RANKEYA DATTA

LOCALIZATION

We will motivate the idea behind localization by reviewing the construction of the rational numbers $\mathbb{Q}$ using the integers $\mathbb{Z}$.

We first define an equivalence relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}\setminus\{0\}$ as follows -
$\forall (a_1, b_1), (a_2, b_2) \in \mathbb{Z} \times \mathbb{Z}\setminus\{0\}$, $(a_1, b_1) \sim (a_2, b_2)$
$\iff b_2 a_1 - b_1 a_2 = 0$ (this is secretly the condition for the equality of two rational numbers $\frac{a_1}{b_1}, \frac{a_2}{b_2}$).

It is easy to verify that $\sim$ as defined is an equivalence relation (i.e., we need to check reflexivity, symmetry and transitivity).

We denote an equivalence class $[(a, b)]_\sim$ of $\frac{\mathbb{Z} \times \mathbb{Z}\setminus\{0\}}{\sim}$ as $\frac{a}{b}$. The set $\frac{\mathbb{Z} \times \mathbb{Z}\setminus\{0\}}{\sim}$ can be given the structure of a ring by defining $+$ and $\times$ as follows -

$\forall \frac{a_1}{b_1}, \frac{a_2}{b_2} \in \frac{\mathbb{Z} \times \mathbb{Z}\setminus\{0\}}{\sim}$,

- $\frac{a_1}{b_1} + \frac{a_2}{b_2} := \frac{b_2 a_1 + b_1 a_2}{b_1 b_2}$
- $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} := \frac{a_1 a_2}{b_1 b_2}$

As an exercise one can check that $+$ and $\times$ as defined on $\dfrac{\mathbb{Z}\times\mathbb{Z}\setminus\{0\}}{\sim}$ is well-defined.

If we assume that $+$ and $\times$ are well-defined, then it is easy to verify that $\dfrac{\mathbb{Z}\times\mathbb{Z}\setminus\{0\}}{\sim}$ is indeed a ring with additive identity $\frac{0}{1}$ and multiplicative identity $\frac{1}{1}$. Moreover, it is easy to show that $\frac{a}{b} = \frac{0}{1} \Leftrightarrow a = 0$, so that

$\frac{a}{b} \neq \frac{0}{1} \Rightarrow a \neq 0 \Rightarrow a \in \mathbb{Z}\setminus\{0\} \Rightarrow \frac{b}{a} \in \dfrac{\mathbb{Z}\times\mathbb{Z}\setminus\{0\}}{\sim}$

$\Rightarrow \frac{a}{b}$ is a unit, since $\frac{a}{b}\cdot\frac{b}{a} = \frac{1}{1}$. Thus, $\dfrac{\mathbb{Z}\times\mathbb{Z}\setminus\{0\}}{\sim}$ is actually a field. We usually denote the field $\dfrac{\mathbb{Z}\times\mathbb{Z}\setminus\{0\}}{\sim}$ as $\mathbb{Q}$. The field $\mathbb{Q}$ is called the <u>field of fractions</u> of $\mathbb{Z}$.

In fact $\mathbb{Z}$ can be replaced by any integral domain D, and the resulting field $\dfrac{D\times D\setminus\{0_D\}}{\sim}$

is called the <u>field of fractions</u> of the domain D, usually denoted as $\mathcal{F}(D)$.

The construction of the field of fractions of a domain is a special instance of a more general process called the localization of a ring A by a multiplicative subset S. We will now define this construction.

Recall, that a subset S of a ring A is <u>multiplicative</u> iff $1 \in S$ and S is closed under multiplication in A.

Define a relation $\sim$ on $A \times S$ as follows–
$\forall (a_1, s_1), (a_2, s_2) \in A \times S$, $(a_1, s_1) \sim (a_2, s_2)$
$\iff \exists s \in S \mid s(s_2 a_1 - s_1 a_2) = 0$.
Again, it is easy to verify that $\sim$ is an equivalence relation. An equivalence class $[(a, s)]_\sim$ is denoted as $\frac{a}{s}$. As in the case of the construction of $\mathbb{Q}$

from $\mathbb{Z}$, one can define the structure of a ring on $\frac{A \times S}{\sim}$ by defining $+$ and $\times$ in the obvious way.

