Computations of Heegaard Floer Homology: Torus Bundles, L-spaces, and Correction Terms

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Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2010

ABSTRACT

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In this thesis we study some computations and applications of Heegaard Floer homology. Specifically, we show how the Floer homology of a torus bundle is always “monic” in a certain sense, extending a result of Ozsváth and Szabó. We also explore the relation between Heegaard Floer homology L–spaces and non-left orderability of three-manifold groups. Finally, we discuss a concordance invariant coming from the Floer homology of ± 1–surgeries.

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Acknowledgments

First, I would like to thank my advisor, Peter Ozsváth, for all he has done: discussed fascinating mathematics, suggested worthwhile research projects, and provided advice for more than just mathematics. It has been an honor and a pleasure to have been his student.

Second, I would like to thank all those who had wonderful mathematical conversations with me: Yinghua Ai, John Baldwin, Jonathan Bloom, Clay Cordova, Jesse Gell-Redman, Danny Gillam, Allison Gilmore, Joshua Greene, Elisenda Grigsby, Matt Hedden, Daniel Krasner, Adam Levine, Robert Lipshitz, John Morgan, Helge Moller Pedersen, Ina Petkova, Dylan Thurston, Liam Watson, and Rumen Zarev. I would also like to thank the anonymous referee for his careful reading of [AP10], which became Chapter 3 of this thesis, as well as Kim Frøyshov for his corrections to and comments on Chapter 5. Thank you to my students who shared their curiosity and enthusiasm with me: Corey Bregman, Leo PeBenito, Rachel Vishnepolsky. Thank you to those who made life outside of mathematics fun: Adam Jacob, Donovan McFeron, Chen-Yun Lin, Joseph Ross, and Dmitry Zakharov.

Most of all, I would like to thank my parents, Jacqueline and David Peters, as well as my sister, Catherine Peters, for their unwaivering support and love and for their dedication to my education. Finally, I would like to thank Alice for making the past four years wonderful.

The author was partially supported by NSF grant DMS-0739392.

To my parents

Chapter 1

Introduction

Heegaard Floer homology was introduced by Ozsváth and Szabó in [OS04e; OS04d]. It provides extremely powerful invariants for closed oriented three-manifolds in the form of a package of abelian groups, denoted collectively by HF°. There is also an extension, called knot Floer homology, due to Ozsváth and Szabó [OS04c] as well as Rasmussen [Ras03], to a homology theory for knots inside closed oriented three-manifolds. Recently, there have even been extensions of Heegaard Floer theory to manifolds with boundary: a theory for sutured manifolds due to Juhasz [Juh06] and a theory for "bordered" three-manifolds by Lipshitz, Ozsváth, and Thurston [LOTa; LOTb].

Heegaard Floer theory has led to many interesting advances in the theory of three and four-dimensional topology, contact geometry, and knot theory. For instance, in the four-dimensional setting, it has been used to reprove Donaldson's diagonalization theorem [OS04a], as well as give restrictions on the topology of symplectic four-manifolds [OS04f]. The three-dimensional theory detects the Thurston semi-norm [OS04b], and gives information about the existence of taut foliations [OS04b]. Knot Floer homology detects the genus of a knot [OS04b; Juh06] and detects fiberdness [Ghi08; Ni09b; Juh08]. In Chapter 2 we provide an overview of some aspects of Heegaard Floer theory.

For fiber bundles with fiber genus greater than one, a computation of Ozsváth and Szabó shows that their Floer homology is "monic" in the "outermost" Spin^c structure (see Theorem 3.1.1 for a precise statement). Ni proved the remarkable fact that the converse holds in [Ni09a]. For a torus bundle, the Floer homology is always infinitely generated as an abelian group, though it was believed that the Floer homology should still be monic in some sense. In Chapter 3, which was joint work with Yinghua Ai, we show how this is indeed possible using twisted coefficients with values in a certain Novikov ring.

A group is called *left-orderable* if there exists a strict total ordering on its elements which is invariant under left-multiplication. Left-orderability of three-manifold groups reflects interesting geometric properties of the corresponding manifolds. For instance, if the fundamental group of a three-manifold is not left-orderable then it cannot possess any $\mathbb{R}$–covered foliations (see Calegari and Dunfield [CD03]). Heegaard Floer theory can also rule out the existence of certain taut foliations on three-manifolds. Specifically, the family of L–spaces (a class of three-manifolds whose Floer homology is "as simple as possible") cannot possess any co-orientable taut foliation [OS04b]. In Chapter 4 we discuss some of the relations between orderability properties of the fundamental group and the class of L–spaces. We then provide some examples in the hyperbolic setting.

A knot K in the three-sphere is called *smoothly slice* if there exists a smoothly embedded disk in the four-ball with boundary K. More generally, two knots K_1 and K_2 are *smoothly concordant* if there exists a smoothly embedded annulus in a thickened three-sphere, $\varphi\colon\ S^1 \times [0,1] \to S^3 \times [0,1]$ such that $\varphi|S^1 \times \{0\} = K_1 \subset S^3 \times \{0\}$ and $\varphi|S^1 \times \{1\}) = K_2 \subset S^3 \times \{1\}$. The connect sum of knots descends to an operation on concordance classes of knots giving the *smooth concordance group*, denoted $\mathcal{C}$. The study of knot concordance is an active area of research, one to which Heegaard Floer theory has applications. For instance, Ozsváth and Szabó [OS03b] and independently Rasmussen [Ras03] used knot Floer homology to define a powerful homomorphism $\tau\colon\ \mathcal{C} \to \mathbb{Z}$. In certain cases, the Heegaard Floer homology of a three-manifold can be equipped with a natural $\mathbb{Q}$–grading. By studying this grading on the branched double cover of a knot, Manolescu and Owens were able to derive a further concordance invariant [MO07]. In Chapter 5, in much the same spirit as Manolescu and Owens, we study a concordance invariant of knots obtained from the grading on the Floer homology of the $+1$–surgery of the knot.

Chapter 2

Heegaard Floer homology

In this chapter, we outline some of the fundamental constructions and properties of Heegaard Floer homology.

2.1 Heegaard diagrams

Let Y be a closed oriented three-manifold. Y may be described by a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ (called a *Heegaard diagram* for Y) where Σ is a oriented genus g surface and $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, ..., \alpha_g\}$, $\boldsymbol{\beta} = \{\beta_1, \beta_2, ..., \beta_g\}$ are collections of closed embedded curves in Σ (called the α and β–curves, or circles, respectfully) such that

1. The α–curves are pairwise disjoint (likewise for the β–curves).
2. The homology classes of the α–curves are linearly independent in $H_1(\Sigma; \mathbb{Z})$ (likewise for the β–curves).
3. The α–curves intersect the β–curves transversely.

These data $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ give rise to a *Heegaard decomposition* for Y (a decomposition into two handlebodies) as follows: start with a thickened surface $\Sigma \times [0, 1]$. Attach thickened disks to $\Sigma \times \{0\}$ along the β–curves and then attach thickened disks to $\Sigma \times \{1\}$ along the α–curves. By condition 2, it follows that we have just constructed a three-manifold with two spherical boundary components. These components are then filled in uniquely by three-balls to give a closed oriented three-manifold (the orientation is chosen to be compatible with the natural orientation on $\Sigma \times [0,1]$ induced by the orientation on Σ). By considering a self-indexing Morse function $f\colon\ Y \to \mathbb{R}$, it follows that any three-manifold admits such a decomposition.

Heegaard decompositions are not unique; there are three moves which do not change the underlying three-manifold. These are *isotopies*, *handleslides*, and *stabilizations*. Isotopies are the simplest to describe: they just move the α– and β–curves in a smooth one-parameter fashion in such a way that all the α–curves remain disjoint (likewise for the β–curves). Stabilization enlarges Σ by connect-summing with a torus T, $\Sigma' = \Sigma \# T$ and replacing $\{\alpha_1, \alpha_2, ..., \alpha_g\}$ with $\{\alpha_1, \alpha_2, ..., \alpha_g, \alpha_{g+1}\}$ and $\{\beta_1, \beta_2, ..., \beta_g\}$ with $\{\beta_1, \beta_2, ..., \beta_g, \beta_{g+1}\}$ where α_{g+1} and β_{g+1} are a pair of transversely intersecting curves in T meeting in a single point. Finally, handleslides take a pair of α–curves (or β–curves) and change one of them to be a "connect-sum" of its previous self and a parallel copy of the other curve. In fact, these three moves and their inverses are sufficient to go between any two Heegaard diagrams for a given three-manifold, by a theorem of Singer [Sin33].

2.1.1 The Heegaard Floer complex

In [OS04e; OS04d], Ozsváth and Szabó introduced Heegaard Floer homology. Starting with a closed oriented three-manifold Y, they associate a *pointed Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. This is just an ordinary Heegaard diagram with the additional choice of a point $z \in \Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$. In the most general construction, Ozsváth and Szabó associate a chain complex $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ (CF^∞ for short) over the module $\mathbb{Z}[U, U^{-1}]$ where U is a formal variable. The Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ gives rise to a pair of tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ inside the symmetric product $\mathrm{Sym}^g(\Sigma)$ via $\alpha_1 \times \alpha_2 \times \cdots \times \alpha_g$ and $\beta_1 \times \beta_2 \times \cdots \beta_g$, respectfully. Since the α– and β–curves are assumed to intersect transversely, the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersect transversely in a finite number of points. CF^∞ is, at the level of generators, the free $\mathbb{Z}[U, U^{-1}]$–module on these intersection points. Generators are either written $[\mathbf{x}, i]$ or $U^i \cdot \mathbf{x}$. The differential on CF^∞ is more difficult to describe. It involves the choice of a generic path of almost complex structures J_t on $\mathrm{Sym}^g(\Sigma)$ and counts certain disks between generators. More specifically, a *Whitney disk* is a map $\varphi\colon\ D^2 \to \mathrm{Sym}^g(\Sigma)$ where D^2 is the unit disk in $\mathbb{C}$ which "connects" a pair of generators $\mathbf{x}$ to $\mathbf{y}$ in the sense that

1. $\varphi(i) = \mathbf{x}$ and $\varphi(-i) = \mathbf{y}$

2. $\varphi(e^{i\theta}) \in \mathbb{T}_\alpha$ if $\cos\theta \leq 0$

3. $\varphi(e^{i\theta}) \in \mathbb{T}_\beta$ if $\cos\theta \geq 0$

For $g > 2$, let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$. For $g = 2$ one takes a further quotient of the Whitney disks as in Ozsváth–Szabó [OS04e] to form $\pi_2(\mathbf{x}, \mathbf{y})$.

The basepoint z gives us two things: a map $n_z\colon \pi_2(\mathbf{x}, \mathbf{y}) \to \mathbb{Z}$ and a map $\mathfrak{s}_z\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$. The first map takes a Whitney disk $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$ to its algebraic intersection number with the subvariety $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$. The map $\mathfrak{s}_z$ works as follows. Let f be a Morse function on Y compatible with the Heegaard decomposition $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. The coordinates of $\mathbf{x}$ determine g trajectories for ∇f[1] which, collectively, connect the index one critical points to the index two critical points. Similarly, z gives a trajectory connecting the index zero critical point with the index three critical point. By removing regular neighborhood of these $g+1$ trajectories, we obtain the complement of a disjoint union of three-balls in Y on which ∇f does not vanish. Further, ∇f has index zero on all the corresponding boundary spheres since the trajectories connect critical points of opposite parity. It follows that ∇f can be extended to a non-vanishing vector field on Y which, by the correspondence of Turaev [Tur97], gives us a Spin^c structure on Y. It is a simple fact that if $\mathfrak{s}_z(\mathbf{x}) \neq \mathfrak{s}_z(\mathbf{y})$ then $\pi_2(\mathbf{x}, \mathbf{y})$ is empty.

Finally, we may define the differential ∂^∞ on CF^∞. It is given by:

$$\partial^\infty[\mathbf{x}, i] = \sum_{\substack{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \mathfrak{s}_z(\mathbf{y}) = \mathfrak{s}_z(\mathbf{x})}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#\widehat{\mathcal{M}}(\phi) \cdot [\mathbf{y}, i - n_z(\phi)]$$

Where here $\mu(\phi)$ denotes the Maslov index of ϕ, the "formal dimension" of the space $\mathcal{M}(\phi)$ of pseudo-holomorphic representatives of the homotopy class of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, and $\widehat{\mathcal{M}}(\phi)$ denotes the quotient of $\mathcal{M}(\phi)$ under the natural action of $\mathbb{R}$ by re-parametrization. Finally, $\#\widehat{\mathcal{M}}(\phi)$ denotes the signed number of points in $\widehat{\mathcal{M}}(\phi)$. Clearly CF^∞ splits according to Spin^c structures. To ensure that the sums appearing in the differential are finite, one must restrict to $\mathfrak{s}$–*admissible* Heegaard diagrams, as defined in Ozsváth and Szabó [OS04e]. In

[1] To form ∇f we can either choose a metric on Y or work with a gradient-like vector field (see Milnor [Mil65].

order to orient the moduli spaces $\widehat{\mathcal{M}}(\phi)$ one needs additional data which we do not get into here. However, if we take Floer homology with $\mathbb{Z}/2$–coefficients, no such data are needed.

There is a natural subcomplex CF^- generated by the $[\mathbf{x}, i]$ with $i < 0$. This gives a quotient complex $CF^+ := CF^\infty/CF^-$. Finally there is the "hat version," $\widehat{CF}$, given by the kernel of the induced map $U\colon\ CF^+ \to CF^+$. By definition, we get a pair of short exact sequences of $\mathbb{Z}[U]$–complexes:

$$0 \longrightarrow CF^- \xrightarrow{\iota} CF^\infty \xrightarrow{\pi} CF^+ \longrightarrow 0 \tag{2.1}$$

and

$$0 \longrightarrow \widehat{CF} \xrightarrow{\iota} CF^+ \xrightarrow{U} CF^+ \longrightarrow 0 \tag{2.2}$$

each of which split according to Spin^c structures on Y.

