The Geometry of Light

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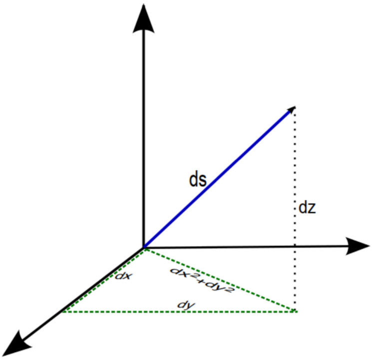
Euclidean Geometry, $\mathbb{R}^4$



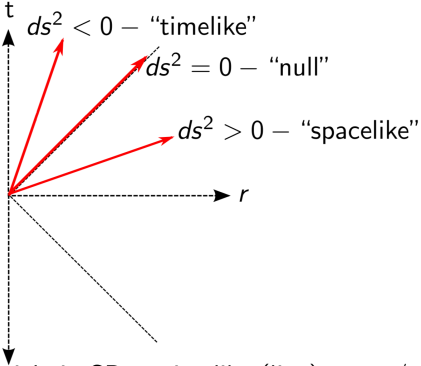
In 3D:
$ds^2 = dx^2 + dy^2 + dz^2$

In 4D:
$ds^2 = dt^2 + dx^2 + dy^2 + dz^2$

Minkowski Geometry, $\mathbb{R}^4_1$

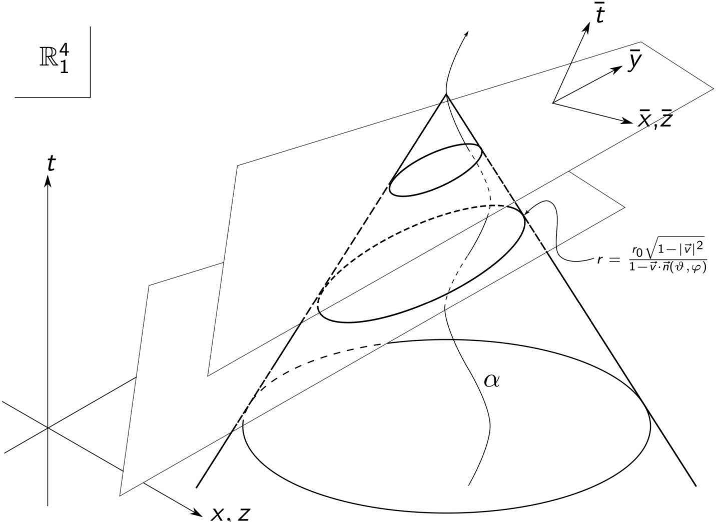
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$(spherical) = -dt^2 + dr^2 + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2)$$

$\xRightarrow{1905}$ Special Relativity (speed of light $c = 1$)



(free falling) particle in SR $\leftrightarrow$ timelike (line) curve $\langle \alpha', \alpha' \rangle < 0$

light-ray in SR $\leftrightarrow$ null line $\langle \beta', \beta' \rangle = 0$



For $\vec{v} \in B^3$ we have a "boost":

$$(t, x, y, z) \xrightarrow{\phi_{\vec{v}}} (\bar{t}, \bar{x}, \bar{y}, \bar{z})$$

observers at rest $\rightarrow$ observers, relative velocity $\vec{v}$

General Relativity

Can generalize to:

$$ds^2 = \begin{pmatrix} dt & dx & dy & dz \end{pmatrix} \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{02} & g_{23} & g_{33} \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

- If signature $(-,+,+,+) \to$ "spacetime"
- If functions $g_{\alpha\beta}(t,x,y,z) \implies$ Curvature.

