The Ubiquity of Graphs

6/12/2022

Introduction to the Topic

Reference: Harris, Hirst, and Mossinghoff:
Combinatorics and Graph Theory

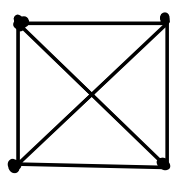
Discrete Math

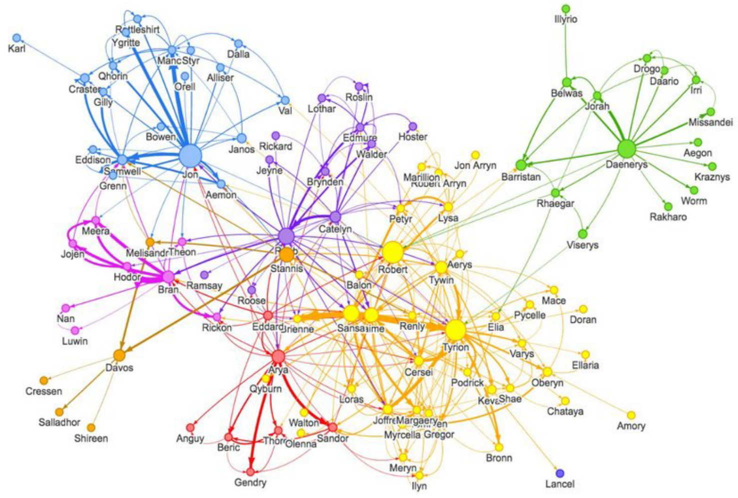
What will we study?

- Graph Theory
- Combinatorics
- Infinite Combinatorics & Graphs

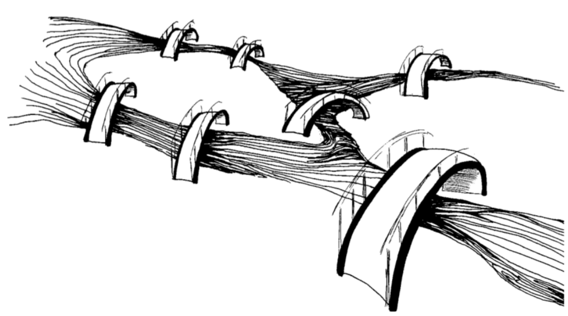
1) Graph Theory

<u>Graphs</u> are a set of vertices with edges between them





The subject originated with the
<u>Königs berg Bridge Problem</u>



← Pregolya River

FIGURE 1.1. The bridges in Königsberg.

1700s - Residents want to walk across bridges, making a route that crosses each bridge exactly once

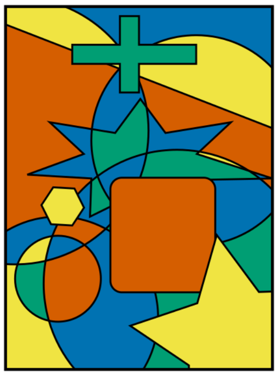
1736 - Euler, learning about this phenomenon, writes an article about it

Graphs can be used to model many things

Example: Four Color Theorem:

4 is the maximum number of colors required to color any map where bordering regions are colored differently

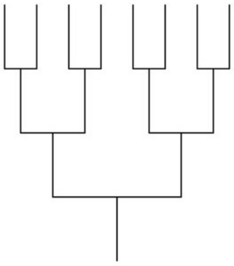
Example of a 4-colored map



Proposed in 1852, unproven until 1976
Used over 1000 hours of computer time

More Examples:

1) Social Networks
2) Roads or Railroads
3) The Internet
4) Games and Tournaments



Ramsey Theory: How many people are required at a gathering so that there must exist either 3 mutual acquaintences or 3 mutual strangers? $\rightsquigarrow R(3,3)$

2) Combinatorics

- Enumerative Combinatorics
  - The science of counting
  - The number of possible arrangements of a set of objects, under some constraints

How many ways to make change for a dollar?
How many ways to put n guests at k tables?

