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Sperner Posets and Young's Lattice

Abstract

Partially ordered sets are ubiquitous in mathematics. In this lecture, we will prove that both (i) the finite Boolean algebra $2^{[n]}$ and (ii) $L(m, n)$, the set of all partitions with at most m parts and largest part at most n, both satisfy the Sperner property: they are graded posets for which no antichain is larger than the largest rank level. Along the way we will encounter the poset of Young diagrams, quotient posets, q-binomial coefficients, and a few proofs characteristic of enumerative combinatorics. As time permits, we explore the probabilistic treatment of Young diagrams in the asymptotic representation theory of the symmetric group.

Combinatorics is the enumerative and extremal study of discrete structures. This area boasts a vast array of applications touching nearly every branch of mathematics. From what I've seen, the concrete objects and methods of combinatorics lend a powerful organized approach to the *quantitative* aspects of any mathematical structure, in the same way that category theory is a way of understanding the intrinsic *qualitative* nature of structures as we have defined them. In this talk, I hope to illustrate the style of the former, as we aren't exposed to it in our coursework. For now, ponder:

- What is the largest collection $\{S_i\}_{i \in I}$ of subsets $S_i \subset [n]$ of $[n] = \{1, 2, \ldots, n\}$ with the property that we never have $S_i \subset S_j$ $\forall i, j \in I$?

1 Posets and the Sperner's Theorem

1.1 Properties and Examples of Posets

A *partially ordered set* $(P, \leq)$ or "poset" is a set P together with a binary relation "$\leq$" which is a subset of $P \times P$ satisfying (i) $x \leq x$ reflexivity (ii) $x \leq y, y \leq x \Rightarrow x = y$ symmetry and (iii) $x \leq y, y \leq z \Rightarrow x \leq z$ transitivity. Write $y \geq x$ if $x \leq y$, $x < y$ if $x \leq y$ and $x \neq y$. We say $m \in P$ is *minimal* if there is no $x \in P$ satisfying $x < m$, and define $M \in P$ *maximal* similarly.

Before we look at some examples, let us give a precise graphical interpretation of some posets. For $x, y \in P$, we say x *covers* y if $x < y$ and there is no $z \in P$ satisfying $x < z < y$. Note that $\mathbb{R}$ has no cover relations; we say that a poset where all order relations come from cover relations and transitivity is *locally finite*. To any locally finite poset P, associate a graph $\Gamma(P)$, the *Hasse diagram*, with vertices P and a directed edge x to y if y covers x. Here are some examples (not all locally finite):

- $\{a, b, c, d, e\}$ with $a \leq c, b \leq c, c \leq d, c \leq e$ (two max, two min).

- $\mathbb{N}$ under usual $\leq$ (min elt 0)
- $\mathbb{N}$ under divisibility $|$: say $n_1 \leq n_2$ if there is some $a \in \mathbb{N}$ such that $n_2 = n_1 \cdot a$ (min elt 1, max elt 0)
- Given a set X, then 2^X under $\subseteq$ (min $\emptyset$, max X). For some set S of size $|S| = n$, without loss of generality take $S = [n]$, then $B_n := 2^S$ is called the (finite) boolean algebra of rank n. Draw the Hasse diagram for $B_3 = 2^{[3]}$.
- The set $\mathbb{T}(X)$ of all topologies on a set X (finer, coarser; indiscrete and discrete tops are min and max). Recall that the product topology on $X \times Y$ is at a nice place in this poset: (coarsest such that projections π_X, π_Y are continuous, finest such that any product $(f, g) : Z \to X, Y$ of continuous maps $f : Z \to X$, $g : Z \to Y$ is continuous).
- Galois connections (field extensions and subgroups of automorphism groups, covering spaces, Hilbert's Nullstellensatz) - examples of poset isomorphisms.

We say that two posets P and Q are *isomorphic* if there is a bijection of sets $\varphi : P \to Q$ such that $x \leq y$ in P iff $\varphi(x) \leq \varphi(y)$ in Q. Draw 5 isomorphism classes of posets on [3] (this number is not known for arbitrary n).