<u>In 1915 Einstein publishes:</u>

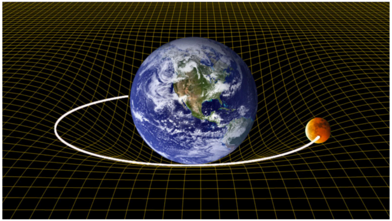
$$\text{Curvature of space} \quad \underset{\text{produces}}{\overset{\text{indicates}}{\rightleftarrows}} \quad \text{Matter content}$$

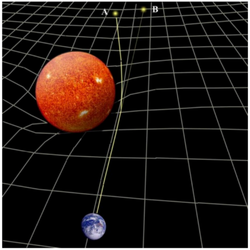
$$\boxed{G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}$$

(free-falling) particle in GR $\leftrightarrow$ (geodesic) timeline curves

light-ray in GR $\leftrightarrow$ null geodesics

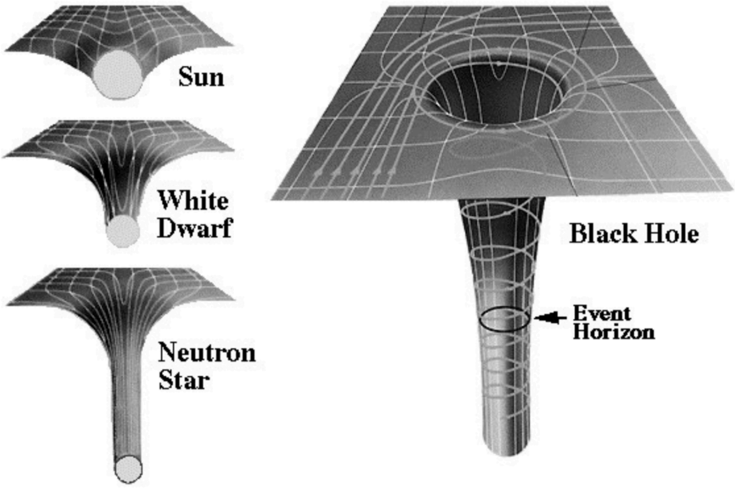
Observations



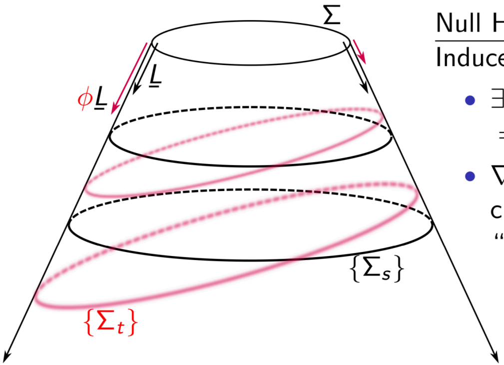


curvature affects dynamics $\Longrightarrow$ i.e. "gravity"

Singularities in Curvature $\stackrel{GR}{\Longrightarrow}$ Extreme Physics!



Null Cones



Null Hypersurface, Ω:

Induced metric is degenerate:

- $\exists \underline{L} \in T\Omega \cap T^{\perp}\Omega$ $\implies$ $\underline{L}$ null!
- $\nabla_{\underline{L}}\underline{L} = 0 \implies$ integral curves geodesic, "light-rays"

Nullcone:

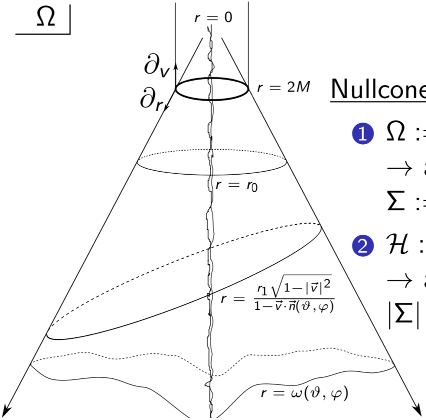
- $\Omega = \{$shoot light-rays off of $\Sigma \cong \mathbb{S}^2$ along $\underline{L}\}$ $\leftrightarrow$ foliation $\{\Sigma_s\} \subset \Omega$, $\underline{L}(s) = 1$
- $\underline{L} \to \phi\underline{L} \implies \{\Sigma_s\} \to \{\Sigma_t\}$ (i.e. $\phi\underline{L}(t) = 1$)
  if $\phi = \phi(\vartheta, \varphi) \implies$ "geodesic foliation" of Ω
  if $\phi = \phi(s, \vartheta, \varphi) \implies$ "general foliation" of Ω