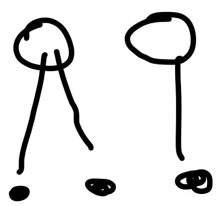
$\rightsquigarrow$ Binomial Coefficients, Permutations, Generating Functions

Ex: How many k-element subsets of an n-element set?
Given by kth coefficient of

$(1+x)^n$

- Existential Combinatorics
  - Studies problems concerning the existence of arrangements that possess some specified property

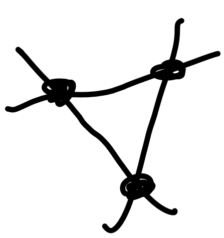
<u>Ex:</u> Pigeonhole Principle:



If more than n objects are distributed among n containers, then some container must contain more than one object

- Constructive Combinatorics
  - The design and study of algorithms for creating arrangements with special properties

<u>Ex:</u> Combinatorial Geometry



<u>Sylvester's Problem:</u>

Given $n \geq 3$ points in the plane which don't all lie on the same line, must there exist a line that passes through exactly two of them?

3) Infinite Combinatorics and Graphs

What happens when the vertex set is infinite in a graph?
Can we "count" infinities? Are there different sizes of infinity?

Graph Theory

Def: A <u>graph</u> consists of a set V and a set E:

- Elements of V are called <u>vertices</u>
- Elements of E, called <u>edges</u> are unordered pairs of vertices

We usually stipulate that V & E are finite

Ex: $V = \{a, b, c, d, e, f, g, h\}$

$E = \{\{a,d\}, \{a,e\}, \{b,c\}, \{b,e\}, \{b,g\}, \{c,f\}, \{d,f\}, \{d,g\}, \{g,h\}\}$

$G = \{V, E\}$

Can be represented visually:

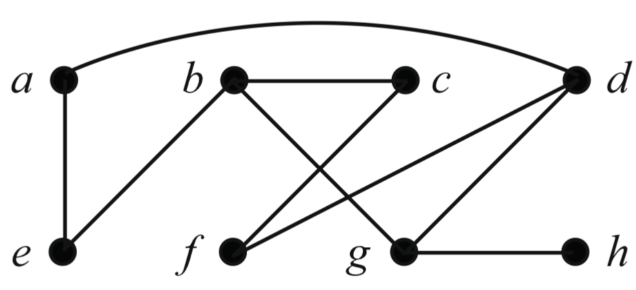


FIGURE 1.2. A visual representation of the graph G.

When thinking of a graph, you should always think about it visually like above

We can alter our definitions in various ways:

(1) If we let E consist of ordered pairs of vertices, we obtain a directed graph

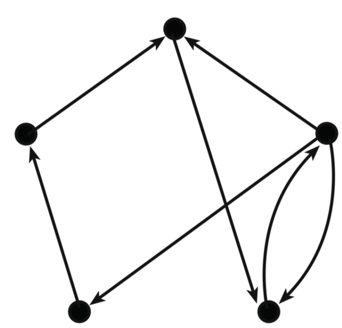


FIGURE 1.3. A digraph.

(2) We get a multigraph if we allow repeated elements in our set of edges (E becomes a multiset)

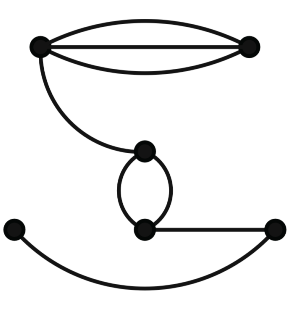


FIGURE 1.4. A multigraph.