We now describe chains and anti-chains in posets in order to describe the Sperner property. A *chain* C in a poset P is a totally ordered subset of P, i.e., $x, y \in C \Rightarrow x \leq y$ or $y \leq x$. A finite chain is said to have *length* n if it has $n+1$ elements: these are of the form $x_0 < \cdots < x_n$. A poset is *graded of rank* n if every maximal chain as length n (a maximal chain being a chain contained in no larger chain). A finite chain $y_0 < \cdots < y_n$ is *saturated* if each y_{i+1} covers y_i. This allows us to define a "height" function on P: if P is graded of rank n and for $x \in P$ every saturated chain of P with top element x has length j, then define $\rho : P \to \mathbb{N}$ by $\rho(x) = j$, call this the *rank* of x. Then writing $P_j = \{x \in P : \rho(x) = j\}$ to be the jth *rank-level*, we have $P = \coprod_{j=1}^n P_j$ when P is graded of rank n. In fact, every maximal chain is of the form $x_0 < x_1 < \cdots < x_n$ with $x_i \in P_i$. Define $p_j = |P_j|$.

- For B_n, $\rho(x) = |x|$ the cardinality of $x \subset [3]$ and $p_j = \binom{n}{j}$. Look at B_3.

As B_n is such a natural poset, some definitions characterizing it are of importance: a graded poset P of rank n is *rank-symmetric* if $p_i = p_{n-i}$ for $0 \leq i \leq n$ and *rank-unimodal* if $p_0 \leq p_1 \leq \cdots \leq p_j \geq p_{j+1} \geq \cdots \geq p_n$ for some $0 \leq j \leq n$.[1]

Just one more definition: an *antichain* in a poset P is a subset $A \subset P$ for which no two elements are comparable, i.e. never have $x, y \in A$ and $x < y$. For instance, the rank-levels P_j are antichains. We'd like to know: **what is the largest antichain in a poset**?

[1]If the sequence p_j is alone or as the coefficients of a polynomial, we say that the corresponding sequence or polynomial is *symmetric* or *unimodal*.

1.2 Sperner's Theorem

Finding the largest antichain in B_n is exactly finding the largest collection of subsets of $[n]$ such that no element of the collection contains another.

- Good guess: All subsets of $[n]$ of size $\lfloor n/2 \rfloor$? This would give a total of $\binom{n}{\lfloor n/2 \rfloor}$ sets in all...

Answer: (Sperner 1927) This is correct. Before we prove this, we can extend this result to any poset by a crucial definition:

Definition 1 *Let $P = \coprod_{i=1}^{n} P_i$ be a graded poset of rank n with rank-levels P_i and let $\mathcal{A}(P)$ denote the set of all antichains on P. We say that P has the Sperner property if*

$$\max\{|A| \ : \ A \in \mathcal{A}(P)\} = \max\{|P_i| \ : \ 0 \leq i \leq n\}.$$

In other words, no antichain is larger than the largest level P_i.

Here's a quick example of a poset that fails to satisfy the Sperner property: draw $a \leq A, a \leq B, a \leq C, b \leq C, c \leq C$. Recall that for any poset, the rank-levels are antichains, and for B_n, $|P_j| = \binom{n}{j}$, hence $j = \lfloor n/2 \rfloor$ is on our list.

Theorem 1 *(Sperner 1927) B_n has the Sperner property.*

Proof (Lubell 1966) First, the number of maximal chains $\emptyset = x_0 < x_1 < \cdots x_n = [n]$ in B_n: there are n choices for x_1, $n-1$ choices for x_2, etc, so there are $n!$ maximal chains in all. Given a fixed $x \in P_i$, the number of maximal chains containing x is $i!(n-i)!$. Now choose $A \in \mathcal{A}(B_n)$. If $x \in A$, let C_x be the set of maximal chains of B_n which contain x. Since A is an antichain, the set of C_x for $x \in A$ are pairwise disjoint. Thus

$$|\bigcup_{x \in A} C_x| = \sum_{x \in A} |C_x| = \sum_{x \in A} (\rho(x)!)(n - \rho(x))!$$

Since the total number of maximal chains in the C_xs cannot exceed the total number $n!$ of maximal chains in B_n, we have $\sum_{x \in A} (\rho(x)!)(n-\rho(x))! \leq n!$ which is

$$\sum_{x \in A} \frac{1}{\binom{n}{\rho(x)}} \leq 1.$$

Since $\binom{n}{i}$ is maximized when $i = \lfloor n/2 \rfloor$, we have

$$\frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq \frac{1}{\binom{n}{\rho(x)}}$$

for all $x \in A$ so $\sum_{x \in A} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq 1$ which is

$$|A| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Since $\binom{n}{\lfloor n/2 \rfloor}$ is the size of the largest level of B_n, it follows that B_n is Sperner. □