Through a careful study of how the complexes CF° change under isotopies, handleslides, and stabilizations, Ozsváth and Szabó proved:

Theorem 2.1.1 (Ozsváth and Szabó [OS04e])**.** *Let Y be a closed oriented three-manifold. Then the homology of the chain complexes $HF^\circ(Y, \mathfrak{s}) := H(CF^\circ)$ are topological invariants of the* Spin^c *three-manifold $(Y, \mathfrak{s})$.*

Here $\circ$ denotes any one of $\widehat{\ }$, $+$, or $-$. Of course, the short exact sequences 2.1 and 2.2 give rise to a pair of $\mathbb{Z}[U]$–equivariant long exact sequences

$$\cdots \longrightarrow HF^-(Y, \mathfrak{s}) \xrightarrow{\iota} HF^\infty(Y, \mathfrak{s}) \xrightarrow{\pi} HF^+(Y, \mathfrak{s}) \longrightarrow \cdots \tag{2.3}$$

and

$$\cdots \longrightarrow \widehat{HF}(Y, \mathfrak{s}) \longrightarrow HF^+(Y, \mathfrak{s}) \xrightarrow{U} HF^+(Y, \mathfrak{s}) \longrightarrow \cdots \tag{2.4}$$

2.1.2 Knot Floer homology

Given a null-homologous knot K in a Spin^c three-manifold, the Heegaard Floer complex $CF^\infty(Y, \mathfrak{s})$ can be endowed with extra structure, as was discovered by Ozsváth and Szabó [OS04c] and independently by Rasmussen [Ras03].

A genus g *marked Heegaard diagram* for a knot K in a three-manifold Y is a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$ such that $\alpha_1, \cdots, \alpha_g$ and $\beta_1, \cdots, \beta_{g-1}$ specify the knot complement, β_g is a meridian for K, together with the choice of a point m on $\beta_g \cup (\Sigma - \cup_i \alpha_i - \cup_{j \neq g} \beta_j)$.

A genus g marked Heegaard diagram leads to a *doubly-pointed Heegaard* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ (ie a Heegaard diagram with the choice of two distinct basepoints $w, z \in \Sigma - \cup_i \alpha_i - \cup_j \beta_j$) as follows: place the basepoints w and z close-by on the two sides of β_g in such a way that a short arc joining w and z meets β_g exactly once at m. Finally order the points w and z so that riding the arc from w to z gives the orientation of K.

Given a null-homologous knot $K \subset Y$ inside a three-manifold with Seifert surface F, one may always find a doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$. Similar to previous constructions, the marked point m gives rise to a map

$$\underline{\mathfrak{s}}_m \colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y_0(K))$$

as defined by Ozsváth and Szabó. Here, $Y_0(K)$ denotes the three-manifold obtained by 0–surgery on K with respect to the surface framing, F. This map is defined as follows: replace the meridian β_g with a longitude λ (from the Seifert surface framing) chosen to wind once along the meridian, never crossing m so that each intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has a pair of "closest points" $\mathbf{x}'$ and $\mathbf{x}''$. Letting $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, w)$ denote the induced Heegaard diagram for $Y_0(K)$, define $\underline{\mathfrak{s}}_m(\mathbf{x}) := \mathfrak{s}'_w(\mathbf{x}') = \mathfrak{s}'_w(\mathbf{x}'')$, where $\mathfrak{s}'_w \colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y_0(K))$ denotes the usual map from intersection points to Spin^c structures on $Y_0(K)$.

The set $\mathrm{Spin}^c(Y_0(K))$ is also denoted $\underline{\mathrm{Spin}}^c(Y)$ and is referred to as the set of *relative* Spin^c *structures on* Y. There is an identification $\underline{\mathrm{Spin}}^c(Y) \cong \mathrm{Spin}^c(Y) \oplus \mathbb{Z}$. Under this identification, the map to the first factor is given by restricting a relative Spin^c structure to $Y - K$ and then extending (uniquely) to Y. The map to the second factor is given by the evaluation $\frac{1}{2}\langle c_1(\underline{\mathfrak{t}}), [\widehat{F}]\rangle$, half of the evaluation of the first Chern class of the relative Spin^c structure on the fundamental class of the capped off surface $\widehat{F}$ in the 0–surgery.

Given the above data, Ozsváth and Szabó define the *knot complex* $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$. It is the free Abelian group on triples $[\mathbf{x}, i, j]$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and $i, j \in \mathbb{Z}$. This is endowed with the differential

$$\partial^\infty[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\, [\mathbf{y}, i - n_w(\phi), j - n_z(\phi)].$$

This chain complex comes with an action by $\mathbb{Z}[U]$ given by

$$U \cdot [\mathbf{x}, i, j] = [\mathbf{x}, i-1, j-1].$$

This complex is also $\mathbb{Z} \oplus \mathbb{Z}$–filtered by the map $[\mathbf{x}, i, j] \mapsto (i, j)$. As in the definition of the three-manifold invariants, admissibility conditions must be met to ensure that the sums appearing in the differential are finite. Again, in order to orient the moduli spaces appearing, one needs additional data. The complex $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ naturally splits into subcomplexes, denoted $CFK^\infty(Y, F, K, \mathfrak{s})$. These chain complexes are generated by triples $[\mathbf{x}, i, j]$ satisfying the constraint that

$$\underline{\mathfrak{s}}(\mathbf{x}) + (i - j)\mathrm{PD}(\mu) = \mathfrak{s}_0$$

where here $\mathrm{PD}(\mu) \in H^2(Y_0(K); \mathbb{Z})$ is the Poincaré dual to the meridian of K and $\mathfrak{s}_0$ is the unique Spin^c structure on $Y_0(K)$ satisfying $\langle c_1(\mathfrak{s}_0), [\hat{F}]\rangle = 0$. When F is clear from context it is usually omitted from the notation $CFK^\infty(Y, F, K, \mathfrak{s})$. Ozsváth and Szabó proved in [OS04c] that the $\mathbb{Z} \oplus \mathbb{Z}$–filtered chain homotopy type of $CFK^\infty(Y, F, K, \mathfrak{s})$ is an invariant of $(Y, F, K, \mathfrak{s})$. The complex depends on the Seifert surface F but only up to a shift in the bifiltration. When $H_1(Y; \mathbb{Z}) = 0$, there is no such dependence.

The bifiltered complex $CFK^\infty(Y, F, K, \mathfrak{s})$ may be viewed as a the ordinary Heegaard Floer complex $CF^\infty(Y, \mathfrak{s})$ along with a new $\mathbb{Z}$–filtration. This is given as follows. Given $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, and an extension $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(Y, K)$, we have an isomorphism

$$\Pi_1 \colon\ CFK^\infty(Y, K, \underline{\mathfrak{s}}) \to CF^\infty(Y, \mathfrak{s})$$

given by

$$\Pi_1[\mathbf{x}, i, j] = [\mathbf{x}, i].$$

The new $\mathbb{Z}$–filtration is defined to the projection of $[\mathbf{x}, i, j]$ to j. Through similar constructions, Ozsváth and Szabó showed how the choice of knot K in a Spin^c three-manifold Y gives rise natural filtrations on all versions of Heegaard Floer homology, $HF^\circ(Y, \mathfrak{s})$.

The chain complex $CFK^\infty(Y, K, \mathfrak{s})$ contains a lot of topological information of the knot K. Just to name a few, it determines the Alexander polynomial [OS04c], the knot genus [OS04b], tells if a knot is fibered [Ni09b], and can be used to give restrictions on the four-genera of knots [OS03b]. Finally, the Heegaard Floer homology of any surgery on K can be determined from $CFK^\infty(Y, K, \mathfrak{s})$ by the so-called "integer surgery formula" of Ozsváth and Szabó [OS08]. We use this constuction in Chapter 5.

Chapter 3

Novikov coefficients and torus bundles

3.1 Introduction

The Heegaard Floer homology groups reflect many interesting geometric properties of three-manifolds. For instance, Ozsváth and Szabó showed that they detect the Thurston semi-norm on a closed oriented three-manifold [OS04b]. As another example, work of Ghiggini [Ghi08] and Ni [Ni09b] shows that knot Floer homology detects fiberedness in knots.

Turning to closed fibered three-manifolds, note that a three-manifold which admits a fibration $\pi\colon Y \to S^1$ has a canonical Spin^c structure, ℓ, obtained as the tangents to the fibers of π.

Theorem 3.1.1 (Ozsváth–Szabó [OS04f])**.** *Let Y be a closed three-manifold which fibers over the circle, with fiber F of genus $g > 1$, and let $\mathfrak{t}$ be a* Spin^c *structure over Y with*

$$\langle c_1(\mathfrak{t}), [F]\rangle = 2 - 2g.$$

Then for $\mathfrak{t} \neq \ell$, we have that

$$HF^+(Y, \mathfrak{t}) = 0$$

while

$$HF^+(Y, \ell) \cong \mathbb{Z}.$$

This is commonly referred to as the fact that the Floer homology of a surface bundle (with fiber genus greater than one) is "monic" in its "top-most" Spin^c structure. In fact, Ni proved a converse to Theorem 3.1.1:

Theorem 3.1.2 (Ni [Ni09a])**.** *Suppose Y is a closed irreducible three-manifold, $F \subset Y$ is a closed connected surface of genus $g > 1$. Let $HF^+(Y, [F], 1-g)$ denote the group*

$$\bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(Y)\\ \langle c_1(\mathfrak{s}),[F]\rangle = 2-2g}} HF^+(Y, \mathfrak{s}).$$

If $HF^+(Y, [F], 1-g) \cong \mathbb{Z}$, then Y fibers over the circle with F as a fiber.

For a fiber bundle with torus fiber F, $HF^+(Y, [F], 0)$ is always infinitely generated as an abelian group[1]. However, as we shall show, if one works with Floer homology and an appropriate version of Novikov coefficients, the Floer homology of a torus bundle is still "monic" in a certain sense. Much is already known about the Floer homology of torus bundles. For instance, Baldwin has computed the untwisted Heegaard Floer homologies of torus bundles with $b_1(Y) = 1$ in [Bal08, Theorem 6.4].

In this chapter, which is based on joint work with Yinghua Ai [AP10], we use Heegaard Floer homology with twisted coefficients in the universal Novikov ring, Λ, of all formal power series with real coefficients of the form

$$f(t) = \sum_{r\in\mathbb{R}} a_r\, t^r$$

such that

$$\#\{r \in \mathbb{R} \,|\, a_r \neq 0,\, r \leq c\} < \infty$$

for all $c \in \mathbb{R}$. Given a cohomology class $[\omega] \in H^2(Y;\mathbb{Z})$, Λ can be given a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module structure, and this gives rise to a twisted Heegaard Floer homology $\underline{HF}^+(Y;\Lambda_\omega)$. This version was first defined by Ozsváth and Szabó in [OS04e, Section 10] and can also be derived from the definition of general twisted Heegaard Floer homology in [OS04d,

[1] This can be seen as follows. First of all, for the three-torus it is true by Ozsváth–Szabó [OS04a, Proposition 8.4]. Torus bundles with $b_1 < 3$ have standard HF^∞ (see Section 5.3 of Chapter 5 for the definition of *standard* HF^∞, and Ozsváth–Szabó [OS04d, Theorem 10.1] for the proof of this statement) and since $HF^+ \cong HF^\infty$ in high degrees, the statement follows.

Section 8]. We will describe this group explicitly in Section 3.2.1. It is worth noting that Heegaard Floer homology with twisted coefficients in a certain Novikov ring has already been studied extensively by Jabuka and Mark [JM08a]. They referred to their construction as "perturbed" Heegaard Floer invariants, by way of analogy with certain constructions in Seiberg-Witten theory. The main theorem we prove in this chapter is the following:

Theorem 3.1.3. *Suppose Y is a closed oriented three-manifold which fibers over the circle with torus fiber F, $[\omega] \in H^2(Y;\mathbb{Z})$ is a cohomology class such that $\omega(F) \neq 0$. Then we have an isomorphism of Λ–modules*

$$\underline{HF}^+(Y;\Lambda_\omega) \cong \Lambda.$$

Remark 3.1.4. *In the setting of Monopole Floer homology, a corresponding version of this theorem was proved by Kronheimer and Mrowka in [KM07, Theorem 42.7.1]. Theorem 3.1.3 was also proved using different methods by Lekili [Lek, Theorem 12]. In fact, Lekili actually proved the stronger statement that for a torus bundle Y, $\underline{HF}^+(Y;\Lambda_\omega)$ is supported in the canonical* Spin^c *structure, ℓ. Also, Ai and Ni proved in [AN09] that the converse of the above theorem also holds, ie the twisted Heegaard Floer homology determines whether an irreducible three-manifold is a torus bundle over the circle.*

This chapter is organized as follows. We provide a review of Heegaard Floer homology with twisted coefficients in Section 3.2, including the most pertinent example, $S^1 \times S^2$. In Section 3.3 we prove a relevant exact triangle for ω–twisted Heegaard Floer homology and prove Theorem 3.1.3.

3.2 Review of twisted coefficients

We recall the construction of Heegaard Floer homology with twisted coefficients, referring the reader to Ozsváth and Szabó [OS04d; OS04b] for more details. Given a closed, oriented three-manifold Y we associate a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, where Σ is an an oriented surface of genus $g \geq 1$ and $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, ...\beta_g\}$ are sets of attaching circles (assumed to intersect transversely) for the two handlebodies in the Heegaard decomposition. These give a pair of transversely intersecting g–dimensional tori

$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ in the symmetric product $\mathrm{Sym}^g(\Sigma)$. Recall that the basepoint z gives a map $\mathfrak{s}_z \colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$. Given a Spin^c structure $\mathfrak{s}$ on Y, let $\mathfrak{S} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be the set of intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}$.

Given intersection points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$. There is a natural map from $\pi_2(\mathbf{x}, \mathbf{x})$ to $H^1(Y;\mathbb{Z})$ obtained as follows: each $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$ naturally gives rise to an associated two-chain in Σ whose boundary is a collection of circles among the α and β–curves. We then close off this two-chain by gluing copies of the attaching disks for the handlebodies in the Heegaard decomposition of Y. The Poincaré dual of this two-cycle is the associated element of $H^1(Y;\mathbb{Z})$.

Given a Spin^c structure $\mathfrak{s}$ on Y and a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y, an *additive assignment* is a collection of maps

$$A = \{A_{\mathbf{x},\mathbf{y}} \colon \pi_2(\mathbf{x}, \mathbf{y}) \to H^1(Y;\mathbb{Z})\}_{\mathbf{x},\mathbf{y} \in \mathfrak{S}}$$

with the following properties:

- when $\mathbf{x} = \mathbf{y}$, $A_{\mathbf{x},\mathbf{x}}$ is the canonical map from $\pi_2(\mathbf{x}, \mathbf{x})$ onto $H^1(Y;\mathbb{Z})$ defined above.
- A is compatible with splicing in the sense that if $\mathbf{x}, \mathbf{y}, \mathbf{u} \in \mathfrak{S}$ then for each $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\phi_2 \in \pi_2(\mathbf{y}, \mathbf{u})$, we have that $A(\phi_1 * \phi_2) = A(\phi_1) + A(\phi_2)$.
- $A_{\mathbf{x},\mathbf{y}}(S * \phi) = A_{\mathbf{x},\mathbf{y}}(\phi)$ for $S \in \pi_2(\mathrm{Sym}^g(\Sigma_g))$.