The ring $\frac{A \times S}{\sim}$ is called the <u>ring of fractions of A</u>

with respect to S, and is denoted as $S^{-1}A$. The process of forming the ring $S^{-1}A$ is called the localization of A at S.

Note that $S^{-1}A$ is usually a ring and not a field.

It should be clear that the construction of $\mathbb{Q}$ from $\mathbb{Z}$, and in general, of $F(D)$ from a domain D are particular instances of the process of localization. (What are the multiplicative subsets in these cases?)

Exercise: Given a ring A and a multiplicative set $S \subset A$,
$$S^{-1}A = 0 \iff 0_A \in S.$$

Exercise: If D is a field, then $F(D) \cong D$.

There is a natural projection map $\pi : A \longrightarrow S^{-1}A$ given by $\pi(a) := \dfrac{a}{1}$. This is clearly a ring homomorphism. Thus, $S^{-1}A$ is an A-algebra, and so, also an A-module. We will show later that $S^{-1}A$ is a very special type of A-algebra which satisfies a certain universal property. But, first it is of utmost importance to discuss an important example of localization, which will also perhaps illustrate why the process is called localization.

An important example of localization:

Recall that a local ring A is a ring with a unique maximal ideal m. Often such a ring is denoted as (A, m, k) where $k = A/m$ is called the residue field.

As the name "localization" suggests, there is an important connection between localization, and the process of forming local rings.

Let A be a ring. Then $\mathrm{Spec}(A)$ is the topological space whose underlying set is the set of all prime ideals of A, and whose topology is the Zariski topology (go back to one of my earlier lectures. I defined what the Zariski topology is).

Now, $\forall$ prime ideals $p \in \mathrm{Spec}(A)$, $A - p$ is a multiplicative set (this follows from the definition of a prime ideal). Thus, one can form the ring of fractions $(A-p)^{-1}A$, which is usually denoted as A_p. We will now show that A_p is a local ring for each $p \in \operatorname{Spec}(A)$. We will state a few lemmas and a proposition (whose proofs we will only give in sketches) which will help us to prove that A_p is indeed local for all $p \in \operatorname{Spec}(A)$.

Recall that if M is an A-module, $\alpha \subset A$ an ideal then $\alpha M = \left\{ \sum_{i=1}^{n} \alpha_i m_i \mid i \in \mathbb{N}, \alpha_i \in \alpha, m_i \in M \right\}$ is an A-submodule of M.

__Lemma 1:__ Let A be a ring, $S \subset A$ a multiplicative set and $\operatorname{Spec}(A)_S := \{ p \in \operatorname{Spec}(A) \mid p \cap S = \phi \}$. Then $\operatorname{Spec}(A)_S$ is a topological space with subspace topology. Prove that the map

$$\varphi : \operatorname{Spec}(A)_S \longrightarrow \operatorname{Spec}(S^{-1}A)$$

$$\varphi(p) \longmapsto p(S^{-1}A)$$

is a continuous map.

__Pf:__ Exercise (Note that the notation $p(S^{-1}A)$ makes sense because $S^{-1}A$ is an A-module)

$\boxtimes$

__Lemma 2:__ The map $\phi : \operatorname{Spec}(S^{-1}A) \longrightarrow \operatorname{Spec}(A)_S$ given by $\forall\, \mathfrak{P} \in \operatorname{Spec}(S^{-1}A)$,

$\phi(\bar{p}) := \{a \in A \mid \exists\, s \in S \text{ s.t. } a/s \in \bar{p}\}$, is a continuous map.

Pf: Exercise (Note you must prove that $\phi(\bar{p})$ as defined is actually an element of $\mathrm{Spec}(A)_S$.

