Week 2

Name: ______________________________
Date: ______________________________

Proof by Contradiction

Proof by contradiction is a powerful indirect proof technique. The core idea is to assume the negation of the statement you wish to prove. From this assumption, you then derive a logical contradiction, which demonstrates that the initial assumption must be false, thereby proving the original statement.
The general steps are:

1. To prove a statement P, assume that P is false (i.e., assume $\neg P$).

2. Using this assumption, logically deduce consequences until you arrive at a contradiction. Examine what you know from this assumption, i.e., how can you manipulate this false expression further? Often it will help to simplify or push the assumption in a simpler direction, where you hope the contradiction will arise.

   Remark. A contradiction occurs when you prove something that is known to be false (e.g., $1 = 0$) or that contradicts one of your initial givens or assumptions (e.g., proving a number is both even and odd).

3. Since the assumption $\neg P$ leads to a contradiction, conclude that the assumption must be false, and therefore, P must be true.

Note: For clarity, it is standard practice to begin a proof by contradiction by stating your intention, for example: "We proceed by contradiction," or "By way of contradiction (BWOC), assume..." or "Assume for the sake of contradiction that..."

Example

Let A, B, and C be sets. Given that $A \cap B \subseteq C$ and $x \in B$, prove that $x \notin A \setminus C$.

Proof. Assume for the sake of contradiction that $x \in A \setminus C$.

By the definition of set difference, our assumption means that $x \in A$ and $x \notin C$.

We are given that $x \in B$. Since we have also deduced that $x \in A$, it follows that $x \in A \cap B$.

Now, we use the other given condition: $A \cap B \subseteq C$. Since x is an element of $A \cap B$, it must also be an element of C. Thus, $x \in C$.

This leads to a contradiction: we have concluded that $x \in C$ and, from our initial assumption, that $x \notin C$. Since a statement cannot be both true and false, our initial assumption must be incorrect.

Therefore, we conclude that $x \notin A \setminus C$. $\square$

Proof by Counterexample

To disprove a universal statement (a statement that claims something is true for **all** cases), one only needs to find a single instance where the statement fails. Such an instance is called a counterexample. If you are unsure whether a statement is true, testing a few specific cases can be a useful strategy—you might find a counterexample.

Example

Prove or disprove the following statement: "All continuous functions are differentiable."

Proof. The statement is false. We disprove it with a counterexample.

Consider the absolute value function, $f(x) = |x|$. This function is continuous for all real numbers. However, at $x = 0$, the function is not differentiable because the limit of the difference quotient does not exist; the left-hand limit is -1 while the right-hand limit is 1.

Since we have found a function that is continuous but not differentiable, the statement is disproved. □

Exercises

Exercise 1. *Prove or disprove: If n is an integer and n^2 is divisible by 4, then n is divisible by 4.*

Exercise 2 (Similar to Freshman's dream)**.** *For $a, b \in \mathbb{R}$, prove or disprove $(a + b)^2 = a^2 + b^2$.*

Exercise 3. *For $x \in \mathbb{R}$, prove or disprove $\dfrac{1}{x+2} = \dfrac{1}{x} + \dfrac{1}{2}$.*

Exercise 4. *For $a, b \in \mathbb{R}$, prove or disprove: if $a^2 - b^2 > 0$, then $a - b > 0$.*

Exercise 5. *Show that if $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.*

Exercise 6. *Show that the sum of a rational real number and an irrational real number is always an irrational number.*

Exercise 7. *(a) Let r be a rational number. Show that $\dfrac{r}{\sqrt{2}}$ is irrational.*

(b) Use part (a) to show that any rational number r can be written as the product of two irrational numbers.

Exercise 8. *Prove that there is no integer pair of solutions (x, y) such that $x^2 = 4y + 2$.*

Exercise 9. *Let p be a prime number. Show that if $p \mid n$, then $p \nmid n + 1$.*

Exercise 10. *Show that there are no positive integers x and y such that $x^2 - y^2 = 1$.*

Homework

Exercise 11 ($\star$)**.** *Prove or disprove: Consider real-valued functions on $[0, 1]$. If the product of two functions is the zero function, then one of the functions is the zero function.*

Exercise 12 ($\star$)**.** *Show that there is no smallest positive real number.*