(3) We get a <u>pseudograph</u> if we allow "loops" e.g. vertices may connect to themselves

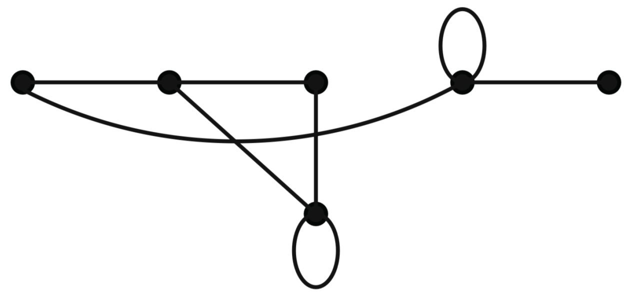
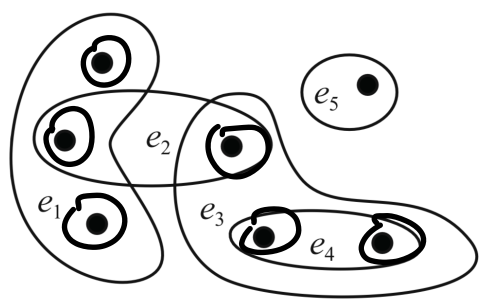


FIGURE 1.5. A pseudograph.

(4) We get a <u>hypergraph</u> if we let our edges be arbitrary subsets of vertices (rather than just pairs)



e_3

FIGURE 1.6. A hypergraph with 7 vertices and 5 edges.

We will usually focus on <u>finite, simple</u> graphs; those without loops or edges

The vertex set of a graph G is denoted $V(G)$, and the edge set denoted $E(G)$.
Denote edge between u,v as uv

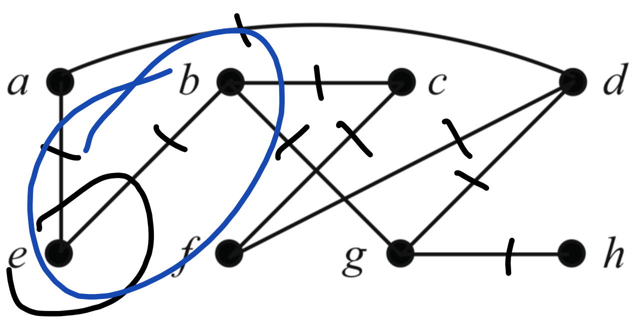
<u>Def:</u> (1) The <u>order</u> of a graph G is the cardinality of V.

(2) The <u>size</u> of G is $|E|$

(3) If $u,v \in V$ and $uv \in E$, then u and v are <u>adjacent</u>. u and v are the <u>end vertices</u> of uv.

(4) If $uv \notin E$, then u and v are <u>nonadjacent</u>

(5) If an edge e has v as an end vertex we say that v is <u>incident</u> with e



Order: 8
Size: 9
Adjacent to b:
Incident to eb: $\{e,b\}$

Def: (1) The open neighborhood of vertex v, denoted by $N(v)$, is

$$N(v) = \{x \in V \mid vx \in E\}$$

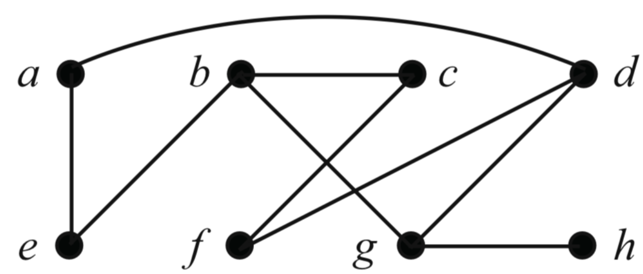
(2) The closed neighborhood of v, denoted $N[v]$, is $\{v\} \cup N(v)$

(3) For arbitrary $S \subset V$, let

$$N(S) = \bigcup_{s \in S} N(s)$$

$$N[S] = S \cup N(S)$$

be the open and closed neighborhoods of S



$N(f) = \{c, d\}$ $\quad$ $N[f] = \{f, c, d\}$

$S = \{a, b, g\}$

$N(S) = \{e, d, c, g, b, h\}$ $\quad$ $N[S] = \{e, d, c, g, b, h, a\}$

$N(a) = \{e, d\}$

$N(b) = \{e, g, c\}$

$N(g) = \{b, h\}$

Def: (1) The <u>degree</u> of $v \in V$, denoted $\deg(v)$, is the number of edges incident with v