$\boxtimes$

Proposition 3: Show that $\mathrm{Spec}(A)_S$ and $\mathrm{Spec}(S^{-1}A)$ are homeomorphic.

Pf: Prove that $\phi \circ \varphi = id_{\mathrm{Spec}(A)_S}$ and

$\varphi \circ \phi = id_{\mathrm{Spec}(S^{-1}A)}$ where φ, ϕ are the maps defined in lemma 1, lemma 2 respectively.

$\boxtimes$

Proposition 4: For all $p \in \mathrm{Spec}(A)$, A_p is a local ring.

Pf: Let $p \in \mathrm{Spec}(A)$. It follows from lemma 1 that $p(A_p)$ is a prime ideal.

Let $m \in \mathrm{Spec}(A_p)$ be any maximal ideal. We know by Zorn's lemma that such an m exists since $p(A_p) \subsetneq A_p$. Since $m \in \mathrm{Spec}(A_p)$, $\phi(m) \in \mathrm{Spec}(A)_{A-p}$. Hence,

$\phi(m) \cap (A-p) = \phi \implies \phi(m) \subset p \implies \phi(m)(A_p) \subset p(A_p)$

By proposition 3, $\phi(m)(A_p) = \varphi \circ \phi(m) = m$. Thus,

$m \subset p(A_p)$, and so by the maximality of m, $m = p(A_p)$.

Hence, $p(A_p)$ is the only maximal ideal of A_p, i.e., A_p is a local ring.

$\boxtimes$

You can now perhaps guess how localization got its name.

We talked earlier about the field of fractions $F(D)$ of a domain D.

Now, it is easy to show that a ring D is an integral domain $\iff$ (0) is a prime ideal in D.

It follows that $F(D)$ is just $D_{(0)}$ when D is a domain, with unique maximal ideal $(0)(D_{(0)}) = (0)$, proving that $F(D)$ is indeed a field.

<u>Exercise:</u> If $R \neq 0$ is a ring such that (0) is a maximal ideal, then R is a field.

There is another important example of localization that we should keep in mind.

Let A be a ring. Then $\forall f \in A$ consider the set

$\{f^n : n \geq 0 \text{ and } n \in \mathbb{Z}\}$. The convention is that $f^0 = 1$. Then, $\{f^n : n \geq 0 \text{ and } n \in \mathbb{Z}\}$ is a multiplicative set, and the localization of A at $\{f^n : n \geq 0 \text{ and } n \in \mathbb{Z}\}$ is denoted as A_f.

We noted before that $S^{-1}A$ is an A-algebra with a natural projection map

$$a \longmapsto \frac{a}{1} : A \longrightarrow S^{-1}A.$$

Formally, given a ring A, an <u>A-algebra</u> is a 2-tuple (B, φ) where B is a ring and $\varphi : A \longrightarrow B$ is a ring map.

We can define the localization $S^{-1}A$ as a universal object in a category.

<u>Def^n</u>: Let A be a ring and $S \subset A$ be a multiplicative set. Define the category $\mathcal{C}_A$ whose objects are A-algebras (B, φ) s.t. $\forall s \in S$, $\varphi(s)$ is a unit in B. Given objects (B_1, φ_1),

$(B_2, \varphi_2) \in \text{ob}\, \mathcal{C}_A$, a morphism of $\mathcal{C}_A$ is a ring map $\phi : B_1 \longrightarrow B_2$ such that the following diagram commutes -

$$\begin{tikzcd} & A \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] \arrow[d, phantom, "\circlearrowright"] & \\ B_1 \arrow[rr, dashed, "\phi"'] & & B_2 \end{tikzcd}$$

Fig 1

A <u>localization of A at S</u> is defined to be an initial object in $\mathcal{C}_A$.

Note that since initial objects are unique up to unique isomorphisms, hence, a localization of A at S is unique up to a unique isomorphism.

<u>Exercise:</u> $(S^{-1}A, \pi)$ where π is the projection map, is an initial object in $\mathcal{C}_A$.