Additive assignments may be constructed with the help of a *complete system of paths* as described in Ozsváth–Szabó [OS04d].

We write elements in the group-ring $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ as finite formal sums

$$\sum_{g \in H^1(Y;\mathbb{Z})} n_g \cdot e^g$$

for $n_g \in \mathbb{Z}$. The *universally twisted Heegaard Floer complex*,

$$\underline{CF}^\infty(Y, \mathfrak{s}; \mathbb{Z}[H^1(Y;\mathbb{Z})], A)$$

is the free $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module on generators $[\mathbf{x}, i]$ for $\mathbf{x} \in \mathfrak{S}$ and $i \in \mathbb{Z}$. The differential, $\underline{\partial}^\infty$, is given by:

$$\underline{\partial}^\infty[x,i] = \sum_{\mathbf{y}\in\mathfrak{S}} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\cdot e^{A(\phi)} \otimes [\mathbf{y}, i-n_z(\phi)]$$

Here $\mu(\phi)$ denotes the Maslov index of ϕ, the formal dimension of the space $\mathcal{M}(\phi)$ of pseudo-holomorphic representatives in the homotopy class of ϕ, $n_z(\phi)$ denotes the intersection number of ϕ with the subvariety $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$, and $\widehat{\mathcal{M}}(\phi)$ denotes the quotient of $\mathcal{M}(\phi)$ under the natural action of $\mathbb{R}$ by reparametrization. To ensure that the sums appearing in the definition of the differential are finite, one must restrict to $\mathfrak{s}$–*admissible* Heegaard diagrams, as defined in Ozsváth–Szabó [OS04e, Section 4.2.2]. Just as in the untwisted setting, this complex admits a $\mathbb{Z}[U]$–action via $U\colon\ [\mathbf{x},i] \mapsto [\mathbf{x}, i-1]$. This gives rise to variants $\underline{CF}^+, \underline{CF}^-$, and $\underline{\widehat{CF}}$, denoted collectively as $\underline{CF}^\circ$. The homology groups of these complexes are the universally twisted Heegaard Floer homology groups $\underline{HF}^\circ$.

More generally, given any $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module M, we may form Floer homology groups with coefficients in M by taking $\underline{HF}^\circ(Y,\mathfrak{s};M)$ as the homology of the complex

$$\underline{CF}^\circ(Y,\mathfrak{s};M,A) := \underline{CF}^\circ(Y,\mathfrak{s};\mathbb{Z}[H^1(Y;\mathbb{Z})],A) \otimes_{\mathbb{Z}[H^1(Y;\mathbb{Z})]} M.$$

For instance, by taking $M=\mathbb{Z}$, thought of as being a trivial $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module (ie, every element acts as the identity), one recovers the ordinary untwisted Heegaard Floer homology, $CF^\circ(Y,\mathfrak{s})$.

In Ozsváth–Szabó [OS04e, Theorem 8.2], it is proved that the homologies defined above are independent of the choice of additive assignment A and are topological invariants of the pair $(Y,\mathfrak{s})$. As in the untwisted setting, these groups are related by long exact sequences:

$$\cdots \longrightarrow \underline{\widehat{HF}}(Y,\mathfrak{s};M) \longrightarrow \underline{HF}^+(Y,\mathfrak{s};M) \xrightarrow{U} \underline{HF}^+(Y,\mathfrak{s};M) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \underline{HF}^-(Y,\mathfrak{s};M) \xrightarrow{\iota} \underline{HF}^\infty(Y,\mathfrak{s};M) \xrightarrow{\pi} \underline{HF}^+(Y,\mathfrak{s};M) \longrightarrow \cdots$$

Of course, the chain complex $\underline{CF}^\circ(Y,\mathfrak{s};M)$ is obtained from the chain complex in the universally twisted case, $\underline{CF}^\circ(Y,\mathfrak{s};\mathbb{Z}[H^1(Y;\mathbb{Z})])$, by a change of coefficients and hence the corresponding homology groups are related by a universal coefficients spectral sequence (see for instance Cartan and Eilenberg [CE99]).

3.2.1 ω–twisted Heegaard Floer homology

In this section we briefly recall the notion of ω–twisted Heegaard Floer homology, following Ozsváth and Szabó [OS04e; OS04b].

We define the *universal Novikov ring* to be the set of all formal power series with real coefficients of the form

$$f(t) = \sum_{r\in\mathbb{R}} a_r\, t^r$$

such that

$$\#\{r\in\mathbb{R}\,|\,a_r\neq 0,\; r\leq c\}<\infty$$

for all $c\in\mathbb{R}$. It is endowed with the following multiplication law, making it into a field:

$$(\sum_{r\in\mathbb{R}} a_r\, t^r)\cdot(\sum_{r\in\mathbb{R}} b_r\, t^r) = \sum_{r\in\mathbb{R}}(\sum_{s\in\mathbb{R}} a_s b_{r-s})\, t^r$$

Furthermore, by fixing a cohomology class $[\omega]\in H^2(Y;\mathbb{R})$ we can give Λ a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module structure via the ring homomorphism:

$$\begin{array}{ccc} \mathbb{Z}[H^1(Y;\mathbb{Z})] & \to & \Lambda \\ \sum a_h\cdot e^h & \mapsto & \sum a_h\cdot t^{\langle h\cup\omega,[Y]\rangle} \end{array}$$

When we have a fixed $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module structure given by a two-form ω in mind, we denote Λ by Λ_ω. This $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module structure gives rise to a twisted Heegaard Floer homology $\underline{HF}^+(Y;\Lambda_\omega)$, which we refer to as *$\omega$–twisted Heegaard Floer homology.* More concretely, it can be defined as follows (see Ozsváth–Szabó [OS04b, Section 3.1]). Choose a weakly admissible pointed Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z)$ for Y and fix a two-cocycle representative $\omega\in[\omega]$. Every Whitney disk ϕ in $\mathrm{Sym}^g(\Sigma)$ (for $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$) gives rise to a two-chain $[\phi]$ in Y by coning off partial α and β–circles with gradient trajectories in the α and β–handlebodies. The evaluation of ω on $[\phi]$ depends only on the homotopy class of ϕ and is denoted $\int_{[\phi]}\omega$ (or sometimes $\omega([\phi])$). The ω–twisted chain complex $\underline{CF}^+(Y;\Lambda_\omega)$ is the free Λ–module generated by $[\mathbf{x}, i]$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and integers $i \geq 0$, endowed with the following differential:

$$\underline{\partial}^+[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\, [\mathbf{y}, i - n_z(\phi)] \cdot t^{\int_{[\phi]} \omega}$$

Its homology is the ω-twisted Heegaard Floer homology $\underline{HF}^+(Y; \Lambda_\omega)$. Notice that this group is both a module for $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ and a module for Λ. Notice also that although the differential depends on the choice of two-cocycle representative $\omega \in [\omega]$, the isomorphism class of the chain complex only depends on the cohomology class. This may be seen as follows: suppose we have cohomologous two-forms ω_1 and ω_2 on Y. Fixing an intersection point $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, define $\Phi\colon\ \underline{CF}^\infty(Y; \Lambda_{\omega_1}) \to \underline{CF}^\infty(Y; \Lambda_{\omega_2})$ by $\Phi([\mathbf{x}, i]) = [\mathbf{x}, i]\, t^{(\omega_2 - \omega_1)[\phi_x]}$ where ϕ_x is any element in $\pi_2(\mathbf{x}_0, \mathbf{x})$ (the choice is irrelevant: the associated domains of any two choices would differ by a periodic domain, which then caps off to a closed surface on which the exact form $\omega_2 - \omega_1$ evaluates to zero). It is then an easy exercise to see that Φ induces a chain isomorphism. An advantage of using this viewpoint is that we avoid altogether the notion of an "additive assignment". It is easy to see (using an argument similar to the previous), that the complex defined above is isomorphic to one obtained by choosing an additive assignment and then tensoring with the $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module Λ_ω.

Suppose $W\colon\ Y_1 \to Y_2$ is a four-dimensional cobordism from Y_1 to Y_2 given by a single two-handle addition and we have a cohomology class $[\omega] \in H^2(W;\mathbb{R})$. Then there is an associated Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ and four-manifold $X_{\alpha\beta\gamma}$ representing W minus a one-complex (see Ozsváth–Szabó [OS06, Proposition 4.3]). Similar to before, a Whitney triangle $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ determines a two-chain in $X_{\alpha\beta\gamma}$ on which we may evaluate a representative, $\omega \in [\omega]$. As before, this evaluation depends only on the homotopy class of $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ and is denoted by $\int_{[\psi]} \omega$. This gives rise to a Λ–equivariant map

$$\underline{F}^+_{W;\omega}\colon\ \underline{HF}^+(Y_1; \Lambda_{\omega|_{Y_1}}) \to \underline{HF}^+(Y_2; \Lambda_{\omega|_{Y_2}})$$

(which is defined only up to multiplication by $\pm t^c$ for some $c \in \mathbb{R}$) defined on the chain level by

$$\underline{f}^+_{W;\omega}[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y}) \\ \mu(\psi)=0}} \#\mathcal{M}(\psi)\, [\mathbf{y}, i - n_z(\psi)] \cdot t^{\int_{[\psi]} \omega}$$

where $\Theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ represents a top-dimensional generator for the Floer homology

$$HF^{\leq 0}(Y_{\beta\gamma}) \cong \wedge^* H^1(Y_{\beta\gamma};\mathbb{Z}) \otimes \mathbb{Z}[U]$$

of the three-manifold determined by the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$, which is a connected sum $\#^{g-1}(S^1 \times S^2)$ (g denotes the genus of Σ), and $\mathcal{M}(\psi)$ denotes the moduli space of pseudo-holomorphic triangles in the homotopy class ψ. This definition may be extended to arbitrary smooth, connected, and oriented cobordisms as in Ozsváth–Szabó [OS06]. These maps may be decomposed as a sum of maps

$$\underline{F}^+_{W;\omega} = \sum_{\mathfrak{s} \in \mathrm{Spin}^c(W)} \underline{F}^+_{W,\mathfrak{s};\omega}$$

which are summed according to Spin^c equivalence classes of triangles, just as in the untwisted setting. This can be extended to arbitrary (smooth, connected) cobordisms from Y_1 to Y_2 as in Ozsváth–Szabó [OS06]. These maps also satisfy a *composition law*: if W_1 is a cobordism from Y_1 to Y_2 and W_2 is a cobordism from Y_2 to Y_3, and we equip W_1 and W_2 with Spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ respectively (whose restrictions agree over Y_2), then putting $W = W_1 \#_{Y_2} W_2$, for any $[\omega] \in H^2(W;\mathbb{R})$ there are choices of representatives for the cobordism maps such that:

$$\underline{F}^+_{W_2,\mathfrak{s}_2;\omega|_{W_2}} \circ \underline{F}^+_{W_1,\mathfrak{s}_1;\omega|_{W_1}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Spin}^c(W) \\ \mathfrak{s}|_{W_i} = \mathfrak{s}_i}} \underline{F}^+_{W,\mathfrak{s};\omega}$$

3.2.2 Example: $S^1 \times S^2$

In this section we calculate twisted Heegaard Floer homologies of $S^1 \times S^2$. We start with the universally twisted version $\underline{\widehat{HF}}(S^1 \times S^2; \mathbb{Z}[t, t^{-1}])$, where we have identified $\mathbb{Z}[H^1(S^1 \times S^2;\mathbb{Z})] \cong \mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials. $S^1 \times S^2$ has a standard genus–one Heegaard decomposition (Σ, α, β) where α is a homotopically nontrivial embedded curve and β is an isotopic translate. For simplicity, we only compute $\underline{\widehat{HF}}$. We make the diagram weakly admissible for the unique torsion Spin^c structure $\mathfrak{s}_0$ by introducing canceling pairs of intersection points between α and β. This gives a pair of intersection points x^+ and x^-. We next need an additive assignment. Notice there is an obvious periodic domain consisting of a pair of (non-homotopic) disks D_1 and D_2 connecting x^+ and x^-. When capped off, the periodic domain gives a sphere representing a generator of $H_2(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$. By taking the Poincaré dual we recover a generator of $H^1(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$. Identifying $H^1(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$, an additive assignment must assign ± 1 to this domain. One way this can be done is by assigning 1 to D_1 and 0 to D_2. We place the basepoint z in the complement of the disks D_1 and D_2. Finally, applying the Riemann mapping theorem, and the argument given in Ozsváth–Szabó [OS04d, page 1169], we see that the complex $\underline{\widehat{CF}}(S^1 \times S^2; \mathbb{Z}[t,t^{-1}])$ is just:

$$0 \longrightarrow \mathbb{Z}[t,t^{-1}] \xrightarrow{1-t} \mathbb{Z}[t,t^{-1}] \longrightarrow 0$$

Here, the first copy of $\mathbb{Z}[t,t^{-1}]$ corresponds to x^+ and the second corresponds to x^-. This complex has homology $\mathbb{Z}$, with trivial $\mathbb{Z}[t,t^{-1}]$–action. This gives the universally twisted Floer homology

$$\underline{\widehat{HF}}(S^1 \times S^2; \mathbb{Z}[t,t^{-1}]) \cong \mathbb{Z},$$

which is supported only in the torsion Spin^c structure $\mathfrak{s}_0$.

Now let us turn to an ω–twisted example. We can view $S^1 \times S^2$ as 0–surgery on the unknot in S^3. Put μ a meridian for the unknot. Then μ defines a curve, also denoted μ, in $S^1 \times S^2$. Put $[\omega] = d \cdot \mathrm{PD}[\mu]$ for an integer d. The complex $\underline{\widehat{CF}}(S^1 \times S^2; \Lambda_\omega)$ is:

$$0 \longrightarrow \Lambda \xrightarrow{t^c(1-t^d)} \Lambda \longrightarrow 0$$

for some $c \in \mathbb{R}$. Notice when $d \neq 0$, $(1-t^d)$ is invertible in Λ. Hence:

$$\underline{\widehat{HF}}(S^1 \times S^2; \Lambda_\omega) = \begin{cases} 0 & \text{when } d \neq 0 \\ \Lambda \oplus \Lambda & \text{when } d = 0 \end{cases}$$

As a final example, we prove a proposition regarding embedded two-spheres in three-manifolds and ω–twisted coefficients.

