Generating Functions and Ehrhart Series

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Abstract

We will begin the talk by introducing an important tool in combinatorics known as the generating function and give some examples of how it can be used for problems such as coin counting and evaluating a formula for the fibonacci sequence. We will then introduce one of the most important tools that will guide us this summer: the Ehrhart Series. For this lecture we will mostly focus on the example of the simplex, basic properties, and giving intuition for how the Ehrhart Series relates the continuous and discrete. Lastly, we will finish the talk by introducing and proving Pick's theorem.

1 Generating Functions

Definition. Given a sequence $\{a_k\}_{k\geq 0}$ the *generating function* for that sequence is the power series:

$$F(z) = \sum_{k\geq 0} a_k z^k$$

Generating functions are useful in that we can oftentimes deduce properties about the sequence a_k from its generating function F. Furthermore, the expressions for F can oftentimes be even simpler than the expression for the series itself.

Example 1. Let $a_k = 1$ for all k. Then the generating function for a_k is:

$$F(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots$$

Example 2. Let $a_k = \binom{n}{k}$ for some fixed n then by the binomial theorem

$$F(z) = (1+z)^n$$

(Note this also holds for n that aren't integers!)

Example 3. Generating functions can also be a useful combinatorial tool. For example suppose we have an infinite number of two denominations of coins worth m_1 and m_2. Suppose we want to try to find the number of ways we can make a value k out of the coins we have. Let C_k be the number of ways we can make k. We claim the generating function for C_k is

$$\left(\sum_{i\geq 0} z^{m_1 i}\right)\left(\sum_{j\geq 0} z^{m_2 j}\right)$$

To see this consider what terms contribute to the z^k term. These will precisely be the pairs (i, j) such that $m_1 i + m_2 j = k$. This problem in general (where we may have more than 2 denominations) is referred to as the *coin exchange problem*. Chapter 1 devotes significant attention to this topic. For the purposes of getting to the more central topics of this book, we are mostly going to omit the discussion.

Example 4. We'll now show how generating functions can also be useful for algebraic purposes by employing them to find a formula for the fibonacci sequence. Let f_k be the fibonacci sequence with $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Let

$$F(z) = \sum_{k\geq 0} f_k z^k$$

be the generating function for the fibonacci sequence. Then

$$\begin{aligned} zF(z) + z^2F(z) &= \sum_{k\geq 1} f_{k-1}z^k + \sum_{k\geq 2} f_{k-2}z^k \\ &= \sum_{k\geq 2} f_k z^k \\ &= -z + F(z) \\ F(z) &= \frac{z}{1-z-z^2} \end{aligned}$$

So we obtain a nice formula for generating function! We now use a partial fraction decomposition:

$$F(z) = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\phi z} - \frac{1}{1-\bar{\phi} z}\right]$$

From which we get

$$f_k = \frac{\phi^k - \bar{\phi}^k}{\sqrt{5}}$$

Example 5. For our final example we will show how roots of unity can be used in concert with generating functions to study specific subsets of terms of a generating function. Suppose we want to find the summation

$$\binom{100}{0} + \binom{100}{4} + \cdots + \binom{100}{96} + \binom{100}{100}$$

To do this we will construct a generating function for the series $a_k = \binom{n}{rk}$ for any r. Plugging in $n = 100, r = 4$ will give us the desired result. Observe that if we have an r-th root of unity ω where $r \neq 1$ then:

$$1 + \omega + \omega^2 + \cdots + \omega^{r-1} = 0$$

Now take an r-th primitive root of unity ω and consider the sum of generating functions:

$$\sum_{t=0}^{r-1} (1 + \omega^t z)^n$$

The coefficient of z^k will be

$$\binom{n}{k}\sum_{t=0}^{r-1}\omega^{tk}$$

If k is not a multiple of r then ω^t is an r-th root of unity distinct from 1. Thus by our result on sums

$$\sum_{t=0}^{r-1}\omega^{tk} = \begin{cases} 0 & r \nmid k \\ r & \text{else} \end{cases}$$

Thus

$$\frac{1}{r}\sum_{t=0}^{r-1}(1+\omega^t z)^n = \sum_{k\geq 0}\binom{n}{rk}z^{rk}$$

In particular if we set $n = 100, r = 4, z = 1$ this tells us

$$\sum_{k=0}^{25}\binom{100}{4k} = \frac{(1+1)^{100}+(1-1)^{100}+(1+i)^{100}+(1-i)^{100}}{4} = 2^{98}-2^{49}$$

2 Ehrhart Series

We begin with some definitions to guide our later work

Definition. A convex polyhedra $\mathcal{P}$ is a *convex polyhedra* if it satisfies either of the following equivalent properties:

1. $\mathcal{P}$ is the smallest set convex set (the *convex hull*) containing some set of points $v_1, \ldots, v_n$. In this case, the smallest subset V of $v_1, \ldots, v_n$ with $\mathcal{P}$ as its convex hull are the *vertices* of $\mathcal{P}$

2. $\mathcal{P}$ is a bounded intersection of finitely many half planes.

Note that this definition implies that $\mathcal{P}$ contains the interiors of the resulting shapes. For instance, the boundary of a triangle would not be a convex polyhedra, only a triangle and its interior.

Definition. Say H is a *supporting hyperplane* if $\mathcal{P}$ lies entirely on one side of H. A *face* of $\mathcal{P}$ is an intersection of $\mathcal{P}$ with a supporting hyperplane. The 0-dimensional faces are the *vertices* and the $d-1$-dimensional faces are the *facets* (if $\mathcal{P}$ has dimension d).

We will now define a generating function that will be central to our studies for the rest of the summer: The Ehrhart series.