Schwarzschild Geom., "Sph. Symmetric Vacuum"

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

$$= -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\vartheta^2 + \sin^2\theta d\varphi^2)$$

$$\text{where:} \quad dv = dt + (1 - \frac{2M}{r})^{-1}dr$$



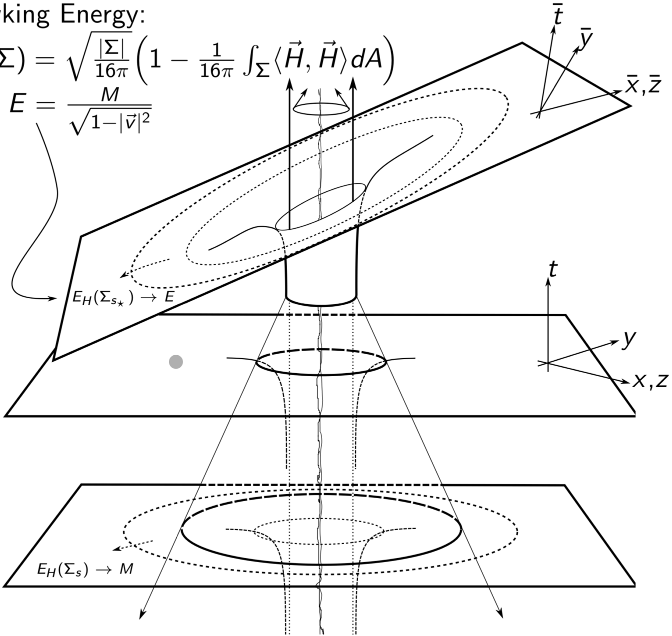
Nullcones

1. $\Omega := \{v = v_0\}$: $D_{\partial_r}\partial_r = 0$
   $\rightarrow$ any $\Sigma \hookrightarrow \Omega$ given by
   $\Sigma := \{r = \omega(\vartheta, \varphi)\}$
2. $\mathcal{H} := \{r = 2M\}$: $D_{\partial_v}\partial_v = 0$
   $\rightarrow$ all $\Sigma \hookrightarrow \mathcal{H}$ have area
   $|\Sigma| = 16\pi M^2! \Longrightarrow$ BH!!

Hawking Energy:

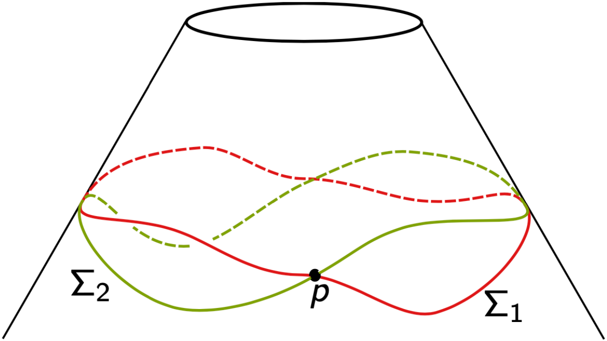
$$E_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \langle \vec{H}, \vec{H}\rangle dA\right)$$

$$E = \frac{M}{\sqrt{1-|\vec{v}|^2}}$$



Null Geometry

We denote $\gamma := ds^2|_\Omega$, $\underline{\chi} := \frac{1}{2}\mathcal{L}_{\underline{L}}\gamma$:

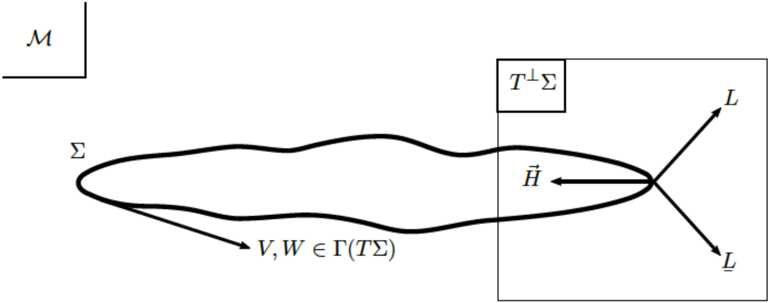


$$\underline{L} \in T\Omega \cap T^\perp\Omega \implies \gamma|_{\Sigma_1}(p) = \gamma|_{\Sigma_2}(p)$$
$$\nabla_{\underline{L}}\underline{L} = 0 \implies \underline{\chi}|_{\Sigma_1}(p) = \underline{\chi}|_{\Sigma_2}(p)$$

i.e. geometry of $\Sigma \subset \Omega$ "intrinsic" to Ω, determined "pointwise".

$\Sigma \subset \Omega$ admits a normal null basis $\{L, \underline{L}\} \subset T^{\perp}\Sigma$:

$$\langle L, L \rangle = \langle \underline{L}, \underline{L} \rangle = 0, \;\; \langle L, \underline{L} \rangle = 2$$



1. "extrinsic" geometry of Σ "intrinsic" to $\Omega \leftrightarrow \underline{\chi} := \frac{1}{2}\mathcal{L}_{\underline{L}}\gamma$
2. "extrinsic" geometry of Σ "extrinsic" to $\Omega \leftrightarrow \chi := \frac{1}{2}\mathcal{L}_{L}\gamma$
3. $\mathrm{tr}\,\underline{\chi}(\chi)\sqrt{\det\gamma} = \underline{L}(L)\sqrt{\det\gamma}$, $\langle \vec{H}, \vec{H} \rangle = \mathrm{tr}\,\underline{\chi}\,\mathrm{tr}\,\chi$
4. $\Sigma \subset \Omega \implies \mathrm{tr}\,\underline{\chi} > 0$,
5. Σ a "Black Hole" $\iff \mathrm{tr}\,\chi = 0 \iff \langle \vec{H}, \vec{H} \rangle = 0$

If a Null Cone Ω "sits" in physically isolated system, $\{\Sigma_s\} \subset \Omega$ foliation along $\underline{L}$:

$$\lim_{s\to\infty} \frac{\gamma|_{\Sigma_s}}{s^2} = \gamma_\infty$$

$$\underline{\text{Uniformization Thm}} \implies \gamma_\infty = \phi_\infty^2 d\sigma^2,$$

$d\sigma^2 = d\vartheta^2 + sin^2\vartheta d\varphi^2$ standard metric on $\mathbb{S}^2$. For:

$$\vec{v} \in \mathbb{R}^3, \ \ |\vec{v}| < 1, \ \ \omega_{\vec{v}} := \frac{\sqrt{1 - |\vec{v}|^2}}{1 - \vec{v} \cdot \vec{n}(\vartheta, \varphi)},$$

we expect foliation $\{\Sigma_t\}$, $t = (\frac{\omega_{\vec{v}}}{\phi_\infty})^2 s + o(s)$, to give

$$\lim_{t\to\infty} E_H(\Sigma_t) = \text{"total energy"} \geq 0.$$

To study $E_H \implies$ studying $\langle \vec{H}, \vec{H} \rangle \implies$ studying $\operatorname{tr} \underline{\chi}, \operatorname{tr} \chi$:

Structure Equations

Along $\{\Sigma_s\}_s$ we have the following propagation equations:

- $\underline{L} \operatorname{tr} \underline{\chi} = -\frac{1}{2} \operatorname{tr} \underline{\chi}^2 - |\hat{\underline{\chi}}|^2 - G(\underline{L}, \underline{L})$,
- $\underline{L} \operatorname{tr} \chi = G(L, \underline{L}) + 2\mathcal{K} - 2\nabla \cdot \zeta - 2|\zeta|^2 - \operatorname{tr} \chi \operatorname{tr} \underline{\chi}$

$\zeta =$"connection" on $T^{\perp}\Sigma_s$, $\mathcal{K} =$"Gauss-Curvature" of Σ_s.

