Minimal surfaces and the isoperimetric inequality

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The Isoperimetric Inequality in $\mathbb{R}^2$

Queen Dido's Problem: Minimize boundary length among all domains in the plane with a given area.

Theorem: Let E be a compact domain in $\mathbb{R}^2$ with smooth boundary ∂E. Then $|\partial E|^2 \geq 4\pi|E|$, and equality holds if and only if E is a disk.

Notation: $|E|$ denotes the area of E, $|\partial E|$ denotes the boundary length.

There are many different proofs. Hurwitz in 1902 found an elegant proof based on Wirtinger's inequality.

Wirtinger's inequality

Theorem: Suppose $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and

$$\int_0^{2\pi} f(s)\,ds = 0.$$

Then

$$\int_0^{2\pi} f(s)^2\,ds \leq \int_0^{2\pi} f'(s)^2\,ds.$$

Equality holds if and only if $f(s) = a\cos(s) + b\sin(s)$.

Proof of Wirtinger's inequality: Write $f(s)$ as a Fourier series.

Hurwitz's proof of the 2D Isoperimetric Inequality

We want to show that $|\partial E|^2 \geq 4\pi|E|$ for every domain E in $\mathbb{R}^2$.

Without loss of generality, we may assume that E is connected. (Otherwise, treat each connected component separately, and sum over all connected components.)

Without loss of generality, we may assume that ∂E is connected. (Otherwise, fill in the holes of E. This will increase area and decrease boundary length.)

Without loss of generality, we may assume that $|\partial E| = 2\pi$. (Otherwise, we dilate the domain E by a suitable factor.)

Without loss of generality, we may assume that the center of mass of ∂E is at the origin, so that $\int_{\partial E} x_i = 0$ for $i = 1, 2$. (Otherwise, we translate the domain E.)

Hurwitz's proof of the 2D Isoperimetric Inequality (continued)

Let $\alpha(s) = (\alpha_1(s), \alpha_2(s))$ denote a parametrization of ∂E by arclength, so that $|\alpha'(s)| = 1$.

Since $|\partial E| = 2\pi$, we can view $\alpha_1(s), \alpha_2(s)$ as 2π-periodic functions.

Since the center of mass of ∂E is at the origin,

$$\int_0^{2\pi} \alpha_i(s)\, ds = 0$$

for $i = 1, 2$. Wirtinger's inequality gives

$$\int_0^{2\pi} \alpha_i(s)^2\, ds \leq \int_0^{2\pi} \alpha_i'(s)^2\, ds$$

for $i = 1, 2$. Sum over $i = 1, 2$:

$$\int_0^{2\pi} |\alpha(s)|^2\, ds \leq \int_0^{2\pi} |\alpha'(s)|^2\, ds = 2\pi.$$

In other words,

$$\int_{\partial E} |x|^2 \leq 2\pi.$$

Hurwitz's proof of the 2D Isoperimetric Inequality (continued)

On the other hand, we can apply the divergence theorem to the position vector field x. This vector field has divergence 2. Therefore, the divergence theorem gives

$$2|E| = \int_E \text{div}(x) = \int_{\partial E} \langle x, \eta \rangle.$$

Here, η denotes the outward-pointing unit normal vector field along ∂E.

Consequently,

$$4\,|E| = 2\int_{\partial E} \langle x, \eta \rangle \leq \int_{\partial E} |x|^2 + \int_{\partial E} |\eta|^2 \leq 4\pi.$$

In the last step, we have used that $\int_{\partial E} |x|^2 \leq 2\pi$ and $\int_{\partial E} |\eta|^2 = |\partial E| = 2\pi$.

To summarize, $|E| \leq \pi$, which implies the isoperimetric inequality.

The isoperimetric inequality in higher dimension

Theorem: Let E be a compact domain in $\mathbb{R}^n$ with smooth boundary ∂E. Then

$$\frac{|\partial E|}{|\partial B^n|} \geq \left(\frac{|E|}{|B^n|}\right)^{\frac{n-1}{n}},$$

and equality holds if and only if E is a ball.

Notation: $|E|$ denotes the n-dimensional volume of E, $|\partial E|$ denotes the $(n-1)$-dimensional measure of the boundary ∂E.

$|B^n|$ denotes the volume of the unit ball in $\mathbb{R}^n$, $|\partial B^n|$ denotes the $(n-1)$-dimensional measure of the boundary of the unit ball.

There are many different proofs. These employ a wide range of techniques: Induction on the dimension, Knothe map, optimal mass transport, symmetrization techniques, geometric measure theory, etc.

Connection with the Brunn-Minkowski inequality

Theorem: Let E and F be compact subsets of $\mathbb{R}^n$, and let $E + F := \{x + y : x \in E,\, y \in F\}$. Then $|E + F|^{\frac{1}{n}} \geq |E|^{\frac{1}{n}} + |F|^{\frac{1}{n}}$.