2 Young Diagrams and q-binomial coefficients

2.1 Young diagrams

A *partition* λ of an integer $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_i \geq 0$, $\lambda_1 \geq \lambda_2 \geq \cdots$, and $\sum_{i \geq 1} \lambda_i = n$. We suppress the 0s at the end from the notation. For example, 5 has seven partitions: 5, 41, 311, 32, 221, 2111, 11111. We can visualize a partition λ as a *Young diagram*.

- Draw for $5 \geq 3 \geq 2 \geq 2 \geq 1$ of 13, English and Russian style.

To every λ, the dual partition λ' from flipping about the main diagonal gives another partition of n. Let $\mathbb{Y}$ denote the set of all partitions, which we now turn into a poset: define $\lambda \leq \mu$ if $\lambda_i \leq \mu_i \; \forall i$. This is an important object because it is both a *lattice* and a *differentiable poset*.[2]

- Draw Young's lattice.

Before we work with these diagrams, let us mention some of their uses. Young diagrams come up all over the place in representation theory, the simplest case of this being representations of $S(n)$. For a finite group G, we know that its conjugacy classes are in bijection with its irreducible representations, but not necessarily canonically. Looking at induction and restriction in the representations of $S(n)$, we are able to get a canonical bijection, because its conjugacy classes are exactly cycle types, which are in bijection with partitions of n. For example, (132)(45) and (145)(23) are conjugate in $S(5)$ and correspond to the partition $3 + 2$ of 5.

The asymptotic representation theory of the symmetric group uses $\mathbb{Y}$ in some beautiful ways. The infinite symmetric group - those permutations of $\mathbb{N}$ leaving almost every n fixed - is the direct limit of the chain $S(1) \subset S(2) \subset \cdots$. $S(\infty)$ is an important case study in non-commutative harmonic analysis, since it is neither compact nor abelian. For example, characters of $S(\infty)$ are exactly non-negative harmonic functions on the graph $\mathbb{Y}$, which are determined by a Poisson integral taken over the boundary of $\mathbb{Y}$ much like the unit circle in $\mathbb{C}$, involving a "Plancherel measure" on $\mathbb{Y}$ and stochastic point processes. By looking at the representations of the infinite dimensional Hecke algebra, it was realized that imposing certain restrictions on the characters of $S(\infty)$ give back exactly the Jones polynomial. Reference: Kerov *Asymptotic Representation Theory of the Symmetric Group*. Also, the Russian way is useful: the cusps on the profile of a (random) tableaux projected down to the x-axis correspond to the spectrum of eigenvalues of certain random matrix ensembles. Also, though I don't know much about this, partitions enumerate monomial ideals, which arise in the study of the Hilbert scheme of points in the plane. Reference to spark your interest: Okounkov *The uses of random partitions*.

[2] A lattice is a poset in which any two elements have a unique supremum and infimum (join and meet). A poset P is differential if it is (1) locally finite and graded with a unique minimal element (2) $x \neq y$ two elts of P and k elts of P covered by both x and y, ten there are exactly k elts of P which cover both x and y (3) $x \in P$ covers k elts of P then x is covered by exactly $k+1$ elts of P. Define $\mathbb{C}\,P = \bigoplus_{x \in P} \mathbb{C}\,x$ then we can define linear transformations here...

2.2 Group Actions and Quotient Posets

Recall that an action of a group G on a set X is a map $G \times X \to X$ satisfying (i) $g \cdot (h \cdot x) = (gh) \cdot x$ and (ii) $e \cdot x = x$ where e is the identity element of the group. Under this action, $G = \coprod_{i \in I} \mathcal{O}_i$ breaks up into *orbits*: the orbit of x is the set $Gx = \{g \cdot x \in X : g \in G\}$. Let X/G denote the set of orbits under this action.

- Quick examples: the tiny cyclic group $\langle g : g^2 = 1\rangle$ acts on a pentagon by reflection about a diagonal, with orbits $\{a\}, \{b, e\}, \{c, d\}$.