Lemma 3.2.1. *Let Y_1 and Y_2 be a pair of closed oriented three-manifolds and fix cohomology classes $[\omega_i] \in H^2(Y_i;\mathbb{Z})$. By the Mayer–Vietoris sequence we get a corresponding cohomology class $\omega_1 \# \omega_2 \in H^2(Y_1 \# Y_2;\mathbb{Z}) \cong H^2(Y_1;\mathbb{Z}) \oplus H^2(Y_2;\mathbb{Z})$. Then we have an isomorphism of Λ–modules:*

$$\underline{\widehat{HF}}(Y_1 \# Y_2; \Lambda_{\omega_1 \# \omega_2}) \cong \underline{\widehat{HF}}(Y_1; \Lambda_{\omega_1}) \otimes_\Lambda \underline{\widehat{HF}}(Y_2; \Lambda_{\omega_2})$$

Proof. This follows readily from the methods of proof of Ozsváth–Szabó [OS04d, Proposition 6.1] and the fact that Λ is a field (so that the Künneth sequence simplifies). $\square$

This allows us to prove:

Proposition 3.2.2. *Let $S \subset Y^3$ be an embedded non-separating two-sphere in a three-manifold Y. Suppose $[\omega] \in H^2(Y;\mathbb{Z})$ is a cohomology class such that $\omega([S]) \neq 0$. Then $\underline{HF}^+(Y;\Lambda_\omega) = 0$.*

Proof. Just as in the untwisted theory, $\underline{HF}^+(Y;M)$ vanishes if and only if $\widehat{\underline{HF}}(Y;M)$ vanishes, so it suffices to show that $\widehat{\underline{HF}}(Y;\Lambda_\omega) = 0$. Notice that Y contains an $S^1 \times S^2$ summand in its prime decomposition. Hence $Y \cong S^1 \times S^2 \# Y'$ for some three-manifold Y'. Now $\omega \in H^2(Y;\mathbb{Z}) \cong H^2(S^1 \times S^2;\mathbb{Z}) \oplus H^2(Y';\mathbb{Z})$ corresponds to classes $\omega_1 \in H^2(S^1 \times S^2;\mathbb{Z})$ and $\omega_2 \in H^2(Y';\mathbb{Z})$ with $\omega_1([S]) \neq 0$. We already know that $\widehat{\underline{HF}}(S^1 \times S^2;\Lambda_{\omega_1}) = 0$ from the above calculation, so the proposition follows from Lemma 3.2.1. $\square$

3.3 An exact sequence for ω–twisted Floer homology

In this section we first prove a long exact sequence for the ω–twisted Heegaard Floer homologies and then use it to prove Theorem 3.1.3. It is interesting to notice that there is a similar exact sequence in Monopole Floer homology with local coefficients; see Kronheimer, Mrowka, Ozsváth, and Szabó [KMOS07, Section 5]. Our proof is a slight modification of the proof of the usual surgery exact sequence in Heegaard Floer homology. A good exposition of the original proof may be found in Ozsváth–Szabó [OS05a].

Let $K \subset Y$ be framed knot in a three-manifold Y with framing λ and meridian μ. Given an integer r, let $Y_r(K)$ denote the three-manifold obtained from Y by performing Dehn surgery along the knot K with framing $\lambda + r\mu$. Let $\nu(K)$ denote a small tubular neighborhood of the knot K and $\eta \subset Y - \nu(K)$ be a closed (not necessarily connected) curve in the knot complement. Then for any integer r, $\eta \subset Y - \nu(K) \subset Y_r(K)$ is a closed curve in the surgered manifold $Y_r(K)$. We denote its Poincaré dual by $[\omega_r] \in H^2(Y_r(K);\mathbb{Z})$. Put $I = [0,1]$. Note that $\eta \times I$ represents a relative homology class in the corbordisms $W_0\colon Y \to Y_0(K)$, $W_1\colon Y_0(K) \to Y_1(K)$ and $W_2\colon Y_1(K) \to Y$ which are given by the natural two-handle additions. So as in Section 3.2.1 it gives rise to homomorphisms between ω–twisted Floer homologies

$$\begin{aligned} \underline{F}^+_{W_0;\mathrm{PD}(\eta\times I)}&\colon \underline{HF}^+(Y;\Lambda_\omega)\to \underline{HF}^+(Y_0(K);\Lambda_{\omega_0}),\\ \underline{F}^+_{W_1;\mathrm{PD}(\eta\times I)}&\colon \underline{HF}^+(Y_0(K);\Lambda_{\omega_0})\to \underline{HF}^+(Y_1(K);\Lambda_{\omega_1}),\\ \underline{F}^+_{W_2;\mathrm{PD}(\eta\times I)}&\colon \underline{HF}^+(Y_1(K);\Lambda_{\omega_1})\to \underline{HF}^+(Y;\Lambda_\omega). \end{aligned}$$ and

Where here $\omega = \mathrm{PD}(\eta)\in H^2(Y;\mathbb{Z})$. We denote the corresponding maps on the chain level by $\underline{f}^+_{W_0;\mathrm{PD}(\eta\times I)}$, $\underline{f}^+_{W_1;\mathrm{PD}(\eta\times I)}$ and $\underline{f}^+_{W_2;\mathrm{PD}(\eta\times I)}$ respectively.

Theorem 3.3.1. *The maps above form an exact sequence of Λ–modules:*

$$\begin{tikzcd} \underline{HF}^+(Y;\Lambda_\omega) \arrow[rr] & & \underline{HF}^+(Y_0(K);\Lambda_{\omega_0}) \arrow[dl] \\ & \underline{HF}^+(Y_1(K);\Lambda_{\omega_1}) \arrow[ul] & \end{tikzcd} \tag{3.1}$$

Furthermore, analogous exact sequences hold for "hat" versions as well.

Proof. Find a Heegaard diagram $(\Sigma_g,\{\alpha_1,\cdots,\alpha_g\},\{\beta_1,\cdots,\beta_g\},z)$ compatible with the knot K. More precisely, K lies in the handlebody specified by the β–curves and β_1 is a meridian for K. For each $i\geq 2$ let γ_i,δ_i be exact Hamiltonian isotopies of the β_i. Let $\gamma_1=\lambda$, $\delta_1=\lambda+\mu$ be the 0–framed and 1–framed longitude of the knot K, respectively. We assume the Heegaard quadruple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta},z)$ is weakly admissible in the sense of Ozsváth–Szabó [OS04e]. It is easy to see that $Y_{\alpha\beta}=Y$, $Y_{\alpha\gamma}=Y_0(K)$, $Y_{\alpha\delta}=Y_1(K)$, and $Y_{\beta\gamma}\cong Y_{\gamma\delta}\cong Y_{\beta\delta}\cong \#^{g-1}S^2\times S^1$.

Following Ozsváth–Szabó [OS05c], we define a map

$$h_1\colon \underline{CF}^+(Y;\Lambda_\omega)\to \underline{CF}^+(Y_1(K);\Lambda_{\omega_1})$$

by counting holomorphic rectangles:

$$h_1([\mathbf{x},i]) = \sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\delta}\ \sum_{\substack{\varphi\in\pi_2(\mathbf{x},\Theta_{\beta\gamma},\Theta_{\gamma\delta},\mathbf{w})\\ \mu(\varphi)=0}} \#\mathcal{M}(\varphi)[\mathbf{w},i-n_z(\varphi)]t^{\int_{[\varphi]}\mathrm{PD}(\eta\times I)}$$

where here $\mathcal{M}(\varphi)$ denotes the moduli space of (maps of) pseudo-holomorphic rectangles into $\mathrm{Sym}^g(\Sigma)$ allowing the conformal structure on the domain to vary. The notation

$\int_{[\varphi]} \mathrm{PD}(\eta \times I)$ in the above formula requires some explanation. The Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ gives rise to a four-manifold $X_{\alpha\beta\gamma\delta}$ (as defined in Ozsváth–Szabó [OS06]) which can be thought of as the complement of two one-complexes in the composite cobordism $Y \to Y_0 \to Y_1$ and therefore we can consider $\mathrm{PD}(\eta \times I)$ as a class in $H^2(X_{\alpha\beta\gamma\delta}; \mathbb{Z})$. Similar to the definition of the cobordism maps, the Whitney rectangle φ determines a two-chain in $X_{\alpha\beta\gamma\delta}$ on which we may evaluate the two-form $\mathrm{PD}(\eta \times I)$, denoted $\int_{[\varphi]} \mathrm{PD}(\eta \times I)$. Similarly we define $h_2\colon \underline{CF}^+(Y_0(K); \Lambda_{\omega_0}) \to \underline{CF}^+(Y; \Lambda_\omega)$ and $h_3\colon \underline{CF}^+(Y_1(K); \Lambda_{\omega_1}) \to \underline{CF}^+(Y_0(K); \Lambda_{\omega_0})$.

We claim that h_1 is a null homotopy of $\underline{f}^+_{W_1;\mathrm{PD}(\eta\times I)} \circ \underline{f}^+_{W_0;\mathrm{PD}(\eta\times I)}$. To see this, we consider the moduli space of holomorphic rectangles of Maslov index one. This moduli space can have 6 kinds of ends:

1. Splicing holomorphic discs at one the four corners of a holomorphic rectangle.

2. Splicing two holomorphic triangles. Triangles may be spliced in two ways: one triangle for $X_{\alpha\beta\gamma}$ and one triangle for $X_{\alpha\gamma\delta}$, or one triangle for $X_{\alpha\beta\delta}$ and one triangle for $X_{\beta\gamma\delta}$

Notice $\mathrm{PD}(\eta \times I)$ is 0 when restricted to the corners $Y_{\beta\gamma}$ and $Y_{\gamma\delta}$: in fact, we can make $\eta \times I$ disjoint from these manifolds since η may be pushed completely into the α–handlebody, U_α, by cellular approximation (see Figure 3.1). Alternatively, since $X_{\alpha\beta\gamma\delta}$ is obtained from the composite cobordism $W_0 \# W_1$ by removing a neighborhood of a one-complex, we can choose this one-complex to be disjoint from the annuli $\eta \times I$.

This implies that

$$\underline{CF}^+(Y_{\beta\gamma}; \Lambda_{\mathrm{PD}(\eta\times I)|_{Y_{\beta\gamma}}}) \cong CF^+(Y_{\beta\gamma}) \otimes_{\mathbb{Z}} \Lambda$$

and all differentials are trivial (informally, we are using an "untwisted" count). For the end coming from splicing two holomorphic triangles, one for $X_{\alpha\beta\delta}$ and one for $X_{\beta\gamma\delta}$, it is also true that $\mathrm{PD}(\eta \times I)$ is 0 when restricted to the four-manifold $X_{\beta\gamma\delta}$ (again, since η may be pushed completely into U_α). Therefore we are counting holomorphic triangles in $X_{\beta\gamma\delta}$ "without twisting". In Ozsváth–Szabó [OS04d] it is shown that the untwisted counting of holomorphic triangles in $X_{\beta\gamma\delta}$ is zero. This leaves three terms remaining.

1. Splicing a disc at corner $Y_{\alpha\beta}$ counted with twisting by $\mathrm{PD}(\eta \times I)|_{Y_{\alpha\beta}} = [\omega]$, which corresponds to $h_1 \circ \partial$.

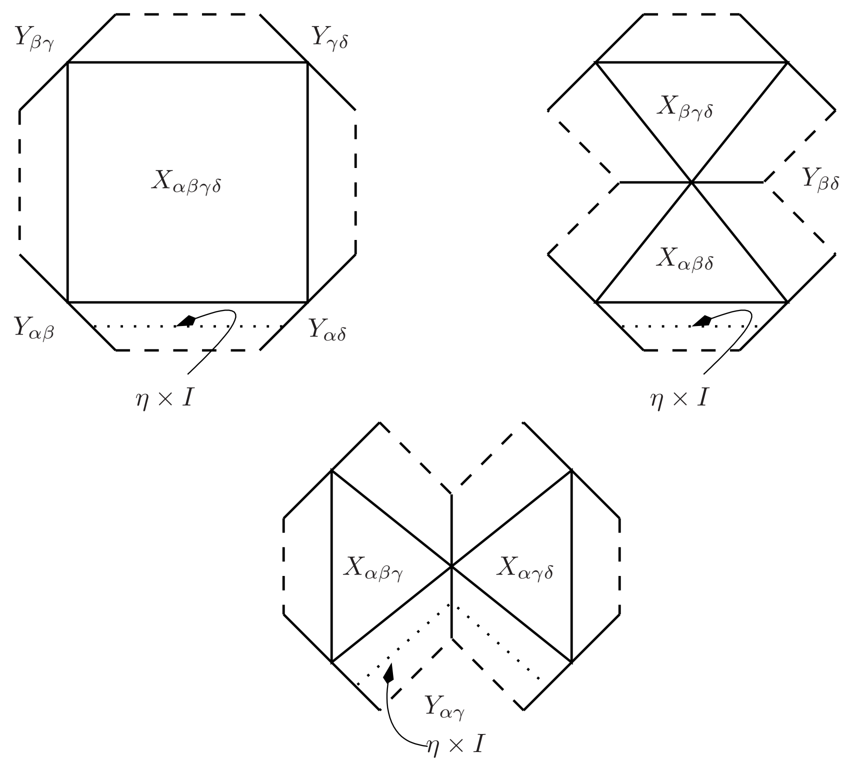

Figure 3.1: Schematics of the four-manifold $X_{\alpha\beta\gamma\delta}$ and its decompositions.

2. Splicing a disc at corner $Y_{\alpha\delta}$ counted with twisting by $\mathrm{PD}(\eta \times I)|_{Y_{\alpha\delta}} = [\omega_1]$, which corresponds to $\partial \circ h_1$.

3. Splicing two holomorphic triangles from $X_{\alpha\beta\gamma}$ and $X_{\alpha\gamma\delta}$ counted with twisting by $\mathrm{PD}(\eta \times I)$, which corresponds to $\underline{f}^+_{W_1;\mathrm{PD}(\eta\times I)} \circ \underline{f}^+_{W_0;\mathrm{PD}(\eta\times I)}$.

