Week 3

Name: ______________________
Date: ______________________

Proof by Induction

A proof by induction is essentially a shortcut to proving an infinite number of cases at once. This is done by a 'chain reaction' similar to a line of dominoes. First you have to prove that the first domino will fall if you push it over. This is called the 'base case' and a proof by induction begins with a proof of this case. Then, we must prove the law that if a domino falls, the next domino in the line will fall too. This is called the inductive hypothesis. Once we have proven these two principles, all cases follow.

Proof by induction is typically utilized in cases where the problem involves the natural numbers, such as a sequence or series. This is because we can *induct* over the natural numbers - start with a base case of $n = 0$ and move forwards from there. The general layout of an inductive proof is as follows:

1. State that you will proceed via an inductive proof.
2. State and prove the base case, usually $n = 0$ or $n = 1$.
3. State the inductive hypothesis, let $n = k$ for some $k \in N$.
4. Prove that $n = k + 1$ follows from the assumption that the inductive case, k, is true. Usually this involves algebraically manipulation to be able to substitute the k case which we know holds.
5. Conclude that since $n = 0$ is true, and since if $n = k$ is true it follows that $n = k + 1$ is true, then the statement is true for all cases.

Example

Show that for every real number $x > -1$ and every natural number n, $(1+x)^n > nx$.

Proof. Let $x > -1$ be arbitrary. We will show by induction that $(1 + x)^n \geq 1 + nx$. It clearly follows that $(1 + x)^n > nx$. Base case: if $n = 0$, then $(1 + x)^n = (1 + x)^0 = 1 = 1 + nx$. Induction step: suppose $(1 + x)^n \geq 1 + nx$. Now, let $n = n + 1$:

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n \\
&\geq (1+x)(1+nx) \\
&= 1 + x + nx + nx^2 \\
&= 1 + (n+1)x
\end{aligned}$$

since $nx^2 \geq 0$. $\square$

Exercises

Prove or disprove the following statements.

1. Show that
$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
2. Show that 2 divides $n^2 + n$ for all positive integers n
3. Prove that $1 + 3 + 5 + ... + (2n - 1) = n^2$ for all $n \in \mathbb{N}$
4. Prove that $1 + 2 + 4 + ... + 2^n = 2^{n+1} - 1$ for all $n \geq 1$.
5. Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$
6. Prove that $n! > 2^n$ for all $n \geq 4$.
7. Prove that the product $n(n+1)(n+2)$ is divisible by 6 for all $n \geq 1$.

Homework

1. Prove (by induction) that the number of subsets of a set with n elements is 2^n. Notice that this means the *cardinality* (or number of elements in) the powerset of a given set is 2^n.