- $\mathbb{R}$ (or S^1) acts on $\mathbb{R}^2$ by rotation about angle of $\alpha \in \mathbb{R}$ radians. This action has orbits given by the concentric circles of radius $r \geq 0$.

An automorphism of a poset is an isomorphism $P \to P$. The symmetric group $S(n)$ acts on B_n by permuting $[n]$, each $\sigma \in S(n)$ of which induces an automorphism of B_n (exercise). For $G \subset S(n)$ a subgroup, the *quotient poset* B_n/G are the orbits of G, and define $\mathcal{O} \leq \mathcal{O}'$ if there are two elements $x \in \mathcal{O}$, $y \in \mathcal{O}'$ with $x \leq y$ in B_n.

- Draw $B_5/\langle(12345)\rangle$.

Proposition 1 *For any subgroup $G \subset S(n)$, the quotient poset B_n/G is graded of rank n, rank-symmetric, rank-unimodal, and Sperner.*

2.3 $L(m, n)$ is Sperner

Let $L(m, n)$ denote the set of all partitions with at most m parts and with largest part at most n. For instance $L(2, 3) = \{\emptyset, 1, 2, 3, 11, 21, 31, 22, 32, 33\}$. $L(m, n)$ is a finite set, and inherits a partial order from $\mathbb{Y}$.

- Draw $L(2, 3)$.

These have a natural geometric interpretation: they are the set of all Young diagrams that fit in a $m \times n$ rectangle, where we fix the northwest corners to agree. Draw $4 \geq 3 \geq 1$ in 4×5 rectangle. It is clear that $L(m, n) \cong L(n, m)$ as posets.

Proposition 2 *$L(m, n)$ is graded of rank mn and rank-symmetric.*

Indeed, the rank of a partition is just $|\lambda|$ the number of boxes. Looking at the complement of λ in $R_{m,n}$ gives rank-symmetry. Our main goal is to prove $L(m, n)$ is rank-unimodal and Sperner. We want to know $p_i(L(m, n))$, the number of elements of $L(m, n)$ of rank i. Though we don't need to, we count $|L(m, n)|$.

Proposition 3 $|L(m, n)| = \binom{m+n}{m}$

Proof: $\binom{m+n}{m}$ counts the number of sequences $a_1, \ldots, a_{m+n}$ where each a_j is either N or E and there are m Ns and hence n E's in total. To each sequence, associate a path in $R_{m,n}$, which makes the bijection apparent. □

To first see how many elements of $L(m,n)$ have rank i, we look at a deformation of our usual numerics by a q-analog, where the limit $q \to 1$ brings us back to where we began.[3] For this indeterminate q, given $j \geq 1$ in $\mathbb{N}$, define

$$[j] := 1 + q + q^2 + \cdots + q^{j-1}$$

so $[1] = 1$, $[2] = 1 + q$, and $[j] \to j$ as $q \to 1$. Then define $[j]! = [1][2]\cdots[j]$ and set $[0]! = 1$, and also

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{[k]!}{[j]![k-j]!}.$$

The expression $\begin{bmatrix} k \\ j \end{bmatrix}$ is called a q-binomial coefficient. You can compute:

Proposition 4 $\begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} k-1 \\ j \end{bmatrix} + q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}$, *whenever* $k \geq 1$, *with the initial conditions* $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} k \\ j \end{bmatrix} = 0$ *if* $j < 0$ *or* $j > k$.

Note that letting $q \to 1$ in this expression gives the defining recurrence for Pascal's triangle. Let us note that for $q = p^n$ a prime power, $\begin{bmatrix} k \\ j \end{bmatrix}$ counts the number of j-dimensional subspaces of a k-dimensional vector space over the finite field $\mathbb{F}_q$. Mention Hecke algebra and counting complete flags in $\mathbb{F}_q^n$.