From the fact that the moduli space must have total end zero, it is clear that the sum of the above three terms is zero, ie h_1 is a homotopy connecting $\underline{f}^+_{W_1;\mathrm{PD}(\eta\times I)} \circ \underline{f}^+_{W_0;\mathrm{PD}(\eta\times I)}$ to the zero map. This shows that $\underline{F}^+_{W_1;\mathrm{PD}(\eta\times I)} \circ \underline{F}^+_{W_0;\mathrm{PD}(\eta\times I)} = 0$ on the homology level. The same argument shows that $\underline{F}^+_{W_2;\mathrm{PD}(\eta\times I)} \circ \underline{F}^+_{W_1;\mathrm{PD}(\eta\times I)} = 0$ and $\underline{F}^+_{W_0;\mathrm{PD}(\eta\times I)} \circ \underline{F}^+_{W_2;\mathrm{PD}(\eta\times I)} = 0$ as well.

At last we prove that the sequence, Equation 3.1, is exact. Using a homological algebra argument as in Ozsváth–Szabó [OS05c] we need to show that $h \circ \underline{f}^+ + \underline{f}^+ \circ h$ is homotopic to the identity map. This can be done by counting holomorphic pentagons and noticing that we have a class $\mathrm{PD}(\eta \times I) \in H^2(X_{\alpha\beta\gamma\delta\beta'})$ similar to before (here $X_{\alpha\beta\gamma\delta\beta'}$ is the complement of three one-complexes in the composite cobordism $Y \to Y_0 \to Y_1 \to Y$) and that $\mathrm{PD}(\eta \times I)$ is zero when restricted to $Y_{\beta\gamma}$,$Y_{\gamma\delta}$,$Y_{\delta\beta'}$,$X_{\beta\gamma\delta}$, $X_{\gamma\delta\beta'}$ and $X_{\beta\gamma\delta\beta'}$, similar to before. This shows that the counts there are "untwisted". From this observation one can easily see that everything in the proof of exactness in [OS05c] can go through to our twisted version. □

In the above theorem, the cohomology classes $[\omega_r]$ are integral. In practice one may need to use real cohomology class as well. In that situation, a given cohomology class $[\omega] \in H^2(Y;\mathbb{R})$ can be expressed as a finite sum

$$[\omega] = \sum a_i \mathrm{PD}(\eta_i)$$

where the η_i are closed curves in the knot complement and $a_i \in \mathbb{R}$. Each η_i can be viewed as a closed curve in $Y_r(K)$, so the expression $\sum a_i \mathrm{PD}(\eta_i)$ also gives a real cohomology class in $Y_r(K)$, denoted by $[\omega_r] \in H^2(Y_r(K);\mathbb{R})$. In the cobordism W_r,

$$\sum a_i \mathrm{PD}(\eta_i \times I)$$

is a real cohomology class in $H^2(W_r;\mathbb{R})$, hence gives rise to homomorphism between ω–twisted Floer homologies. With this understood, it is easy to see that an analogue of Theorem 3.3.1 still holds.

Remark 3.3.2. *The exact sequence in Theorem 3.3.1 depends on the actual curve η, not just its Poincaré dual $[\omega] \in H^2(Y;\mathbb{Z})$. In fact if we take another closed curve $\eta' = \eta + k \cdot \mu$ (where μ is a meridian of K), this doesn't change $[\omega]$, but may change $[\omega_0], [\omega_1]$ and the exact sequence. For example, take $K \subset S^3$ to be the unknot and $\eta = k \cdot \mu$ in the knot complement, then $[\omega_0]$ is k times the generator of $H^2(S^2 \times S^1;\mathbb{Z})$. When $k \neq 0$, the corresponding exact sequence for the hat version is*

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{\widehat{HF}}(S^3_0(K);\Lambda_{\omega_0}) \arrow[r] \arrow[d, "\cong"] & \underline{\widehat{HF}}(S^3_1(K);\Lambda_{\omega_1}) \arrow[r] \arrow[d, "\cong"] & \underline{\widehat{HF}}(S^3;\Lambda_{\omega}) \arrow[r] \arrow[d, "\cong"] & \cdots \\
& 0 \arrow[r] & \Lambda \arrow[r, "1-t^k"] & \Lambda &
\end{tikzcd}$$

Clearly it depends on k. When $k = 0$, the exact sequence is obtained from the corresponding exact sequence for untwisted Heegaard Floer homology by tensoring with Λ.

In [OS04b], Ozsváth and Szabó used another version of twisted Floer homology,

$$\underline{HF}^+(Y;[\omega]),$$

which is defined by using the $\mathbb{Z}[H^1(Y;\mathbb{Z})]$–module $\mathbb{Z}[\mathbb{R}]$. The ω–twisted Floer homology we used in this chapter can be viewed as a completion of $\underline{HF}^+(Y;[\omega])$. It is easy to see that there is a similar exact sequence in their context. More precisely, we have the following exact sequence:

$$\begin{tikzcd} \underline{HF}^+(Y;[\omega]) \arrow[rr] & & \underline{HF}^+(Y_0(K);[\omega_0]) \arrow[dl] \\ & \underline{HF}^+(Y_1(K);[\omega_1]) \arrow[ul] & \end{tikzcd} \tag{3.2}$$

With the above exact sequences in place, we can now prove Theorem 3.1.3. We merely mimic Ozsváth and Szabó's proof of [OS04f, Theorem 5.2].

Proof of Theorem 3.1.3. For a given cohomology class $[\omega] \in H^2(Y;\mathbb{Z})$ with $\omega(F) = d \neq 0$, choose a closed curve $\eta \subset Y$ such that its Poincaré dual $\mathrm{PD}(\eta)$ equals the image of $[\omega]$ in $H^2(Y;\mathbb{R})$. Since the mapping class group of a torus is generated as a monoid by right-handed Dehn twists along non-separating curves (see Humphries [Hum77] or Ozsváth–Szabó [OS04f, Theorem 2.2]), we can connect Y to the three-manifold $S^3_0(T)$ which is obtained from S^3 by performing 0–surgery on the right-handed trefoil, by a sequence of torus bundles

$$\pi_i\colon\ Y^i \to S^1$$

and cobordisms

$$Y = Y^0 \xrightarrow{W_0} Y^1 \xrightarrow{W_1} \cdots \xrightarrow{W_{n-1}} Y^n = S^3_0(T)$$

such that the monodromy of Y^{i+1} differs from that of Y^i by a single right-handed Dehn twist along a non separating knot K_i which lies in a fiber F_i of π_i. The curve $\eta \subset Y$ induces curves $\eta_i \subset Y^i$ which can be assumed disjoint from the K_i. In this way, we get a sequence of cohomology classes $\omega_i = \mathrm{PD}(\eta_i) \in H^2(Y^i;\mathbb{Z})$ such that $\omega_i(F_i) = d \neq 0$. The cobordism W_i is obtained by attaching a single two-handle to $Y^i \times I$ along the knot K_i with framing -1 (with respect to the framing K_i inherits from the fiber F_i). Since η_i is disjoint from K_i,

$\eta_i \times I$ defines a relative homology class $[\eta_i \times I] \in H_2(W_i, \partial W_i; \mathbb{Z})$ and hence its Poincaré dual gives rise to homomorphisms between ω–twisted Floer homologies:

$$\underline{F}^+_{W_i;\mathrm{PD}(\eta_i \times I)} \colon \underline{HF}^+(Y^i; \Lambda_{\omega_i}) \to \underline{HF}^+(Y^{i+1}; \Lambda_{\omega_{i+1}})$$

We claim that these maps are all isomorphisms. Notice that $Y^{i+1} = (Y^i)_{-1}(K_i)$ where the 0–framing of K_i is defined to be the framing K_i inherits from the fiber, F_i. Now consider $(Y^i)_0(K_i)$. This manifold contains a two-sphere S_i (which is obtained from F_i by surgering along K_i) and also an induced curve η_i such that $\eta_i \cdot S_i = d \neq 0$, therefore $\underline{HF}^+((Y^i)_0(K_i); \Lambda_{\mathrm{PD}(\eta_i)}) = 0$ by Proposition 3.2.2. The exact sequence, Equation 3.1, now proves the claim.

This shows that

$$\underline{HF}^+(Y; \Lambda_\omega) \cong \underline{HF}^+(S^3_0(T); \Lambda_{\mathrm{PD}(\eta)})$$

where η is the induced curve in $S^3_0(T)$. We now identify the latter group. For simplicity we write $\omega = \mathrm{PD}(\eta)$. Identifying $\mathbb{Q}[H^1(S^3_0(T); \mathbb{Z})]$ with $\mathbb{Q}[t, t^{-1}]$, Ozsváth and Szabó show in [OS04a] that there is an identification of $\mathbb{Q}[t, t^{-1}]$–modules:

$$\underline{HF}^+_k(S^3_0(T); \mathbb{Q}[t, t^{-1}]) \cong \begin{cases} \mathbb{Q} & \text{if } k \equiv -1/2 \pmod 2 \text{ and } k \geq -1/2 \\ \mathbb{Q}[t, t^{-1}] & \text{if } k = -3/2 \\ 0 & \text{otherwise} \end{cases}$$

Where the left hand group is the universally twisted Heegaard Floer homology of $S^3_0(T)$, $\mathbb{Q}[H^1(S^3_0(T); \mathbb{Z})]$ acts on $\mathbb{Q}$ by the identity, and $\mathbb{Q}[t, t^{-1}]$ is a module over itself in the natural way. By definition:

$$\underline{CF}^+(S^3_0(T); \Lambda_\omega) = \underline{CF}^+(S^3_0(T); \mathbb{Q}[t, t^{-1}]) \otimes_{\mathbb{Q}[t,t^{-1}]} \Lambda_\omega$$

Notice $\mathbb{Q}[t, t^{-1}]$ is a principal ideal domain, so by the universal coefficients theorem (see for instance of Hilton and Stammbach [HS70, Theorem 2.5]) there is an exact sequence:

$$\begin{aligned} 0 \to \underline{HF}^+(S^3_0(T); \mathbb{Q}[t, t^{-1}]) &\otimes_{\mathbb{Q}[t,t^{-1}]} \Lambda_\omega \\ &\to \underline{HF}^+(S^3_0(T); \Lambda_\omega) \to \mathrm{Tor}_1^{\mathbb{Q}[t,t^{-1}]}(\underline{HF}^+(S^3_0(T), \Lambda) \to 0 \end{aligned}$$

We need only compute $\mathrm{Tor}_1^{\mathbb{Q}[t,t^{-1}]}(\mathbb{Q}, \Lambda_\omega)$. Start with the free $\mathbb{Q}[t, t^{-1}]$–resolution of $\mathbb{Q}$:

$$0 \longrightarrow \mathbb{Q}[t, t^{-1}] \xrightarrow{1-t} \mathbb{Q}[t, t^{-1}] \longrightarrow \mathbb{Q} \longrightarrow 0$$

Tensoring this complex over $\mathbb{Q}[t,t^{-1}]$ with Λ_ω and augmenting gives the complex

$$0 \longrightarrow \Lambda \xrightarrow{1-t^d} \Lambda \longrightarrow 0$$

where here $d = \langle \mathrm{PD}(\eta), F\rangle$ for F the torus fiber. Since we are working over Λ and $d \neq 0$, the middle map is an isomorphism and we see that $\mathrm{Tor}_q^{\mathbb{Q}[t,t^{-1}]}(\mathbb{Q},\Lambda_\omega) = 0$ for all q. From the above exact sequence, we obtain an isomorphism of Λ–modules $\underline{HF}^+(S^3_0(T);\Lambda_\omega) \cong \Lambda$. Therefore

$$\underline{HF}^+(Y;\Lambda_\omega) \cong \Lambda \qquad \square$$

It is worth noting that alternate proofs of this theorem as well as Proposition 3.2.2 are possible through the use of inadmissible diagrams, which have been explored by Wu in [Wu09] as well as by Lekili in [Lek].

Chapter 4

L-Spaces and left-orderings of the fundamental group

4.1 Introduction

A three-manifold Y is called an L–space if it is a rational homology three-sphere and its hat version of Heegaard Floer homology is "as simple as possible" in the sense that the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y;\mathbb{Z})|$[1]. The class of L–spaces includes all lens spaces and is closed under connected sum as well as orientation reversal. According to a theorem of Némethi, a three-manifold obtained as a plumbing of disk bundles over spheres is an L–space if and only if it is the link of a rational surface singularity [N 05]. In particular, any three-manifold with spherical geometry is an L–space, a fact which was first established by Ozsváth and Szabó [OS05b, Proposition 2.3]. According to a theorem of Ozsváth and Szabó, an L–space cannot have a co-orientable taut foliation [OS04b]. This provides a nice bridge between the world of pseudo-holomorphic curve invariants and the geometry of three-manifolds. Though there is not yet a classification of L–spaces, there is a complete answer in the case of Seifert fibered spaces with base orbifold $S^2(\alpha_1, \cdots, \alpha_n)$, according to the following theorem of Lisca and Stipsicz [LS07] which states

Theorem 4.1.1 (Lisca-Stipsicz [LS07]). *Let M be an oriented Seifert fibered rational ho-*

[1] For a rational homology three-sphere, we always have $|H_1(Y;\mathbb{Z})| \leq \text{rank } \widehat{HF}(Y)$.

mology three-sphere with base S^2. Then the following statements are equivalent:

1. *M is an L–space*

2. *Either M or $-M$ carries no positive transverse contact structures*

3. *M carries no transverse foliations*

4. *M carries no taut foliations.*

Moreover, the existence of transverse foliations is completely understood and has a simple combinatorial answer given in terms of the Seifert invariants, as was shown by work of Eisenbud, Hirsch, Jankins, Neumann, and Naimi (see [EHN81], [JN85a], [JN85b], [Nai94]).

A group G is called *left-orderable* if it may be given a strict total ordering $\prec$ which is left-invariant, ie $g \prec h$ if and only if $fg \prec fh$ for any $f, g, h \in G$. Orderability properties of the fundamental group of have interesting consequences for the topology of three-manifolds. For instance, Calegari and Dunfield showed that three-manifolds with non-left-orderable fundamental group do not support co-orientable $\mathbb{R}$–covered foliations[2] [CD03]. Though in general there is not a complete understanding of when a three-manifold group is left-orderable, Boyer, Rolfsen, and Wiest provide the answer in the case of Seifert fibered spaces [BRW05]:

Theorem 4.1.2 (Boyer-Rolfsen-Wiest [BRW05])**.** *The fundamental group of a compact, connected, Seifert fibered space M is left-orderable if and only if $M \cong S^3$ or one of the following two sets of conditions holds:*

1. $\mathrm{rank}_{\mathbb{Z}} H_1(M;\mathbb{Z}) > 0$ *and* $M \ncong \mathbb{R}P^2 \times S^1$*;*

2. *M is orientable, the base orbifold of M is of the form $S^2(\alpha_1, \alpha_2, ..., \alpha_n)$, $\pi_1(M)$ is infinite, and M admits a transverse foliation.*

[2] By definition, an $\mathbb{R}$–covered foliation of a three-manifold is a codimension one foliation such that the pulled-back foliation on the universal cover is the product foliation of $\mathbb{R}^3$ by horizontal $\mathbb{R}^2$'s. For a closed manifold, this implies tautness. See Calegari [Cal00a; Cal00b; Cal99] for a discussion of some of the properties of $\mathbb{R}$–covered foliations.