Asymptotics

In an isolated system:

- $\operatorname{tr} \underline{\chi} = \frac{2}{s} - \frac{\theta}{s^2} + o(s^{-2})$
- $\operatorname{tr} \chi = \frac{2\mathcal{K}_\infty}{s} - \frac{\theta}{s^2} + o(s^{-2})$

A closer look at $\mathrm{tr}\,\underline{\chi}$

$$\underline{L}\,\mathrm{tr}\,\underline{\chi} = -\tfrac{1}{2}\,\mathrm{tr}\,\underline{\chi}^2 - |\hat{\underline{\chi}}|^2 - G(\underline{L},\underline{L})$$

1. We define $a = s^2\,\mathrm{tr}\,\underline{\chi} - 2s$:

$$\implies a = -\underline{\theta} + o(1) \;\&\; \underline{L}a = -\frac{1}{2}\frac{a^2}{s^2} - s^2(|\hat{\underline{\chi}}|^2 + G(\underline{L},\underline{L}))$$

$$\implies \boxed{\underline{\theta} = \int_0^\infty \frac{1}{2}\frac{a^2}{u^2} + u^2(|\hat{\underline{\chi}}|^2 + G(\underline{L},\underline{L}))du}$$

2. Recall $\underline{L}(dA_s) = \mathrm{tr}\,\underline{\chi}\,dA_s$:

$$\implies dA_{s_2} = e^{\int_{s_1}^{s_2}\mathrm{tr}\,\underline{\chi}du}dA_{s_1} = (\frac{s_2}{s_1})^2 e^{\int_{s_1}^{s_2}\frac{a}{u^2}du}dA_{s_1}$$

$$\implies dA_\infty = \frac{e^{\int_s^\infty \frac{a}{u^2}du}}{s^2}dA_s$$

$$\implies \boxed{dA_s = s^2 e^{-\int_s^\infty \frac{a}{u^2}du}dA_\infty}$$

$$E_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int \langle \vec{H}, \vec{H}\rangle dA\right)$$

Consequently, we have:

$$\frac{E_H(\Sigma_s)}{\sqrt{\frac{|\Sigma_\infty|}{4\pi}}} =$$

$$\frac{s}{2}\left(1 - \frac{1}{16\pi}\int (\frac{2}{s} - \frac{\underline{\theta}}{s^2})(\frac{2\mathcal{K}_\infty}{s} - \frac{\theta}{s^2})s^2 e^{-\int_s^\infty \frac{a}{u^2}du} dA_\infty\right) + o(1)$$

$$= \frac{1}{16\pi}\int \left(2\mathcal{K}_\infty \frac{1 - e^{-\int_s^\infty \frac{a}{u^2}du}}{s^{-1}} + (\mathcal{K}_\infty \underline{\theta} + \theta)e^{-\int_s^\infty \frac{a}{u^2}du}\right) dA_\infty$$

$$+ o(1)$$

$$\Longrightarrow \boxed{\lim_{s\to\infty} E_H(\Sigma_s) = \frac{1}{16\pi}\sqrt{\frac{|\Sigma_\infty|}{4\pi}}\int (\theta - \mathcal{K}_\infty \underline{\theta}) dA_\infty.}$$

$$\underline{\underline{L}\operatorname{tr}\chi = G(L,\underline{L}) + 2\mathcal{K} - 2\nabla\cdot\zeta - 2|\zeta|^2 - \operatorname{tr}\chi\operatorname{tr}\underline{\chi}}$$