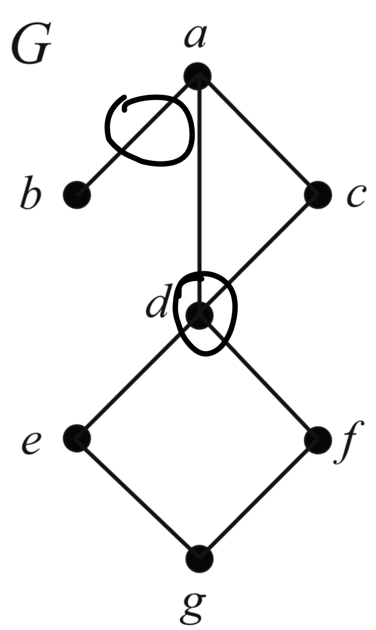
(2) The <u>maximum degree</u> of G, denoted $\Delta(G)$, is

$$\Delta(G) = \max\{\deg(v) \mid v \in V\}$$

(3) The <u>minimum degree</u> $\delta(G)$ is

$$\delta(G) = \min\{\deg(v) \mid v \in V\}$$

Rmk: In simple graphs, $\deg(v) = |N(v)|$



$\deg(a) = 3$

$\Delta(G) = 4$

$\delta(G) = 1$

Q: How many people at Columbia have an odd number of friends?

<u>Thm:</u> In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges.
Consequently, the number of vertices with odd degree is even.

<u>Pf:</u> Let $S = \sum_{v \in V} \deg(v)$. Since each edge is exactly 2 vertices, when counting S we count each edge exactly twice. Thus $S = 2|E|$.

Since S is always even, the number of vertices with odd degree is always even, for else S would be odd. $\square$

Partial Answer: An even number of people have an odd number of friends.

$V = \{\text{students at Columbia}\}$
$E = \{uv \mid u \text{ is friends with } v\}$

Examples:

(1) Complete Graph of order n, K_n

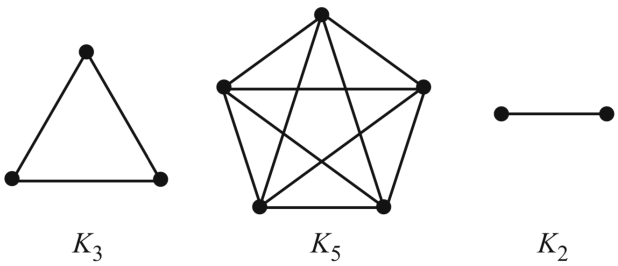


FIGURE 1.11. Examples of complete graphs.

Every vertex adjacent to every other vertex.

Q: How many edges in K_n?

$\deg(v) = n-1$

$\sum_v \deg(v) = n(n-1)$

$|E| = \frac{n(n-1)}{2}$

$\binom{n}{2} = \frac{n(n-1)}{2}$

(2) Empty graph on n vertices E_n

$E = \emptyset$

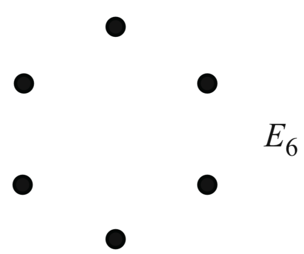
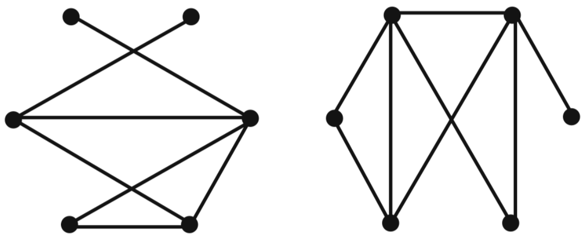


FIGURE 1.12. An empty graph.