Putting together Theorems 4.1.1 and 4.1.2, the remark that spherical manifolds are L–spaces, and the fact that closed Seifert fibered three-manifolds with finite fundamental group are spherical, we see that the class of Seifert fibered L–space is almost exactly the class of Seifert fibered three-manifolds with non-left-orderable fundamental group (union the three-sphere). The only place where they could presumably differ is the case of Seifert fibered L–spaces with base $\mathbb{R}P^2$. However, they were proved to agree in this case by Boyer and Watson in [Wat09]. In another direction, Greene has informed me [Gre] of a quick proof that the branched double cover of an alternating knot (which is always an L–space) has non-left-orderable fundamental group. Given these facts, it is natural then to explore the connection between L–spaces and non-left-orderable three-manifold groups further. Examples of infinite families of hyperbolic manifolds with non-left-orderable fundamental group are provided by the work or Roberts, Shareshian, and Stein [RSS03]. These manifolds were shown to be L–spaces by the work of Baldwin [Bal07]. In their paper [CD03], Calegari and Dunfield determined that of the 128 closed hyperbolic manifolds of volume < 3 which are $\mathbb{Z}/2$–homology spheres, at least 44 of them have non-left-orderable fundamental group. Dunfield later showed that all of these are in fact L–spaces [Dun]. Further examples of non-left-orderable three-manifold groups are provided by a paper of Dabkowski, Przytycki, and Togha [DPT05]. They prove

Theorem 4.1.3 (Dabkowski-Przytycki-Togha [DPT05])**.** *Let $\Sigma_n(L)$ denote the n–fold branched cyclic cover of the oriented link L, where $n > 1$. Then the fundamental group, $\pi_1(\Sigma_n(L))$, is not left-orderable in the following cases:*

1. *$L = T_{(2',2k)}$ is the torus link of type $(2, 2k)$ with the anti-parallel orientation of strings, and n is arbitrary.*

2. *$L = P(n_1, n_2, ..., n_k)$ is the pretzel link of the type $(n_1, n_2, ..., n_k)$, $k > 2$, where either $n_1, n_2, ..., n_k > 0$ or $n_1 = n_2 = \cdots = n_{k-1} = 2, n_k = -1$ and $k > 3$. The multiplicity of the covering is $n = 2$.*

3. *$L = L_{[2k,2m]}$ is the two-bridge knot of type $p/q = 2m + \frac{1}{2k} = [2k, 2m]$, where $k, m > 0$,*

and n is arbitrary.

4. *$L = L_{[n_1,1,n_3]}$ is the two-bridge knot of type $p/q = n_3 + \frac{1}{1+\frac{1}{n_1}}$, where n_1 and n_3 are odd, positive integers. The multiplicity of the covering is $n \leq 3$.*

In this chapter we show that

Theorem 4.1.4. *All of the manifolds in Theorem 4.1.3 are Heegaard Floer homology L–spaces.*

The manifolds in Theorem 4.1.3, items (1) and (2), $\Sigma_3(4_1)$ from (4), and the covers of the trefoil from (4) are Seifert fibered and are hence covered by our previous remarks. For the other cases, which are hyperbolic, we realize them as branched *double* covers of of quasi-alternating links in the three-sphere. This allows us to apply a theorem of Ozsváth and Szabó which states that the branched double cover of a quasi-alternating link in S^3 is an L–space. We also give an independent proof for the manifolds in Theorem 4.1.3, item (1) also by realizing them as branched double covers of alternating links.

4.1.1 Further questions

We provide a list of unanswered questions which the author finds fascinating.

1. We have a question of Ozsváth and Szabó: is it true in general that a closed, oriented, and irreducible three-manifold is an L–space if and only if it has no co-orientable taut foliation?

2. Need a three-manifold with non-left-orderable fundamental group be an L–space? What about the converse?

3. Given a knot or link, when is its n–fold cyclic cover an L–space? For instance, it follows from Baldwin's classification of L–spaces among three-manifolds admitting genus one, one boundary component open books [Bal08] that $\Sigma_n(3_1)$ is an L–space if and only if $n \leq 5$ and $\Sigma_n(4_2)$ is an L–space for every n.

4. Which manifolds on the Hodgson-Weeks census are L–spaces? Dunfield informs me that at least 3,000 of the 11,000 census manifolds are L–spaces [Dun].

5. Is every L–space the branched double cover of a link in S^3?

6. Give some description of hyperbolic L–spaces.

7. Connections to contact geometry: Lisca and Stipsicz recently solved the existence problem for tight contact structures on Seifert fibered three-manifolds [LS09]:

   Theorem 4.1.5 (Lisca-Stipsicz [LS09])**.** *A Seifert fibered three-manifold admits a tight contact structure if and only if it is not orientation preserving diffeomorphic to the result of* $(2n-1)$*-surgery along the* $(2, 2n+1)$*-torus knot* $T_{2,2n+1} \subset S^3$ *for some* $n \in \mathbb{N}$.

   In proving this theorem, their classification of Seifert fibered L–spaces proved essential. On the other hand, toroidal three-manifolds are known to admit infinitely many different contact structures (see [CGH03], [HKM04]). Little is known, however, about the existence of tight contact structures on hyperbolic three-manifolds. Some information is provided by work of Baldwin [Bal07], and it is known that the Weeks manifold admits tight contact structures [Sti08]. Futhermore, any co-orientable taut foliation (of which there are many—see for instance Roberts, Shareshian and Stein [RSS03]) may be perturbed to a tight contact structure by a theorem of Eliashberg and Thurston [ET98]. What can one say about tight contact structures on the manifolds from items (3) and (4) in Theorem 4.1.3?

4.2 Background

4.2.1 A surgery presentation of the branched double cover of a link in S^3

We begin with a review an algorithm that takes a diagram for a knot or link and produces a surgery presentation for its branched double cover, which is described in Ozsváth–Szabó [OS04b]. Given a diagram of a link $D(K)$ pick an edge at random to mark. Then checkerboard color the plane. This allows us to produce the *black graph* of $D(K)$, denoted

$B(D(K))$: it is a planar graph whose vertices are in one-to-one correspondence with the black regions in our checkerboard coloring of the plane, and whose edges correspond to crossings in the diagram. The edges are further decorated by an incidence number $\mu(e) = \pm 1$ given by the rule of Figure 4.1. The vertices are then weighted by the sum of the incidences of the incident edges $w(v) = -\sum_{e \text{ incident to } v} \mu(e)$. We then form the *reduced black graph* $\widetilde{B}(D(K))$ by deleting the vertex which corresponds to the region touching the marked edge and then deleting all edges which are incident to this vertex. We then draw a surgery diagram as follows: for each vertex of $\widetilde{B}$ we draw a planar unknot (such that all are unlinked). For each edge between two vertices we add a right/left-handed clasp between the corresponding unknots according to the incidence of the edge (see Figure 4.2) or, equivalently, we perform ∓ 1 surgery, respectively, on an unknot which links the two components as shown in Figure 4.2. If we chose to draw clasps, we frame each unknotted component by the weight on its corresponding vertex. If we chose to draw linking ± 1 curves, we then 0–frame each of the original unknots coming from the vertices of the reduced black graph and add small linking unknots of framing ± 1 in such a way that the sum of all the framings of the linking unknots to this component is minus the vertex marking of the corresponding vertex. This gives a surgery presentation for $\Sigma_2(K)$ (see Figure 4.3 for an example).

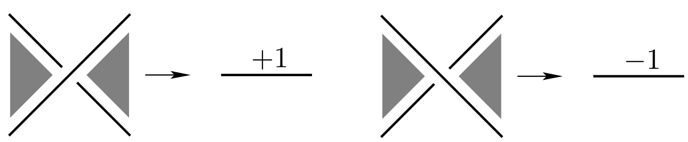

Figure 4.1: Incidence assignment rules.

Warning: in this chapter, we will occasionally work with decorated graphs such as the graph labeled $\widetilde{B}(D(K))$ in Figure 4.3. Though aesthetically similar, these diagrams are generally not the same as plumbing graphs (in the sense of Neumann [Neu81], for instance). There is one exception, however: when our graph is a tree and we delete the edge markings (they are irrelevant) then we actually do have a plumbing description of our manifold.

One may visualize the involution on this manifold, giving rise to our link: line up the 0–framed circles on a line and then put in the ± 1–framed unknots each intersecting the axis of symmetry in two points in such a way that the whole diagram has a symmetry about

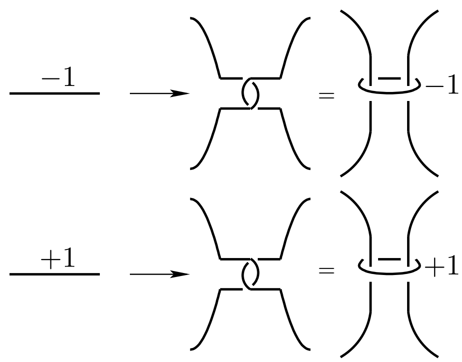

Figure 4.2: Creating clasps out of incidences.

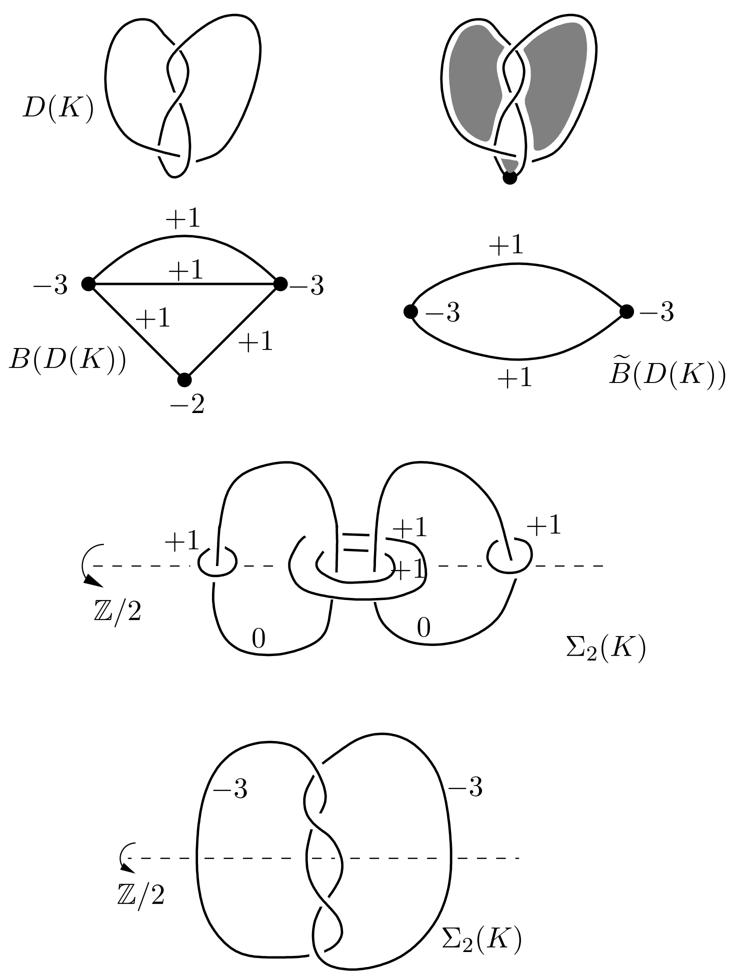

Figure 4.3: Going from a diagram of the figure eight to a surgery presentation of its branched double cover.

an axis, as shown in the example, Figure 4.3. The complement of the surgered solid tori has obvious branch locus: the axis drawn minus its intersections with the solid tori. The involution of S^3 about this axis may be extended to a hyperelliptic involution of the surgery tori fixing longitudes and meridians set-wise (but reversing their orientation). It is easy to see that the quotient of the complement of the surgery solid tori under the involution is a ball minus a collection of disjoint sub-balls, one for each surgery torus (for instance, take as fundamental domain the "upper half space" cut out by half solid tori). This shows that the quotient orbifold is indeed topologically S^3. We now determine its branch locus. The branch locus in each of the solid tori is pair of arcs, each of which is isotopic (rel boundary) to a "half" of the corresponding framing curve in the surgery diagram. For instance, if we only had $n-1$ 0–framed curves (no ± 1-surgeries), then the downstairs branch locus coming from the "outside" (the complement of the solid tori) is a collection of n arcs in the three–sphere. Isotoping the downstairs branch loci (rel boundary) of the solid tori connects these arcs in such a way that we get a collection of n unknots. In a similar way, we can analyze what happens with the introduction of ± 1 surgeries, as above. By pushing the branch loci into the "outside" we see that $+1$ surgeries correspond to the introduction of right-handed "crossings" between the corresponding unknots from above and that -1 surgeries correspond to the introduction of left-handed crossings.

4.2.2 Quasi alternating knots.

In [OS05c] Ozsváth and Szabó defined the class of *quasi-alternating* links—it is the smallest collection $\mathcal{Q}$ of links such that

- The unknot is in $\mathcal{Q}$.
- If the link L has a diagram with a crossing c such that
  1. both resolutions of c, L_0 and L_∞ as in Figure 4.4, are in $\mathcal{Q}$,
  2. $\det(L) = \det(L_0) + \det(L_\infty)$,

  then L is in $\mathcal{Q}$.

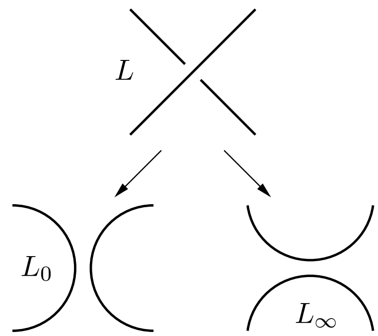

Figure 4.4: Resolving a crossing.

As in Champanerkar and Kofman [CK09], we shall call such a crossing c as above a *quasi-alternating crossing* of L and say that L is *quasi-alternating at c*.

