AN ELEMENTARY PROOF OF THE UNCOUNTABILITY OF REAL NUMBERS

MAX LIPYANSKIY

The following fundamental property of real numbers is key in many analytic arguments involving the real line:

A bounded non-decreasing sequence of real numbers has a least upper bound in the real numbers.

Our proof of the uncountability of real numbers has the following characteristics:

i) It is rigorous and yet avoids more elaborate constructions such as compactness and measure.

ii) It follows most directly from the fundamental property of real numbers which is also used to prove connectedness and compactness of the unit interval. In this manner the proof avoids decimal expansions which are commonly used in an elementary proof.

THE PROOF

Assume that there exists a bijection $f : \mathbb{N} \to \mathbb{R}$. We construct a real number s_∞ not in the image of f thus contradicting the hypothesis. Define $a_1 = f(1) - 1$.
For $n > 1$, let

$$a_n = min\{f(k) - \sum_{i=1}^{n-1} \frac{a_i}{4^{i-1}}\} \text{ where } 1 \leq k \leq n \text{ and } f(k) > \sum_{i=1}^{n-1} \frac{a_i}{4^{i-1}}.$$

Let $s_n = \sum_{i=1}^{n} \frac{a_i}{4^{i-1}}$ and $s_\infty = lim_{n\to\infty} s_n$.

Lemma: $\{s_n\}$ is a well defined, bounded and strictly increasing sequence such that if $s_n < f(n)$, then $s_\infty < f(n)$.
Suppose s_n is defined and $s_n < f(n)$. One has

$$a_{n+1} \leq f(n) - s_n \text{ and } s_{n+1} \leq s_n + \frac{f(n)-s_n}{4^n} < f(n).$$

It follows by repeating this argument that for $n' > n$,

$$s_{n'} \leq s_n + \sum_{i=n+1}^{n'} \frac{f(n)-s_n}{4^{i-1}} < s_n + \sum_{i=n+1}^{\infty} \frac{f(n)-s_n}{4^{i-1}}.$$

Since $n \geq 1$, we see by evaluating the geometric series that

$$s_n + \sum_{i=n+1}^{\infty} \frac{f(n)-s_n}{4^{i-1}} = s_n + \frac{f(n)-s_n}{4^n - 4^{n-1}} \leq s_n + \frac{f(n)-s_n}{3}.$$

Thus,

$$s_{n'} < s_n + \frac{f(n)-s_n}{3} = f(n) - \frac{2(f(n)-s_n)}{3}.$$

We conclude that $s_\infty < f(n)$. Since $s_1 < f(1)$, by the preceding argument $\{s_n\}$ is well-defined, strictly increasing and bounded above by $f(1)$.

To complete the proof suppose $s_\infty = f(m)$. Then $s_m < f(m)$ since the sequence $\{s_n\}$ is strictly increasing. By the lemma, $s_\infty < f(m)$. This is a contradiction.