(3) Complement of G, $\overline{G}$

$V(\overline{G}) = V(G)$

$E(\overline{G})$ = all edges not present in G



G $\overline{G}$

FIGURE 1.13. A graph and its complement.

(4) <u>Regular</u> graphs are graphs with every vertex the same degree

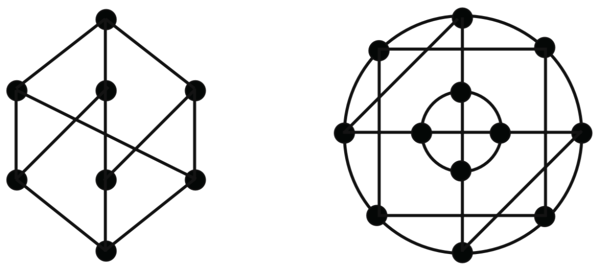
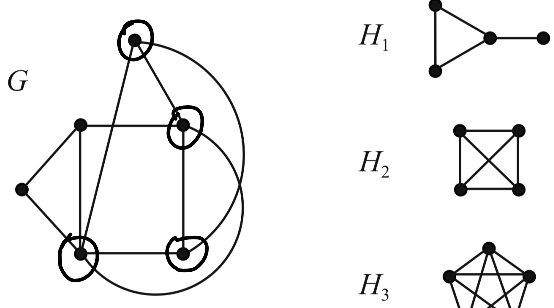


FIGURE 1.14. Examples of regular graphs.

K_n is regular with degree $n-1$

(5) A graph H is a <u>subgraph</u> of G is $V(H) \subset V(G)$ and $E(H) \subset E(G)$



<u>Not</u> a subgraph

FIGURE 1.17. H_1 and H_2 are subgraphs of G.

Walks & Connectivity

Def: (1) A walk is a sequence of vertices
$v_1, v_2, \ldots, v_k$ s.t. $v_i v_{i+1} \in E$ for all i.
We call it a $v_1 - v_k$ walk, and v_1, v_k are the end vertices of the walk.

(2) If the edges in a walk are distinct, it is a trail

(3) If the vertices are distinct, it is a path

(4) The length of a walk is its number of edges, counting repetitions

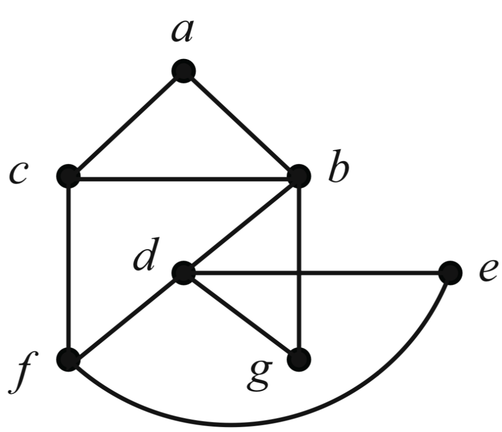


FIGURE 1.7.

acfcbd is a walk of length 5

bacbd is a trail, but not a path

dgbacf is a path and a trail

Every path is a trail, but not every trail is a path
(different vertices $\Rightarrow$ different edges)

__Def:__ (1) A __cycle__ is a path with an edge $v_k v_1$

(2) A __circuit__ is a trail that begins and ends at the same vertex

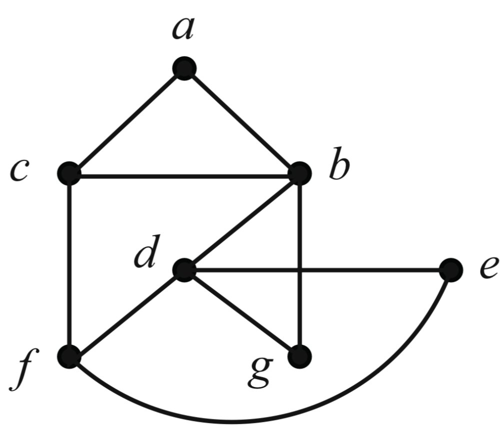


FIGURE 1.7.