Definition. The *lattice point enumerator* or *discrete volume* of $\mathcal{P}$ is

$$\mathcal{L}_{\mathcal{P}}(t) = \#(t\mathcal{P}\cap\mathbb{Z}^d)$$

The *Ehrhart Series* is the generating function of the lattice point enumerator:

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t\geq 1}\mathcal{L}_{\mathcal{P}}(t)z^t$$

Before we give some of the properties we'll begin by giving a result that gives an intuition as to why we should care about the Ehrhart series.

Lemma 1.

$$vol\ \mathcal{P} = \lim_{t\to\infty} \frac{1}{t^d}\mathcal{L}_{\mathcal{P}}(t)$$

Proof. Note that

$$\frac{1}{t^d}\mathcal{L}_{\mathcal{P}}(t) = \frac{1}{t^d}\#\left(\mathcal{P}\cap\left(\frac{1}{t}\mathbb{Z}\right)^d\right)$$

Imagine placing a box with sides $\frac{1}{t}$ at each point of $\left(\frac{1}{t}\mathbb{Z}\right)^d$ insides $\mathcal{P}$. Then the above expression is simply the sum of the boxes. As t shrinks the amount of area not covered by the boxes tends to 0. Thus the above lemma holds. □

In fact, as we'll see in later lectures an even stronger connection with the area function holds – one that is directly computable.

2.1 Examples

Theorem 2.1. *Let $\mathcal{P}$ be a unit d-cube. Then:*

$$\mathcal{L}_{\mathcal{P}}(t) = (t+1)^d$$

and satisfies $\mathcal{L}_{\mathcal{P}^\circ}(t) = (-1)^d\mathcal{L}_{\mathcal{P}}(-t)$ where $\mathcal{P}^\circ$ is the interior of $\mathcal{P}$. Lastly, we can write

$$Ehr_{\mathcal{P}}(z) = \frac{\sum_{k=1}^d A(d,k)}{(1-z)^{d+1}}$$

Theorem 2.2. *Let $\mathcal{P}$ be the unit d-simplex. Then:*

$$\mathcal{L}_{\mathcal{P}}(t) = \binom{t+d}{d}$$

and satisfies $\mathcal{L}_{\mathcal{P}^\circ}(t) = (-1)^d\mathcal{L}_{\mathcal{P}}(-t)$. Lastly, we can write

$$Ehr_{\mathcal{P}}(z) = \frac{1}{(1-z)^{d+1}}$$

Proof. Note that

$$\mathcal{L}_{\mathcal{P}}(t) = \#\{m_1 + \cdots + m_{d+1} = t\}$$

By a standard combinatorial argument, this is $\binom{t+d}{d}$. Now notice by the logic used for the coin exchange problem

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) &= \left(\sum_{m_1\geq 0} x^{m_1}\right)\left(\sum_{m_1\geq 0} x^{m_2}\right)\cdots\left(\sum_{m_{d+1}\geq 0} x^{m_1}\right)\\ &= \frac{1}{(1-z)^{d+1}}\end{aligned}$$

□

2.2 Pick's Theorem

We'll end with one of the simplest theorems relating point counting to areas:

Theorem 2.3. *(Pick's Theorem) Let $\mathcal{P}$ be a convex polygon with coordinates at integer pairs. If I is the number of interior points of $\mathcal{P}$ and B is the number of boundary points then the area of $\mathcal{P}$ is*

$$[\mathcal{P}] = -1 + I + \frac{B}{2}$$

Proof. We follow the proof from Computing the Continuous Discretely. Suppose $\mathcal{P}$ is the union of convex polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ which are disjoint aside from sharing a single side. Then we claim if Pick's theorem holds for $\mathcal{P}_1$ and $\mathcal{P}_2$ then it holds for $\mathcal{P}$. By disjointness

$$[\mathcal{P}_1] + [\mathcal{P}_2] = [\mathcal{P}]$$

and so

$$[\mathcal{P}] = -2 + I_1 + I_2 + \frac{B_1 + B_2}{2}$$

Let S be the side shared by $\mathcal{P}_1, \mathcal{P}_2$ with vertices v_1, v_2. Then the interior points of P will be the interior points I_1, I_2 plus the points of S minus v_1, v_2. Now notice every point is being counted precisely once in $I_1 + I_2$ and $\frac{B_1+B_2}{2}$ except for those on S each of which we are adding two halves for. As all of the points except for v_1, v_2 become interior points that means we are only over counting for v_1, v_2. However, we are overcounting by precisely 1. Thus:

$$-2 + I_1 + I_2 + \frac{B_1 + B_2}{2} = -2 + I + \frac{B}{2} + 1 = -1 + I + \frac{B}{2}$$

as desired. By similar reasoning we can also show that if the theorem holds for $\mathcal{P}_1, \mathcal{P}$ then it holds for $\mathcal{P}_2$. Now notice for any polygon with greater than or equal to 3 sides, we can simply split it into two convex polygons with fewer sides each. Thus it suffices to prove the theorem for triangles.

Now given any triangle we can adjoin right triangles to it to form a rectangle. Thus by the additive property it suffices to prove the theorem for rectangles and right triangles with sides parallel to the axes. As two isomorphic right triangles can be combined to form a rectangle, by additivity it suffices to prove the theorem for right triangles which can be shown via a computation. □

Corollary 2.1. *Let $\mathcal{P}$ be an integral convex polygon with B points on its boundary and area A. Then*

$$L_{\mathcal{P}}(t) = At^2 + \frac{1}{2}Bt + 1$$

and

$$Ehr_{\mathcal{P}}(z) = \frac{(A - \frac{B}{2} + 1)z^2 + (A + \frac{B}{2} - 2)z + 1}{(1-z)^3}$$