Subgroup of a free group is free: a topological proof

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Free groups

- The *free group* F_n of rank n is the group with n generators and no non-trivial relations.
- In other words, F_n is the set of *reduced words* consisting of $\{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.

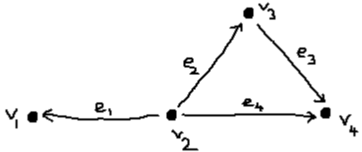
- The goal of this talk is to understand the subgroups of F_n, including a proof of the following theorem:

Theorem (Nielsen–Schreier)

Every subgroup of a free group is free.

Graphs

- A (directed) *graph* Γ is a pair (V, E), where V is a set of *vertices* and E is a set of *edges* which are ordered pairs of vertices.



- An *edge path* is a list of edges $e_1, \ldots, e_k$ where the terminal vertex of e_i is the initial vertex of e_{i+1} for $i = 1, \ldots, k-1$.
- A *loop* based at $v \in V$ is an edge path $e_1, \ldots, e_k$ where v is the initial vertex of e_1 and the terminal vertex of e_k.

Fundamental groups

- Let X be a topological space.
- A *loop* based at $x_0 \in X$ is a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.
- Two loops γ_1, γ_2 based at x_0 are *homotopy-equivalent* if there exists a continuous interpolation between them; in other words, a continuous map $h : [0, 1] \times [0, 1] \to X$ with $h(0, \cdot) = h(1, \cdot) = x_0$ and $h(\cdot, 0) = \gamma_1$ and $h(\cdot, 1) = \gamma_2$.
- Homotopy-equivalence is an equivalence relation.
- The set of homotopy-equivalence classes of loops based at $x_0 \in X$ forms a group under 'concatenation', called the *fundamental group* and denoted $\pi(X; x_0)$.

Fundamental group of a graph

- Given a graph Γ, define the *reduction* of a loop $e_1, \ldots, e_k$ by repeatedly removing adjacent pairs of inverse edges.
- Define two loops based at $v \in V$ to be *equivalent* if their reductions are the same.
- The *fundamental group* $\pi(\Gamma; v)$ of the graph Γ is the set of equivalence classes of loops, under 'concatenation'.

Fundamental group of a graph (cont.)

Theorem

Let $\Gamma = (V, E)$ *be a finite connected graph, and let* $n = 1 - |V| + |E|$*. Then* $\pi(\Gamma, v) \cong F_n$ *for any* $v \in V$*.*

Proof. Induction on $|V|$. $\square$

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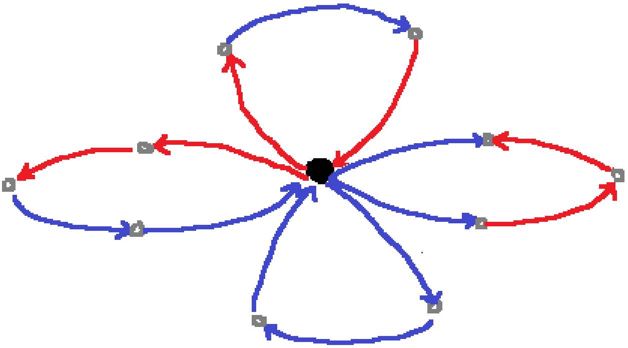
The main theorem

Theorem

Every finitely-generated subgroup of a free group is free.

- We will illustrate the proof with the subgroup $H \subset F_2$ generated by $\{aba,\ a^2b^2,\ ba^2b^{-1},\ b^3\}$.

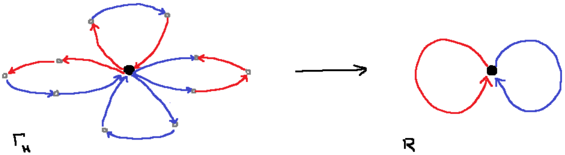
Setup of the proof



- Every loop in Γ_H naturally corresponds to an element of H, but not necessarily bijectively.

Setup of the proof

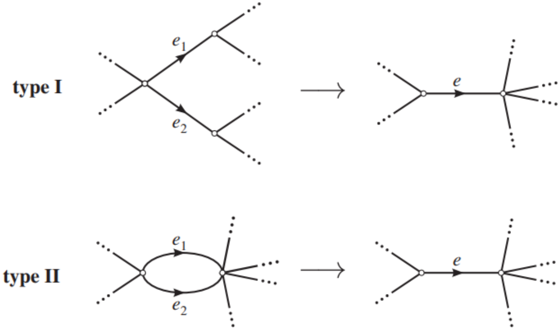
- A *graph map* $\Gamma_1 \to \Gamma_2$ maps the vertices and edges of Γ_1 to the vertices and edges of Γ_2, preserving adjacency.
- A map $\Gamma_1 \to \Gamma_2$ induces a homomorphism $\pi(\Gamma_1; v) \to \pi(\Gamma_2; w)$.



- $r : \Gamma_H \to R$ is a graph map.
- The image of the induced homomorphism $\rho : \pi(\Gamma_H; v) \to \pi(R; w)$ is H.
- Can we modify Γ_H to make ρ injective?