So far, we have just been looking at localization of rings. But, the method of localization can be generalized to modules over a ring as well. Since every ring is trivially a module over itself, the localization of a ring then becomes a special instance of the localization of a module. We will first define the localization of a module concretely, and then categorically.

Localization of a module (Concrete definition):

Let M be a module over a commutative ring A and $S \subset A$ be a multiplicative set. We define a relation $\sim$ on $M \times S$ as we have done before –

$\forall\ (m_1, s_1), (m_2, s_2) \in M \times S,\ (m_1, s_1) \sim (m_2, s_2)$
$\iff \exists\, s \in S \mid s(s_2 m_1 - s_1 m_2) = 0$.

Again, as an exercise, one can verify that $\sim$ is an equivalence relation. An equivalence class, $[(m,s)]_\sim$ is denoted as $\dfrac{m}{s}$. We can define an A-module structure on $\dfrac{M \times S}{\sim}$ as follows –

- $\dfrac{m_1}{s_1} + \dfrac{m_2}{s_2} := \dfrac{s_2 m_1 + s_1 m_2}{s_1 s_2}$ (addition)
- $a \cdot \dfrac{m}{s} := \dfrac{am}{s}$. (scalar multiplication)

As an exercise one can verify that the operations of addition and scalar multiplication are well defined.

The A-module $\underset{\sim}{\underline{M \times S}}$ is denoted as $S^{-1}M$ and is called the localization of M at S.

Exercise: Verify that $S^{-1}M$ is an $S^{-1}A$ module with module action given by

$$\frac{a}{s} \cdot \frac{m'}{s'} := \frac{am'}{ss'}$$

Localization of a module (Categorical definition):

Given an A-module M, and a multiplicative set $S \subset A$, we would like to define the localization of M at S categorically, in a manner similar to the categorical definition of the localization of a ring.

If $\mu : A \times M \to M$ denotes the module action, then $\forall\, a \in A$, define $\mu_a : M \to M$ to be left multiplication by a, i.e., $\mu_a(m) := am$. Then, given $a, a' \in A$,

$\mu_{aa'} = \mu_a \circ \mu_{a'}$. If $a \in A$ is a unit, i.e., a^{-1} exists, then $\mu_a \circ \mu_{a^{-1}} = \mu_{aa^{-1}} = \mu_1 = id_M = \mu_1 = \mu_{a^{-1}a} = \mu_{a^{-1}} \circ \mu_a$. Thus, μ_a is an isomorphism with

$\mu_a^{-1} = \mu_{a^{-1}}$.

Since there are no units in a module, we would like to use the maps μ_a (where μ_a is an isomorphism), instead, to define the localization of a module categorically.

As before, let M be an A-module and $S \subset A$ a multiplicative set. Define the category $\mathcal{C}_M$ whose objects are 2-tuples (N, φ) where N is an A-module with module action μ^N and $\varphi : M \to N$ is an A-linear map, such that $\forall\, s \in S$, the left multiplication map $\mu_s^N : N \to N$ is an isomorphism. Given objects $(N_1, \varphi_1), (N_2, \varphi_2) \in \mathrm{ob}\, \mathcal{C}_M$, a morphism of $\mathcal{C}_M$ is an $S^{-1}A$-linear map $\phi : N_1 \to N_2$ such that the following diagram commutes –

$$\begin{tikzcd} & M \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] \arrow[d, phantom, "\circlearrowleft"] & \\ N_1 \arrow[rr, dashed, "\phi"'] & {} & N_2 \end{tikzcd}$$

Fig 2.

A <u>localization of M at S</u> is an initial object in $\mathcal{C}_M$. The condition that $\forall s \in S$, μ_s^N is an iso. secretly means that N is an $S^{-1}A$ module with action given by $a/s \cdot n := (\mu_s^N)^{-1}(an)$.

Exercise: Prove that $\pi : M \longrightarrow S^{-1}M$

$$m \longmapsto \frac{m}{1}$$

is an A-linear map and $(S^{-1}M, \pi)$ is an initial object in $\mathcal{C}_M$.