gdbcabg is __circuit__

__edbacfe__ is __cycle__

__More examples of graphs__

C_n = cycle on n vertices

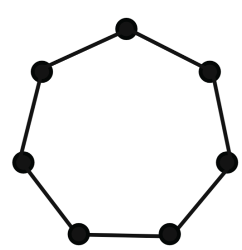


FIGURE 1.15. The graph C_7.

P_n = path on n vertices

FIGURE 1.16. The graph P_6.

<u>Def:</u> A graph is <u>connected</u> if every pair of vertices can be joined by a path

Each maximal connected piece of a graph is called a <u>connected component</u>

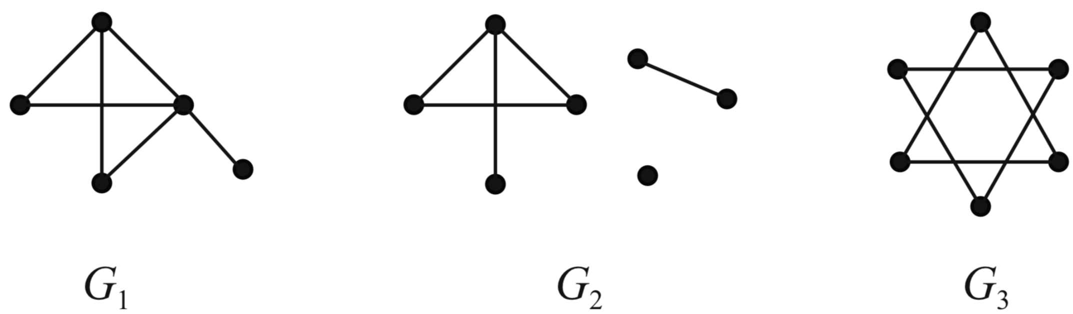


FIGURE 1.9. Connected and disconnected graphs.

	G_1	G_2	G_3
Connected:	Yes	No	No
# Components:	1	3	2

I'll end with a few cool graph theoretic proofs of popular theorems.

Source: V. Yegnanarayan, "Graph Theory to Pure Mathematics: Some Illustrative Exomples"

<u>Fermat's Little Theorem</u>: Let p be a prime and $a \in \mathbb{N}$ such that $p \nmid a$. Then $a^p - a$ is divisible by p.

<u>Proof</u>: Let $G=(V,E)$, where

$V =$ all sequences $(a_1, \ldots, a_p)$ s.t. $1 \leq a_i \leq a$, with $a_i \neq a_j$ for some $i \neq j$.

Then $|V| = a^p - a$.

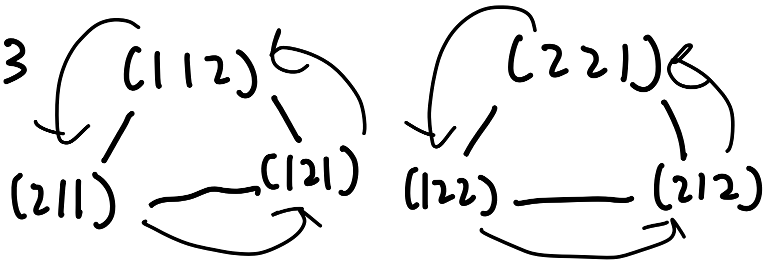
For all $u = (u_1, \ldots, u_p) \in V$, let $uv \in E$ iff. $v = (u_p, u_1, \ldots, u_{p-1})$. Then each vertex has degree 2; have v_1u and uv_1.

Hence each component of G is a cycle of length p. Then the number of components is $(a^p - a)/p$.