Proved by Minkowski for convex sets, and in full generality by Lusternik (1935). Idea: Assume E and F are unions of finitely many boxes, argue by induction on the number of boxes.

Corollary: Let E be a compact subset of $\mathbb{R}^n$. Let

$$N_r(E) := E + rB^n = \{x + ry : x \in E,\, y \in B^n\}$$

denote the set of all points which have distance at most r from the set E. Then $|N_r(E)|^{\frac{1}{n}} \geq |E|^{\frac{1}{n}} + |B^n|^{\frac{1}{n}}\, r$.

For r small, $|N_r(E)| = |E| + |\partial E|\, r + O(r^2)$. Sending $r \to 0$ in the Brunn-Minkowski inequality gives $\frac{1}{n}\, |E|^{\frac{1}{n}-1}\, |\partial E| \geq |B^n|^{\frac{1}{n}}$, which is equivalent to the isoperimetric inequality.

Connection with the calculus of variations (Bernoulli, Euler, Lagrange)

We can think of the isoperimetric problem as a constrained variational problem. We minimize one functional (boundary area) while keeping the value of another functional (volume) fixed.

Idea: Apply a first derivative test.

Let E be a compact domain, and let V be a vector field. Let φ_s be a one-parameter family of diffeomorphisms such that $\varphi_0(x) = x$ and $\frac{d}{ds}\varphi_s(x)\Big|_{s=0} = V(x)$. Define $E_s := \varphi_s(E)$.

First order change in volume:

$$\frac{d}{ds}|E_s|\Big|_{s=0} = \int_{\partial E} \langle V, \nu \rangle.$$

First order change in boundary area:

$$\frac{d}{ds}|\partial E_s|\Big|_{s=0} = \int_{\partial E} H \, \langle V, \nu \rangle,$$

Here, H is the **mean curvature** of ∂E, ν is the unit normal, and $\langle V, \nu \rangle$ is the normal velocity.

The mean curvature and its geometric meaning

Mean curvature is the L^2 gradient of surface area.

Alternatively, the mean curvature can be characterized as the sum of the principal curvatures.

Consider an n-dimensional hypersurface Σ in $\mathbb{R}^{n+1}$ given as a graph of a height function u:

$$x \in \Sigma \iff x_{n+1} = u(x_1, \ldots, x_n).$$

If $\nabla u(0) = 0$ (i.e. if the tangent plane to Σ at 0 is horizontal), then the principal curvatures at 0 are the eigenvalues of the Hessian $D^2u(0)$. The mean curvature at 0 equals the trace of the Hessian $D^2u(0)$, i.e. the Laplacian $\Delta u(0)$.

However, this is **not** true for $\nabla u(0) \neq 0$. Mean curvature is a **nonlinear** operator. This is forced by the fact that the mean curvature must be invariant under rigid motions in $\mathbb{R}^{n+1}$.

Mean curvature and minimal surfaces

Definition: We say that a hypersurface Σ in $\mathbb{R}^{n+1}$ is a **minimal surface** if $H = 0$. This means that, if we deform Σ (while fixing the boundary $\partial\Sigma$), then the area is unchanged to first order.

Definition: We say that a hypersurface Σ in $\mathbb{R}^{n+1}$ is a **constant mean curvature surface** if $H = c$. This means that, if we deform Σ in such a way that the enclosed volume stays constant, then the area is unchanged to first order.

Many explicit examples of minimal surfaces in $\mathbb{R}^3$ are known, the simplest ones being the catenoid and the helicoid.

By the solution of Plateau's problem, every closed curve in $\mathbb{R}^3$ bounds at least one minimal surface.

The isoperimetric inequality on a 2D minimal surface in $\mathbb{R}^3$

Torsten Carleman (Upsala University, 1921): Does the isoperimetric inequality hold for minimal surfaces?

Theorem (Carleman): Let Σ be a compact two-dimensional **minimal** surface in $\mathbb{R}^3$ with boundary $\partial\Sigma$. If $\partial\Sigma$ is **connected**, then $|\partial\Sigma|^2 \geq 4\pi|\Sigma|$, and equality holds if and only if Σ is a flat disk.

This can be proved by generalizing Hurwitz's proof of the 2D isoperimetric inequality.

This classical proof uses in a crucial way the assumption that $\partial\Sigma$ is connected.

Proof of Carleman's theorem

Without loss of generality, we may assume that $|\partial\Sigma| = 2\pi$. We may further assume that the center of mass of $\partial\Sigma$ is at the origin. In other words, $\int_{\partial\Sigma} x_i = 0$ for $i = 1, 2, 3$.