The localization functor:

Recall that given an A-module M, and a multiplicative set $S \subset A$, $S^{-1}M$ is an A-module.

Let $\varphi : M \to N$ be an A-linear map. Then we get an induced map $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ given by $S^{-1}\varphi\left(\frac{m}{s}\right) := \frac{\varphi(m)}{s}$. (Check that this is well-defined).

Thus, we get a functor $S^{-1}\_ : A\text{-mod} \longrightarrow A\text{-mod}$ such that $\forall\, M \in ob(A\text{-mod})$, $S^{-1}\_(M) := S^{-1}M$ and $\forall\, \varphi \in Hom(M,N)$, $S^{-1}\_(\varphi) := S^{-1}\varphi$. This functor is called the localization functor.

If $M \xrightarrow{\phi} N \xrightarrow{\varphi} P$ is an exact sequence of A-modules, then it is an easy exercise to verify that the sequence

$$S^{-1}M \xrightarrow{S^{-1}\phi} S^{-1}N \xrightarrow{S^{-1}\varphi} S^{-1}P$$

is also exact. Thus, the localization functor preserves exact sequences, and so is an example of an <u>exact functor</u>, i.e., a functor that preserves exact sequences.

We will conclude this lecture note by giving another construction of the localization of a ring over a multiplicative set. I came across it in Mel Hochster's notes on Commutative Algebra.

Let R be a ring and $S \subset R$ be a multiplicative set. For all $s \in S$, let x_s be an indeterminate, and let $R[x_s : s \in S]$ be the ring of polynomials in the indeterminates x_s with coefficients in R. Define $A := R[x_s : s \in S]$. Let I be the ideal generated by elements of the form $sx_s - 1$ for all $s \in S$. We have natural projections $R \hookrightarrow A \twoheadrightarrow A/_I$. Let

$\pi : R \to A/_I$ denote the composition of these projection maps. Thus, $\forall r \in R$, $\pi(r) = r + I$. We will show that $A/_I \cong S^{-1}R$.

Let $\varphi : R \to T$ be a ring homomorphism such that $\forall s \in S$, $\varphi(s)$ is a unit in T.

Define the map $\overline{\varphi} : A \longrightarrow T$ by mapping each x_s to $\varphi(s)^{-1}$ and each $r \in R$ to itself. $\overline{\varphi}$ is clearly a ring homomorphism. It is easy to verify that $I \subset \ker \overline{\varphi}$ so that the induced ring map $\overline{\overline{\varphi}} : A/_I \longrightarrow T$ given by $\overline{\overline{\varphi}}(z+I) = \overline{\varphi}(z)$ (where $z \in A$) is well defined.
Now, $\forall r \in R$, $\overline{\overline{\varphi}} \circ \pi(r) = \overline{\overline{\varphi}}(r+I) = \overline{\varphi}(r) = \varphi(r)$. Thus, $\overline{\overline{\varphi}} \circ \pi = \varphi$. Also, if $\gamma : A/_I \longrightarrow T$ is a ring map such that $\gamma \circ \pi = \varphi$, then $\forall r \in R$, $\gamma(r+I) = \varphi(r)$. Thus, $\forall s \in S$, $\gamma(sx_s + I) = \gamma(1+I)$ (Recall $sx_s - 1 \in I$) $= \varphi(1) = 1 \Rightarrow \forall s \in S$, $\gamma(s+I)\gamma(x_s+I) = 1$ $\Rightarrow \forall s \in S$, $\varphi(s)\gamma(x_s+I) = 1 \Rightarrow \forall s \in S$, $\gamma(x_s+I) = \varphi(s)^{-1}$. Thus, $\gamma = \overline{\overline{\varphi}}$, and so $\overline{\overline{\varphi}}$ is unique.

By the universal property of $S^{-1}R$ mentioned earlier, $A/_I \cong S^{-1}R$.