Proposition 5 *Let* $p_i(m,n)$ *denote the number of elements of* $L(m,n)$ *of rank* i. *Then*

$$\sum_{i \geq 0} p_i(m,n) q^i = \begin{bmatrix} m+n \\ m \end{bmatrix}.$$

This is a *generating function.* Note that the sum on the left is truly finite since $p_i(m,n) = 0$ for $i > mn$, and that as $q \to 1$, we get $\sum_{i \geq 0} p_i(m,n) = |L(m,n)| = \binom{m+n}{m}$ as before. *Proof:* let $P(m,n)$ denote the LHS. We will show that (i) $P(0,0) = 1$ and $P(m,n) = 0$ if $m < 0$ or $n < 0$ and also (ii) $P(m,n) = P(m,n-1) + q^n P(m-1,n)$. These completely determine $P(m,n)$, and substituting $k = m+n$ and $j = m$ here shows that $\begin{bmatrix} m+n \\ m \end{bmatrix}$ also satisfy this recurrence and set of initial conditions. First (i) is true since $L(0,n)$ consists of a single point $\emptyset$ the empty partition, so $\sum_{i \geq 0} p_i(0,n) q^i = 1$, while $L(m,n)$ is empty when $m < 0$ or $n < 0$, so $P(m,n) = 0$ here. The beef is proving the recurrence, which is equivalent to proving that the coefficient of each q_i satisfies $p_i(m,n) = p_i(m,n-1) + p_{i-n}(m-1,n)$. A partition λ of i fitting in $R_{m,n}$ either contains the upper-right hand corner of $R_{m,n}$ (there are $p_i(m,n-1)$ such partitions) or it does contain it (deleting the first row, there are $p_{i-n}(m-1,n)$ such partitions of $i-n$ fitting in $R_{(m-1),n}$). $\square$

[3]This sort of method has huge appearances: hypergeometric series, fractals and fractal measures, entropy of chaotic dynamical systems, elliptic integrals, quantum groups and q-deformed superalgebras, and string theory.

Consider $S(mn)$ the rearrangements of the boxes of $R_{m,n}$. Define a subgroup $G_{mn} \subset S(mn)$ as follows: $\pi \in G$ is allowed to (i) permute the elements of each row of R_{mn} in any way and then (ii) permute the rows themselves. We have m rows of size n, which gives $n!^m$ permutations preserving the rows, so $|G_{mn}| = m!n!^m$. This group is called the *wreath product* of $S(n)$ and $S(m)$. We have a pretty lemma:

Lemma 1 *Every orbit $\mathcal{O}$ of the action of G_{mn} on the boolean algebra B_R contains exactly one Young diagram D.*

This is to say that $\mathcal{O}$ has exactly one subset $D \subset R_{mn}$ with D left-justified and $\lambda_1 \geq \cdots \geq \lambda_m$. *Proof*: Let S be some subset of $R_{m,n}$ with α_i elements in row i. If $\pi \in G_{mn}$ and $\pi \cdot S$ has β_i elements in row i, then $\beta_1, \ldots, \beta_m$ is a permutation of $\alpha_1, \ldots, \alpha_m$. There is a unique permutation satisfying $\lambda_1 \geq \cdots \geq \lambda_m$, which is the only possible Young diagram D in the orbit $\pi \cdot S$. This can be achieved: by left-justifying rows, then arranging in weakly-decreasing order, we've got D_λ. $\square$

Theorem 2 *The quotient poset $B_{R_{mn}}/G_{mn}$ is isomorphic to $L(m,n)$.*

Each element of B_R/G_{mn} contains a unique Young diagram D_λ, so the map φ : B_R/G_{mn} by $D_\lambda \mapsto \lambda$ is a bijection. We need to show that for $D_\lambda \in \mathcal{O}$, $D_{\lambda*} \in \mathcal{O}^*$, we have $D \subseteq D^*$ in the quotient iff $\lambda \leq \lambda^*$ in $L(m,n)$. Proof: if $\lambda \leq \lambda^*$, then obviously $D_\lambda \subseteq D_{\lambda^*}$. Conversely, given some $D \in \mathcal{O}$ and $D^* \in \mathcal{O}^*$ satisfying $D \subseteq D^*$, look at row lengths: $\lambda_i \leq \lambda_i^*$. $\square$

Corollary 1 *The posets $L(m,n)$ are rank-symmetric, rank-unimodal, and Sperner.*

The Sperner property of $L(m,n)$ can be stated in simple terms: the largest possible collection C of Young diagrams fitting in an $m \times n$ rectangle such that no diagram in C is contained in another diagram in C is obtained by taking all the diagrams of size $\frac{1}{2}mn$.

Citation: Richard Stanley's notes *Topics in Algebraic Combinatorics* (his website).