Functions: Basic Definitions and Exercises

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1 Basic Definitions

Definition 1.1. A **function** f from a set D to a set C is a rule which assigns every element of D to a single element of C. We write $f : D \to C$, where D is the *domain* and C the *codomain*.

Definition 1.2. The **image** of $f : D \to C$ is the set

$$\operatorname{im}(f) = \{f(x) : x \in D\} \subseteq C.$$

Definition 1.3. A function $f : D \to C$ is

- **injective (one-to-one)** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- **surjective (onto)** if $\operatorname{im}(f) = C$.
- **bijective** if it is both injective and surjective.

Definition 1.4. If $f : D \to C$ is bijective, its **inverse function** $f^{-1} : C \to D$ is defined by

$$f^{-1}(y) = x \quad \text{whenever } f(x) = y.$$

Definition 1.5. If $f : D \to E$ and $g : E \to C$ are functions, their **composition** is

$$g \circ f : D \to C, \quad (g \circ f)(x) = g(f(x)).$$

2 Exercises

1. Define $f(x) = x^2$ and $g(x) = \sin(x)$ as functions $\mathbb{R} \to \mathbb{R}$. Prove that these are well-defined. Give examples of domains or codomains where these rules would not define functions.
2. Let S be any set. Define a function $\emptyset \to S$ and a function $S \to \emptyset$.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ with $f(0) = 0$. Prove that f is unique.
4. Prove that the functions $f : \{1\} \to \mathbb{R}$, $f(x) = x - 1$, and $g : \{1\} \to \mathbb{R}$, $g(x) = \log x$, are equal.
5. Show that any two functions $\varnothing \to X$ are equal (why?).
6. Prove that $f : X \to Y$ is injective if and only if there exists a function $g : Y \to X$ such that $g(f(x)) = x$ for all $x \in X$.
7. Let $g : Y \to X$. Show that if there exists $f : X \to Y$ such that $g(f(x)) = x$ for all $x \in X$, then g is surjective. (★) Is the converse true?
8. Let $f : X \to Y$, $g : Y \to Z$.
   (a) Show that if f and g are injective, then $g \circ f$ is injective.
   (b) Show that if f and g are surjective, then $g \circ f$ is surjective.