The class of quasi-alternating links extends the class of alternating links in the sense that if a link admits a connected alternating diagram then it is quasi-alternating. In [OS05c], Ozsváth and Szabó show that if a link L is quasi-alternating, then its branched double cover is an L–space[3].

Quasi-alternating knots may be "generated" by the following construction of Kofman and Champanerkar [CK09]. Consider a crossing c as above as a two-tangle with marked endpoints. Let $\epsilon(c) = \pm 1$ according to whether the over strand has positive or negative slope. We say that a rational two-tangle $\tau = C(a_1, ..., a_m)$ *extends* c if τ contains c and $\epsilon(c) \cdot a_i \geq 1$ for $i = 1, ..., m$. They prove

Theorem 4.2.1 (Champanerkar–Kofman [CK09])**.** *Let L be a quasi-alternating link with quasi-alternating crossing c and L' be obtained by replacing c with an alternating rational tangle τ that extends c. Then L' is quasi-alternating at any crossing of τ.*

The reason that the branched double cover of a quasi-alternating link in S^3 is an L–spaces follows from the following construction of L–spaces due to Ozsváth and Szabó [OS05b]. Fix a closed, oriented three-manifold Y and let K be a framed knot in Y. Then we have manifolds Y_0 and Y_1, obtained by 0–surgery and +1–surgery on K, respectfully. We call the ordered triple (Y, Y_0, Y_1) a *triad* of three-manifolds. Suppose that Y, Y_0, Y_1 are

[3]Not every L–space arises in such a way—the Brieskorn sphere $\Sigma(2, 3, 5)$ is an example.

all rational homology three–spheres and $|H_1(Y;\mathbb{Z})| = |H_1(Y_0;\mathbb{Z})| + |H_1(Y_1;\mathbb{Z})|$. It follows from the surgery exact triangle in Heegaard Floer homology that if Y_0 and Y_1 are L–spaces, then so is Y. The discussion in Section 4.2.1 shows that the branched double cover of a link and the branched double covers of its two resolutions at a crossing fit into a triad. The previously mentioned theorem leads then to the recursive definition of quasi-alternating links. A further consequence of the exact triangle shows that if $K \subset S^3$ is a knot in the three–sphere such that $S^3_r(K)$ is an L–space for some rational number $r > 0$ (with respect to the Seifert framing) then $S^3_s(K)$ is an L–space for any rational $s > r$.

4.3 Proof of Theorem 4.1.4

4.3.1 The manifolds in Theorem 4.1.3, item (1)

Here we consider the manifolds $\Sigma_n(L)$ where $L = T_{(2',2k)}$ is the torus link of type $(2, 2k)$ with the anti-parallel orientation of strings, and n is arbitrary.

Consider the standard genus 0, one-boundary component open book decomposition of S^3. Now consider an unlink $\widetilde{L}$ of two components which meets each page of the open book in two points. After performing $-\frac{1}{k}$–surgery on the binding of this open book, the unlink $\widetilde{L}$ becomes the torus link $L = T_{(2,2k)}$. Now orient L so that $L = T_{(2',2k)}$. Each page of this open book meets L in exactly two points. With this orientation, the n–fold strongly cyclic branched cover of the disk branched along two points is the n–times punctured sphere S_n. The covering transformations consist of rotations through an axis which meets S_n in two points through angles which are multiples of $2\pi/n$ and cyclically permute the boundary components (see Figure 4.5).

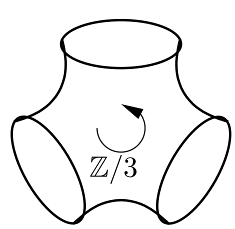

Figure 4.5: A three-fold branched covering of the disk branched over two points downstairs.

The open book of S^3 with disk pages lifts in the n–fold branched cover to an open book with page S_n and trivial monodromy—an open book decomposition of $\#^{n-1}S^2 \times S^1$. This open book decomposition is visualized in Figure 4.6. Performing $-\frac{1}{k}$–surgery on the binding downstairs lifts to $-\frac{1}{k}$–surgery on the binding upstairs (with respect to the page framings). This gives us the plumbing graph in the left hand side of Figure 4.7 (this is reached by beating one's head on Figure 4.6). After a sequence of blow ups and blow downs, we reach the final plumbing graph, the right hand side of Figure 4.7 which, by the algorithm described in the Section 4.2, may be realized as a branched double cover of an alternating, hence quasi-alternating knot (see Figure 4.8).

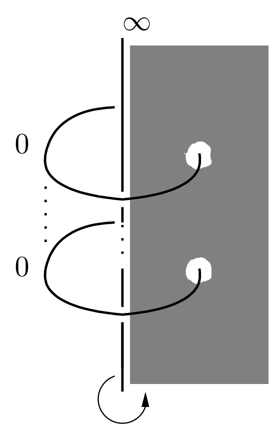

Figure 4.6: An open book decomposition of $\#^{n-1}S^2 \times S^1$ (there are $n-1$ 0–framed unknots).

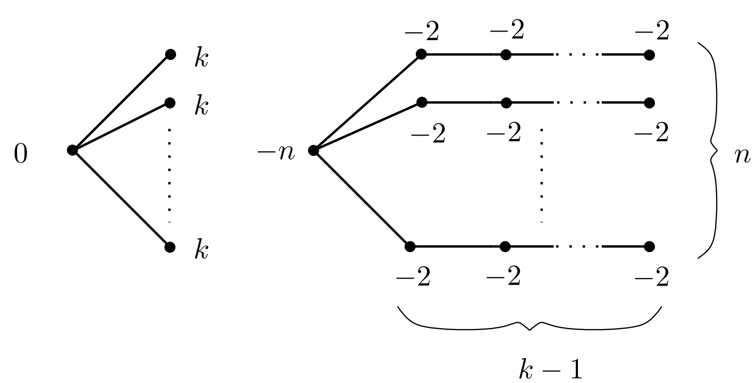

Figure 4.7: Two plumbing trees representing the manifolds from (1). Both have n branches.

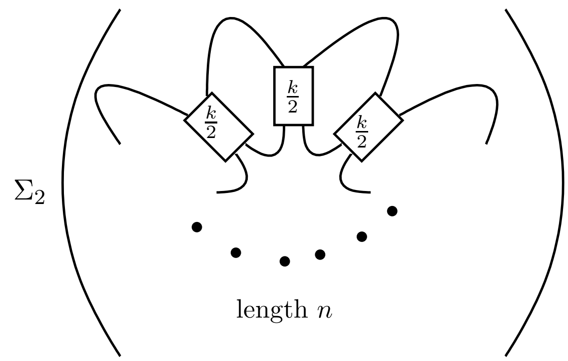

Figure 4.8: Realizing as a branched double cover. A box marked with a half-integer $p/2 \in \frac{1}{2}\mathbb{Z}$ means we do p half twists in this region (the direction in which we twist should be clear from context).

4.3.2 The manifolds in Theorem 4.1.3, item (3)

We consider $\Sigma_n(L)$ for $L = L_{[2k,2m]}$ the two-bridge knot of type $p/q = 2m + \frac{1}{2k} = [2k, 2m]$, where $k, m > 0$, and n is arbitrary. Surgery presentations of these manifolds may obtained by applying the "Montesinos trick" or by appealing to a construction of Mulazzani and Vesnin [MV02].

Consider the three-manifold $T_{n,m}(1/q_j; 1/s_j)$ defined by the surgery diagram in Figure 4.9, with $2mn$ components, joined up to form a necklace. Mulazzani and Vesnin prove that $T_{n,m}(1/q_j; 1/s_j)$ is the n–fold cyclic branched covering[4] of the two-bridge knot corresponding to the Conway parameters $[-2q_1, 2s_1, ..., -2q_m, 2s_m]$.

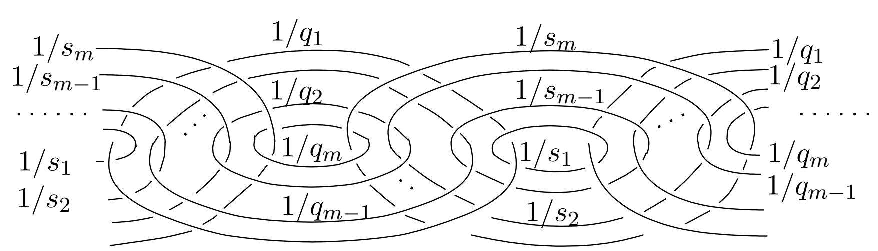

Figure 4.9: The three-manifold $T_{n,m}(1/q_j; 1/s_j)$. The link is n–periodic.

[4]The covering transformations are generated by the n–fold rotational symmetry of the diagram.

Thus our manifolds have surgery presentation in Figure 4.10 (a). Rolfsen twisting about each of the $\frac{1}{m}$–framed components give Figures 4.10 (b) and 4.10 (c). After some Kirby calculus we get Figure 4.11 which we may realize as the branched double cover of the alternating knot shown in Figure 4.12 (add a ghost vertex at the center of the necklace).

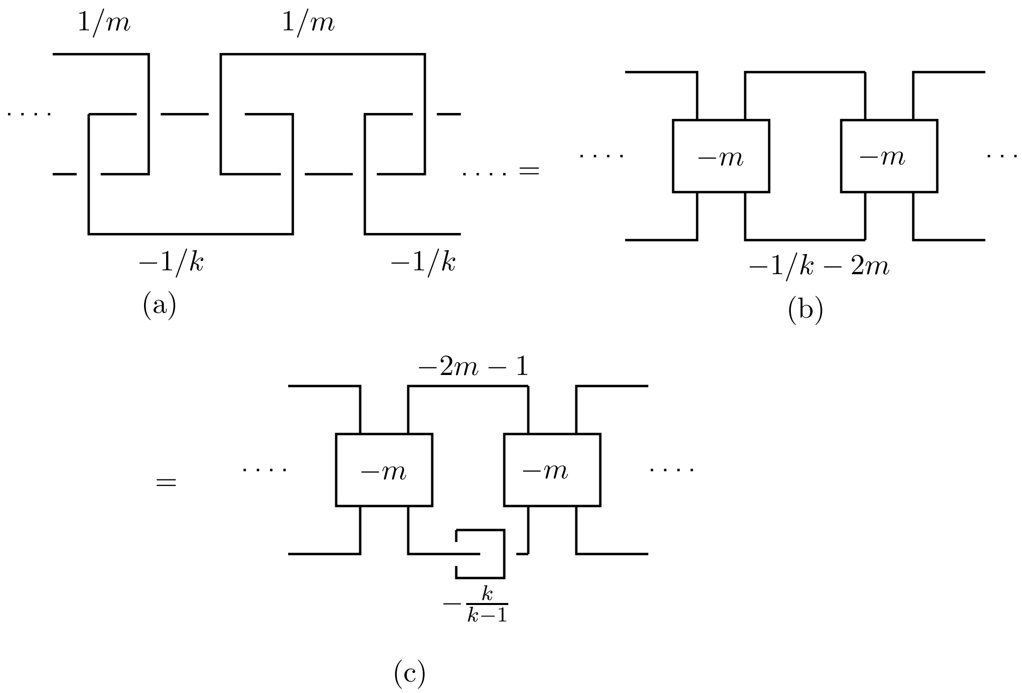

Figure 4.10: Three views of the three-manifold $\Sigma_n(L_{[2k,2m]})$. Each diagram is joined up to form a necklace of length $2n$.

4.3.3 The manifolds in Theorem 4.1.3, item (4)

Finally, we consider the manifolds $\Sigma_k(L)$ where $L = L_{[n,1,m]}$ is the two-bridge knot of type $p/q = m + \frac{1}{1+\frac{1}{n}}$, where n and m are odd, positive integers. The multiplicity of the covering is $n \leq 3$. For $k = 2$, these manifolds are lens spaces, and hence L–spaces, so we consider the case of $k = 3$. To construct surgery diagrams for these manifolds, we could again appeal to the work of Mulazzani and Vesnin. Instead, we use the "Montesinos trick" (see, for instance, Rolfsen [Rol03]), as in Figures 4.13, 4.14, and 4.15.

After some blow ups and blow downs, we see that our surgery diagram can be represented

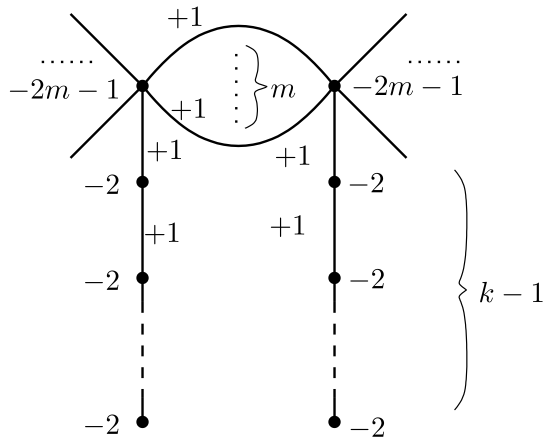

Figure 4.11: This is not a plumbing diagram! It describes a way of writing a Kirby diagram, as described in Section 4.2.1.

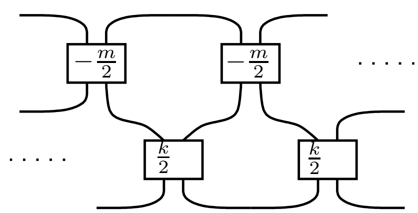

Figure 4.12: A link over which the manifold $\Sigma_n(L_{[2k,2m]})$ is a two-fold branched cover. This link is n–periodic, closed up to form a necklace.

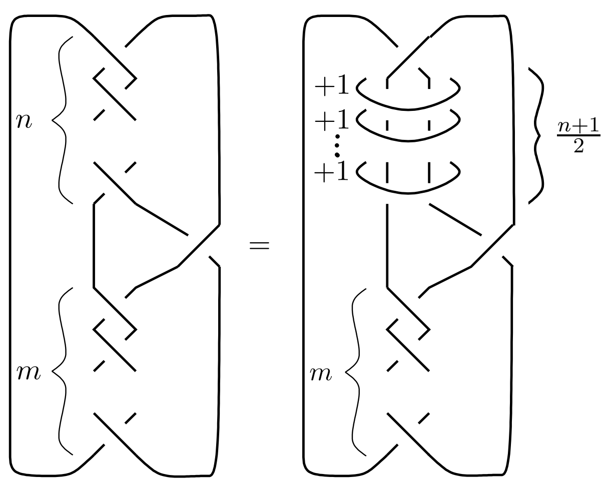

Figure 4.13: The knots $L_{[n,1,m]}$.