Since $(a^p-a)/p$ is a whole number, $p \mid a^p - a$. $\square$

Ex; $a=2$, $p=3$

$2^3 - 2 = 6$



Def: A graph is <u>bipartite</u> if its vertex set can be partitioned into two sets X and Y such that every edge has one end vertex in X and the other in Y.

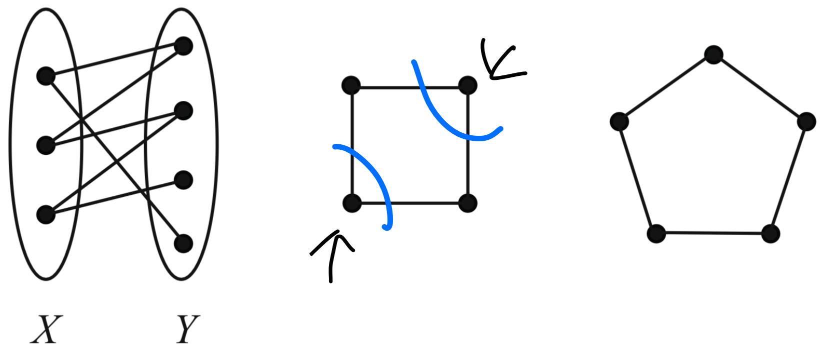
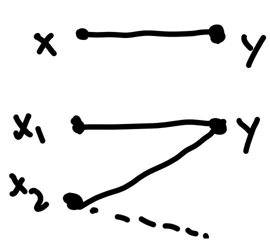


FIGURE 1.19. Two bipartite graphs and one non-bipartite graph.

Thm: (Cantor-Shröder-Bernstein) Let A, B be two sets. If there is an injective mapping $f: A \to B$ and an injective mapping $g: B \to A$, then there is a bijection between A and B.

Proof: Assume $A \cap B = \emptyset$. We will define a bipartite graph G with partition A and B and $V = A \cup B$. Let $xy \in E$ iff. either $f(x) = y$ or $g(y) = x$ for $x \in A$, $y \in B$.

Since f, g injective, $1 \le \deg(v) \le 2$ for all $v \in V$.



$f(x) = y$, $g(y) = x$

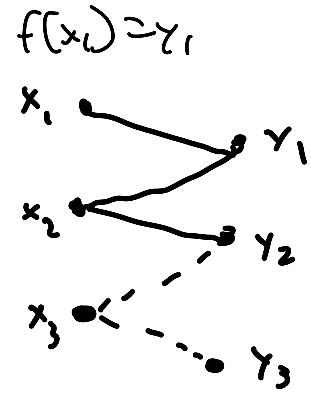
$f(x_1) = y$

$g(y) = x_2$

Decompose G into connected components; 3 types.

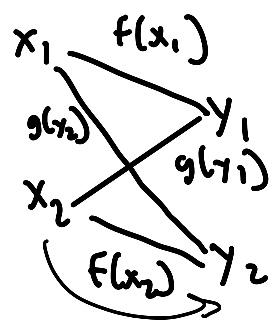
1) one-way infinite path

$x_0 - x_1 - x_2 - \cdots\cdots$



(cannot be finite, for else contradicts injectivity)

2) Cycle of even length with >2 vertices



3) Edge

$f(x_1)=y_1$,
$g(y_1)=x_1$

For each component, there is a set of edges E_1 s.t. each vertex in the component is incident to exactly one of these edges.

Using these edges, construct a bijection $A \to B$.

Define $f'(x)=y$ iff. $xy \in E_1$.

f' is injective and surjective by condition above. $\square$

Ex: If f, g are inverses, get case 3.

Ex: Case 1 is $\mathbb{Z} \xrightarrow{\times 2} 2\mathbb{Z}'$, $2\mathbb{Z}' \hookrightarrow \mathbb{Z}$

$\mathbb{Z} \qquad 2\mathbb{Z}'$

$1 \to 2'$

$2 \to 4'$

$4 \to 8'$

$\cdots \to$