By assumption, $\partial\Sigma$ is connected. Using Wirtinger's inequality in the same way as above, we obtain

$$\int_{\partial\Sigma} |x|^2 \leq 2\pi.$$

On the other hand, applying the divergence theorem to the position vector field in $\mathbb{R}^3$ gives

$$2\,|\Sigma| - \int_{\Sigma} H\,\langle x, \nu\rangle = \int_{\partial\Sigma} \langle x, \eta\rangle.$$

Here, H denotes the mean curvature, ν denotes the unit normal to Σ in $\mathbb{R}^3$, and η denotes the co-normal to $\partial\Sigma$ in Σ.

On a minimal surface, H vanishes. Hence

$$4\,|\Sigma| = 2\int_{\partial\Sigma} \langle x, \eta\rangle \leq \int_{\partial\Sigma} |x|^2 + \int_{\partial\Sigma} |\eta|^2 \leq 4\pi.$$

Thus, $|\Sigma| \leq \pi$. From this, the isoperimetric inequality follows.

The isoperimetric inequality for hypersurfaces in $\mathbb{R}^{n+1}$

Theorem (B. 2019): Let Σ be a compact n-dimensional hypersurface in $\mathbb{R}^{n+1}$ with boundary $\partial\Sigma$. Then

$$\frac{|\partial\Sigma| + \int_\Sigma |H|}{|\partial B^n|} \geq \left(\frac{|\Sigma|}{|B^n|}\right)^{\frac{n-1}{n}},$$

and equality holds if and only if Σ is a flat disk.

Corollary (B. 2019): Let Σ be a compact n-dimensional **minimal** hypersurface in $\mathbb{R}^{n+1}$ with boundary $\partial\Sigma$. Then

$$\frac{|\partial\Sigma|}{|\partial B^n|} \geq \left(\frac{|\Sigma|}{|B^n|}\right)^{\frac{n-1}{n}},$$

and equality holds if and only if Σ is a flat disk.

No assumptions on the topology of Σ are needed.

A Brunn-Minkowski-type inequality for minimal hypersurfaces in $\mathbb{R}^{n+1}$

Corollary (B. 2019): Let Σ be a compact n-dimensional **minimal** hypersurface in $\mathbb{R}^{n+1}$ with boundary $\partial\Sigma$. Let E be a compact subset of Σ, and let

$$N_r(E) = E + rB^{n+1} = \{x + ry : x \in E,\, y \in B^{n+1}\}$$

denote the set of all points in ambient space $\mathbb{R}^{n+1}$ which have distance at most r from the set E. If $\partial\Sigma \cap N_\rho(E) = \emptyset$, then

$$|\Sigma \cap N_r(E)|^{\frac{1}{n}} \geq |E|^{\frac{1}{n}} + |B^n|^{\frac{1}{n}}\, r$$

for $0 < r < \rho$.

Note: The Brunn-Minkowski-type inequality can be deduced from the isoperimetric inequality for minimal surfaces.

Conversely, the Brunn-Minkowski-type inequality implies the isoperimetric inequality for minimal surfaces by sending $r \to 0$.

The special case when E consists of a single point

Let Σ be a compact n-dimensional **minimal** hypersurface in $\mathbb{R}^{n+1}$ with boundary $\partial\Sigma$.

Let p be an arbitrary point on Σ. Let $B_r(p)$ denote the ball of radius r around p in ambient space $\mathbb{R}^{n+1}$, and suppose that $\partial\Sigma \cap B_\rho(p) = \emptyset$.

Applying the Brunn-Minkowski-type inequality with $E = \{p\}$ gives

$$|\Sigma \cap B_r(p)|^{\frac{1}{n}} \geq |B^n|^{\frac{1}{n}} \, r$$

or

$$|\Sigma \cap B_r(p)| \geq |B^n| \, r^n$$

for $0 < r < \rho$. This recovers a classical theorem in minimal surface theory.

Idea of proof

The proof of the isoperimetric inequality for hypersurfaces relies on the Alexandrov-Bakelman-Pucci maximum principle in PDE theory, building on earlier work of Alexandrov, Bakelman, Trudinger, Cabré.

Alternatively, we can prove the result using techniques from optimal mass transport. A novel feature is that we need to consider transport problems between spaces of different dimension: we consider the optimal transport map from $(B^{n+1}, \frac{1}{\sqrt{1-|\xi|^2}}\, d\xi)$ to $(\Sigma^n, d\text{vol})$.

Further applications

A similar inequality holds if we replace the ambient space $\mathbb{R}^{n+1}$ by a noncompact manifold with nonnegative curvature. In that case, the estimate depends on the volume growth at infinity of the ambient manifold.

A similar technique gives a sharp logarithmic Sobolev inequality for hypersurfaces (B. 2019). Here, we consider the optimal transport map from $(\mathbb{R}^{n+1}, (4\pi)^{-\frac{n+1}{2}} \, e^{-\frac{|\xi|^2}{4}} \, d\xi)$ to $(\Sigma^n, d\text{vol})$.