Is it possible for sequences a_n and b_n to diverge while the sequence $c_n = a_n + b_n$ converges? Prove your answer!
6. Given sequences a_n and b_n of $\mathbb{R}$ converge to a and b respectively, prove that the sequence $c_n = a_n + b_n$ converges. What does it converge to?

7. Given sequences a_n and b_n of $\mathbb{R}$ converge to a and b respectively, prove that the sequence $c_n = max(a_n, b_n)$ converges. What does it converge to?

8. Given an arbitrary $x \in \mathbb{R}$, let $a_1 = x$ and $\forall n > 1$, $a_n = \frac{1}{2}a_{n-1}$ What is $\lim_{n\to\infty} a_n$?

9. $a_n = \dfrac{sin(n)}{n}$ What is $\lim_{n\to\infty} a_n$?

10. Given f(x) is a polynomial of finite degree, prove that either f is constant or $\lim_{n\to\infty} f(x)$ is ∞ or $-\infty$.