Graph folding

- Given a graph Γ, we can *fold* any two distinct edges with the same initial vertex, as follows:



Source: Figure 4.5 (page 72)

- The induced homomorphism $\pi(\Gamma; v) \to \pi(\Gamma_f; v)$ is bijective for a type I fold, and surjective but not injective for a type II fold.

Factoring graph maps

- Suppose $r : \Gamma_1 \to \Gamma_2$ is a graph map, which maps two distinct edges e_i, e_j with the same initial vertex in Γ_1 to the same edge in Γ_2.
- Let $f : \Gamma_1 \to \Gamma_f$ denote folding (i.e., identifying) e_i and e_j.
- Then r factors through Γ_f; there exists a unique $r' : \Gamma_f \to \Gamma_2$ such that $r = (r' \circ f)$.

Immersions

- Now suppose $r : \Gamma_1 \to \Gamma_2$ is an *immersion*, which means no two distinct edges with the same initial vertex in Γ_1 are mapped to the same edge in Γ_2.
- Then if $e_1, \ldots, e_k$ is a reduced edge path in Γ_1, then e_i^{-1} and e_{i+1} are distinct edges that share an initial vertex, so they map to distinct edges in Γ_2.
- Reduced edge paths in Γ_1 are mapped to reduced edge paths in Γ_2.
- Hence non-trivial loops in Γ_1 are mapped to non-trivial loops in Γ_2, so the induced homomorphism $\rho : \pi(\Gamma_1; v) \to \pi(\Gamma_2; w)$ is injective.

Putting it together...

Lemma

For any graph map $r : \Gamma_1 \to \Gamma_2$*, we can write* $r = r' \circ (f_m \circ \cdots \circ f_1)$*, where each* f_i *is a fold, and* r' *is an immersion.*

Proof: Repeatedly factor r through folding; if r' is not factorable through folding, then r' is an immersion. $\square$

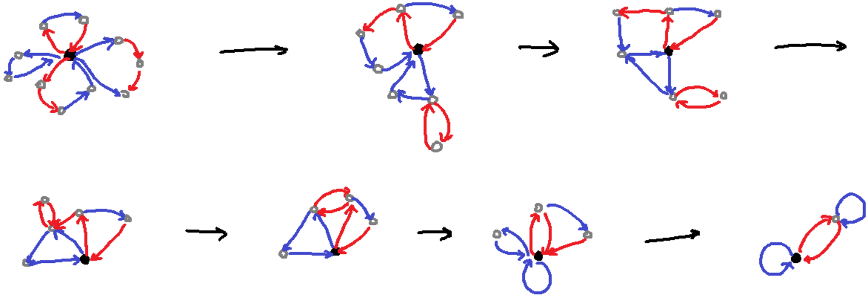
Proof of the main theorem

Theorem

Every finitely-generated subgroup of a free group is free.

Proof: Apply the lemma to the graph map $r : \Gamma_H \to R$. There must be a graph $\Gamma' = (f_m \circ \cdots \circ f_1)\,\Gamma_H$ such that $r' : \Gamma' \to R$ is an immersion.
Let ρ, ρ', φ be the induced homomorphisms of r, r', $(f_m \circ \cdots \circ f_1)$.
The image of $\rho = (\rho' \circ \varphi)$ is H, and φ is surjective, so the image of ρ' is H. Meanwhile, since r' is an immersion, ρ' is injective.
Hence ρ' gives an isomorphism from $\pi(\Gamma'; v)$ to H, so H is free. $\square$

Illustration



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New generators, membership problem

- From the graph Γ', we see that H is generated by $\{b, a^2, aba\}$ with no non-trivial relations.
- We can write these in terms of the original generators $\{aba, a^2b^2, ba^2b^{-1}, b^3\}$.

$$\begin{aligned} b &= (ba^2b^{-1})(b^3)(a^2b^2)^{-1} \\ a^2 &= (a^2b^2)(b^3)^{-1}(ba^2b^{-1})(b^3)(a^2b^2)^{-1} \\ aba &= aba \end{aligned}$$

- We can also easily determine if a given element of F_2 is in H.

$$aba^{-1}b^{-1} \in H, \qquad ab \notin H$$

Finite-index subgroups

Theorem

A subgroup $H \subseteq F_n$ has finite index iff for each vertex v in Γ', there are n edges with initial vertex v and n edges with terminal vertex v.
In this case, the index of H in F_n is the number of vertices of Γ'.

The cosets H_i correspond to $\{$reduced edge paths in Γ' from v_1 to $v_i\}$, where $v_1 = w$ is the central vertex of Γ'.

Normal subgroups

- Define Γ' to be *vertex-transitive* if for any vertex v_i, there is an automorphism of Γ' which maps v_1 to v_i, and a_j-edges to a_j-edges.

Theorem

A subgroup $H \subseteq F_n$ is normal iff Γ' is vertex-transitive.

Corollary

If $H \subseteq F_n$ is finitely-generated and normal, then H has finite index in F_n.