1. We observe:

$$\begin{aligned}
&\underline{L}(8\pi s - \int \operatorname{tr}\chi dA_s) \\
&\qquad = 8\pi - \int \underline{L}\operatorname{tr}\chi dA_s + \operatorname{tr}\chi\underline{L}(dA_s) \\
&\qquad = 8\pi - \int G(L,\underline{L}) + 2\mathcal{K} - 2\nabla\cdot\zeta - 2|\zeta|^2 dA_s \\
&\qquad = \int 2|\zeta|^2 + G(\underline{L},-L)dA_s \\
\implies &\lim_{s\to\infty}(8\pi s - \int \operatorname{tr}\chi dA_s) \\
&\qquad = \boxed{-\int \operatorname{tr}\chi_0 dA_0 + \int_0^\infty \int 2|\zeta|^2 + G(\underline{L},-L)dA_u du}
\end{aligned}$$

2 Considering asymptotics:

$$
\begin{aligned}
&8\pi s - \int \operatorname{tr}\chi dA_s \\
&= \int \left(2\mathcal{K}_\infty s - (\frac{2\mathcal{K}_\infty}{s} - \frac{\theta}{s^2})(s^2 e^{-\int_s^\infty \frac{a}{u^2}du})\right) dA_\infty + o(1) \\
&= \int \left(2\mathcal{K}_\infty \frac{1 - e^{-\int_s^\infty \frac{a}{u^2}du}}{s^{-1}} + \theta e^{-\int_s^\infty \frac{a}{u^2}du}\right) dA_\infty + o(1) \\
&\Longrightarrow \lim_{s\to\infty}(8\pi s - \int \operatorname{tr}\chi dA_s) = \boxed{\int(-2\mathcal{K}_\infty \underline{\theta} + \theta) dA_\infty}
\end{aligned}
$$

Combining both expressions for $\lim_{s\to\infty}(8\pi s - \int \operatorname{tr}\chi dA_s)$:

$$
\boxed{\int(\theta - 2\mathcal{K}_\infty \underline{\theta}) dA_\infty = -\int \operatorname{tr}\chi_0 dA_0 + \int_0^\infty \int 2|\zeta|^2 + G(\underline{L}, -L) dA_u du}
$$

Combining everything:

$$\frac{\lim_{s\to\infty} E_H(\Sigma_s)}{\frac{1}{16\pi}\sqrt{\frac{|\Sigma_\infty|}{4\pi}}} = \int (\theta - \mathcal{K}_\infty \underline{\theta}) dA_\infty$$
$$= \int (\theta - 2\mathcal{K}_\infty \underline{\theta}) dA_\infty + \int \mathcal{K}_\infty \underline{\theta} dA_\infty$$
$$= -\int \operatorname{tr} \chi_0 dA_0$$
$$+ \int \int_0^\infty \Big(\big(2|\zeta|^2 + G(\underline{L}, -L)\big) u^2 e^{-\int_u^\infty \frac{a}{t^2} dt}$$
$$+ \big(\frac{a^2}{2u^2} + u^2(|\hat{\underline{\chi}}|^2 + G(\underline{L}, \underline{L}))\mathcal{K}_\infty \Big) du dA_\infty .$$

Consequently, choosing $t = (\frac{\omega_{\vec{v}}}{\phi_\infty})^2 s + o(s) \implies \mathcal{K}_\infty = 1$. Choosing Σ_0 to be a Black Hole, or a point, yields non-negative total energy. **Proof by: Piotr T Chruciel and Tim-Torben Paetz 2014 Class. Quantum Grav. 31 102001**

The End

THANK YOU!!

Pictures

- Einstein equation: manyworldstheory.com
- Sun and light: http://www.zamandayolculuk.com/cetinbal/HTMLdosya1/RelativityFile.htm
- Earth and Moon: https://einstein.stanford.edu/MISSION/mission1.html
- Sun-WD-NS-BH: https://s-media-cache-ak0.pinimg.com/originals/07/6c/df/076cdf115caedd07f08cea6252c13783.jpg