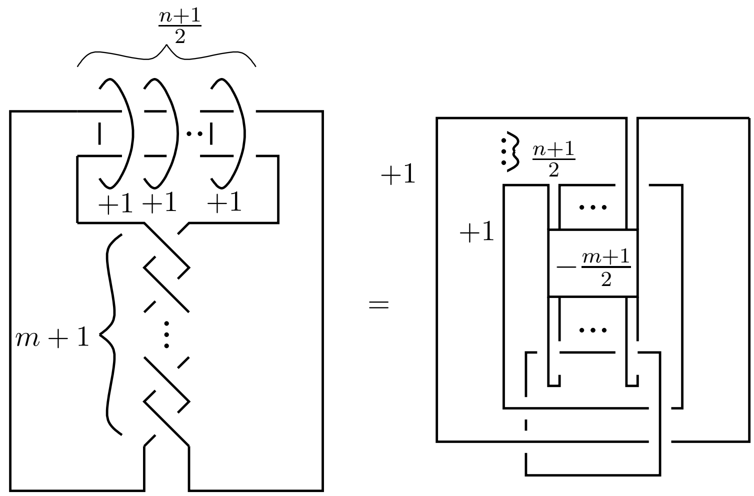

Figure 4.14: Another view of the knots $L_{[n,1,m]}$.

by the graph in Figure 4.16, which we claim is (usually) the reduced black graph of a quasi-alternating knot. Indeed, consider the family of links $K(p, q)$ $(p, q \in \mathbb{N}_{\geq 0})$ shown below. Then our most recent diagram is the branched double cover of the link $K(\frac{n+1}{2}, \frac{m+1}{2})$ (add a ghost vertex at the center, connected to the closest 2's by three -1–marked edges). $K(1, 1)$ is the three-braid $(\sigma_1\sigma_2)^3$ which, though not quasi-alternating, has an L–space as branched double cover (see [Bal08]). This manifold can be realized as the triple branched cover of the trefoil knot, which is the spherical space form $\mathbb{S}^3/Q_8$ (Q_8 denoting the quaternion group). We claim that $K(p, q)$ is quasi-alternating if $pq > 1$. By a symmetry of the diagram for $K(p, q)$, we may assume that $p > 1$. Using Champanerkar and Kofman's Theorem 4.2.1, it is enough to see that the link L (shown in Figure 4.18) is quasi-alternating at the circled crossing. Figures 4.19 and 4.20 show that the two resolutions L_0 and L_∞ admit connected alternating diagrams and hence are quasi-alternating.

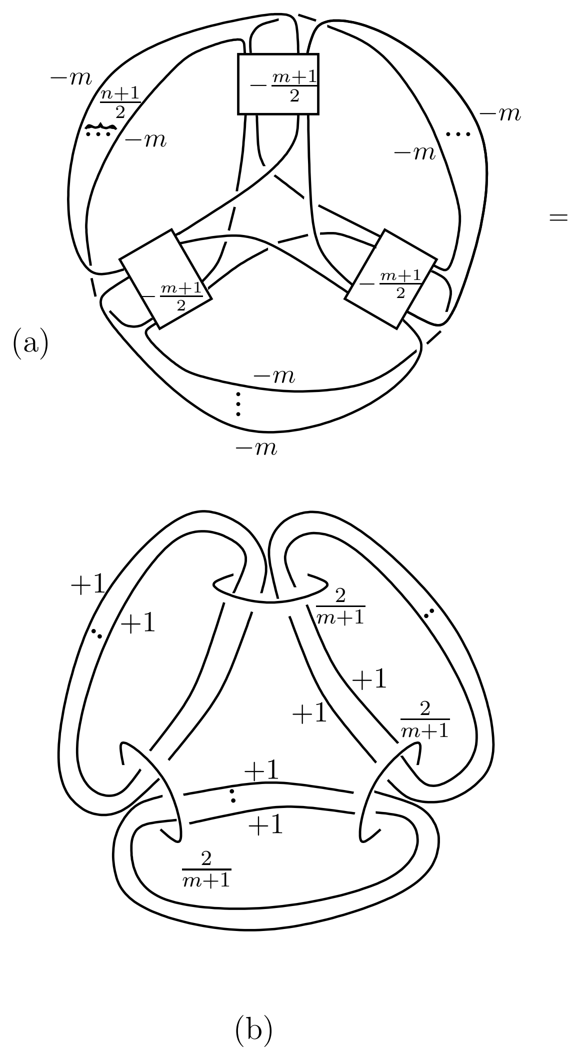

Figure 4.15: The manifold $\Sigma_3(L_{[n,1,m]})$. Notice in (a) how the $+1$–framed unknots became $-m$–framed unknots.

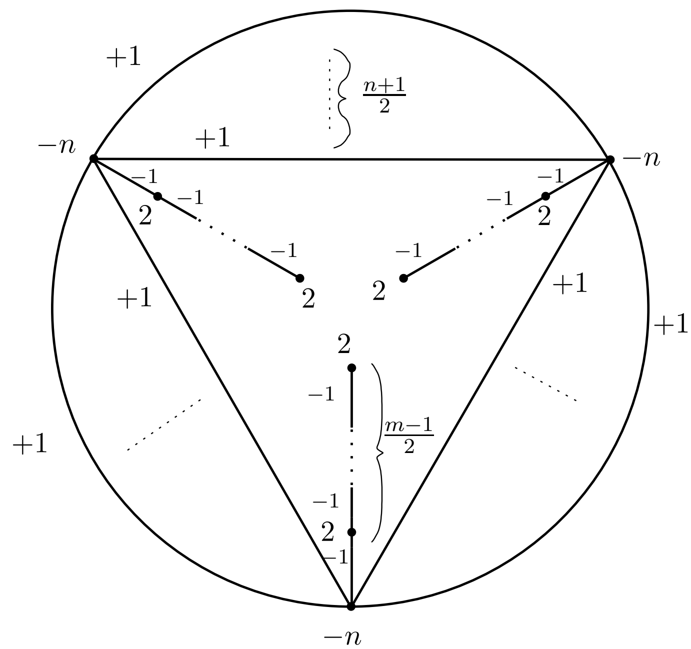

Figure 4.16: Another view of the manifold $\Sigma_3(L_{[n,1,m]})$.

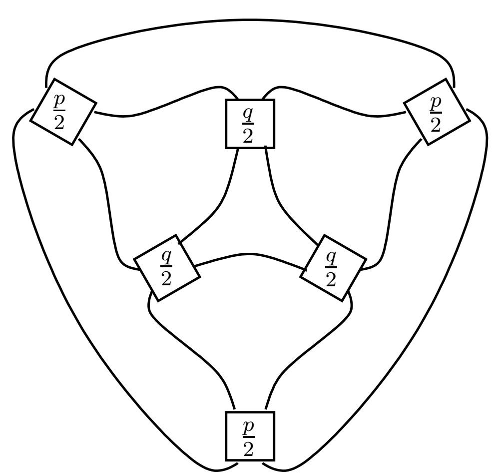

Figure 4.17: The family of links $K(p, q)$.

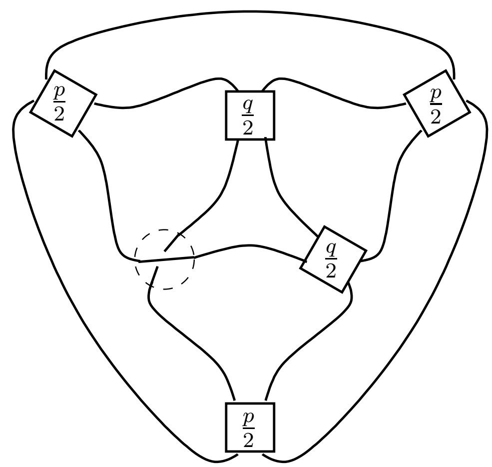

Figure 4.18: The link L.

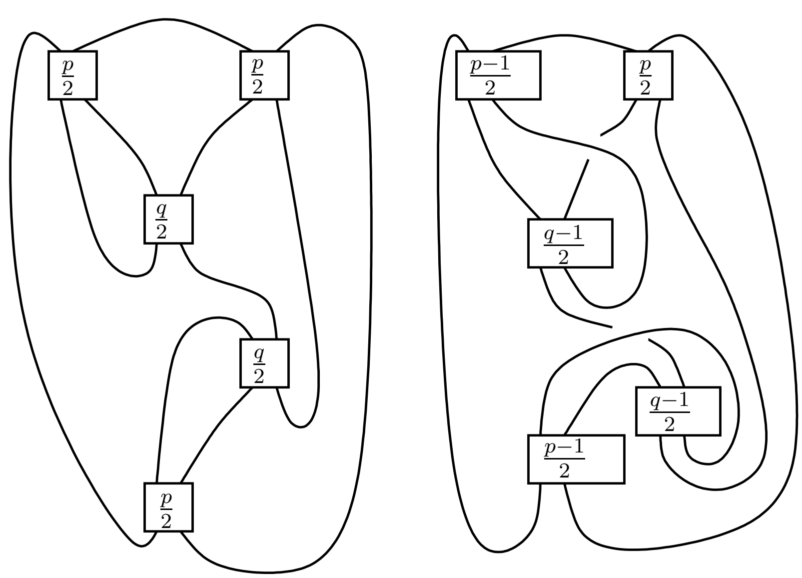

Figure 4.19: The 0–resolution, L_0.

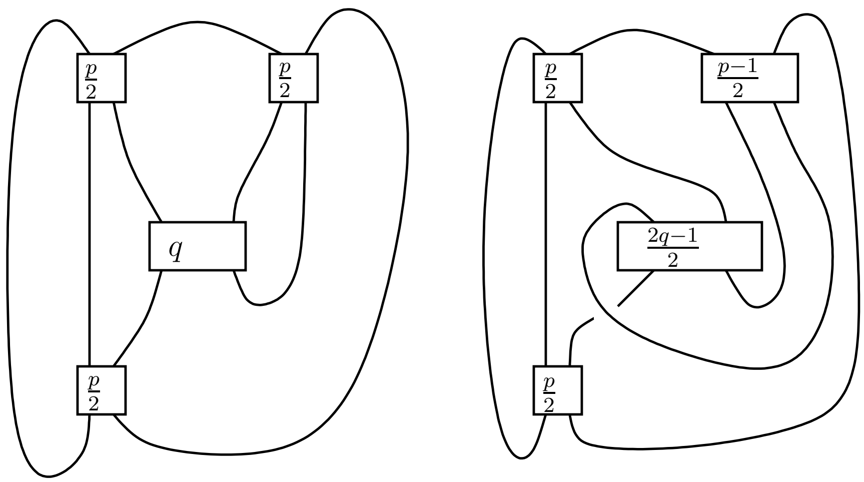

Figure 4.20: The ∞–resolution, L_∞.

Let

$$A = \begin{vmatrix} 1-2p & p & p & & & & & & & & \\ p & 1-2p & p & & & \cdots & & -1 & & & \\ p & p & 1-2p & -1 & & & & & & & \\ & & -1 & 2 & -1 & & & & & & \\ & & & -1 & 2 & & & & & & \\ & \vdots & & & & \ddots & -1 & & & & \\ & & & & & -1 & 2 & & & & \\ & -1 & & & & & & 2 & -1 & & \\ & & & & & & & -1 & 2 & & \\ & & & & & & & & & \ddots & -1 \\ & & & & & & & & & -1 & 2 \end{vmatrix}$$

Where both strings of 2's have length $q-1$. Then $|A| = \det L$.

Now let

$$B = \begin{vmatrix}
-2p & p & p & & & & & & & & \\
p & 1-2p & p & & & \cdots & & -1 & & & \\
p & p & 1-2p & -1 & & & & & & & \\
 & & -1 & 2 & -1 & & & & & & \\
 & & & -1 & 2 & & & & & & \\
 & \vdots & & & & \ddots & -1 & & & & \\
 & & & & & -1 & 2 & & & & \\
 & -1 & & & & & & 2 & -1 & & \\
 & & & & & & & -1 & 2 & & \\
 & & & & & & & & & \ddots & -1 \\
 & & & & & & & & & -1 & 2
\end{vmatrix}$$

Where again both strings of 2's have length $q-1$. Then $|B| = \det L_\infty$. Finally let

$$C = \begin{vmatrix}
1-2p & p & & & \cdots & & -1 & & & \\
p & 1-2p & -1 & & & & & & & \\
 & -1 & 2 & -1 & & & & & & \\
 & & -1 & 2 & & & & & & \\
 & \vdots & & & \ddots & -1 & & & & \\
 & & & & -1 & 2 & & & & \\
-1 & & & & & & 2 & -1 & & \\
 & & & & & & -1 & 2 & & \\
 & & & & & & & & \ddots & -1 \\
 & & & & & & & & -1 & 2
\end{vmatrix}$$

Where again the first string of 2's is length $s-1$ and the second string is of length $t-1$. Then $|C| = \det L_0$. Clearly we have $A = B + C$. We claim that $B, C > 0$ so that $|A| = |B| + |C|$

and L is quasi-alternating. We start with B. Consider the more general determinant

$$B(p,q,r) = \begin{vmatrix}
-2p & p & p & & & & & & & & \\
p & 1-2p & p & & & \cdots & & -1 & & & \\
p & p & 1-2p & -1 & & & & & & & \\
 & & -1 & 2 & -1 & & & & & & \\
 & & & -1 & 2 & & & & & & \\
 & \vdots & & & & \ddots & -1 & & & & \\
 & & & & & -1 & 2 & & & & \\
 & -1 & & & & & & 2 & -1 & & \\
 & & & & & & & -1 & 2 & & \\
 & & & & & & & & & \ddots & -1 \\
 & & & & & & & & & -1 & 2
\end{vmatrix}$$

Where the first string of 2's is of length q and the second of length r. Then $B = B(p, q-1, q-1)$. We claim that $B(p,q,r) > 0$ whenever $p > 1$. A simple calculation shows that $B(p,q,r)$ satisfies the recurrences:

$$B(p,q,r) = 2B(p,q-1,r) - B(p,q-2,r),\ q > 1$$

$$B(p,q,r) = 2B(p,q,r-1) - B(p,q,r-2),\ r > 1$$

These lead to the solution:

$$\begin{aligned}
B(p,q,r) = {} & rq(B(p,1,1) - B(p,0,1) - B(p,1,0) + B(p,0,0)) \\
& + r(B(p,0,1) - B(p,0,0)) + q(B(p,1,0) - B(p,0,0)) + B(p,0,0)
\end{aligned}$$

A little computer assistance then shows that:

$$B(p,q,r) = rq(0) + r(3p^2) + q(3p^2) + (-2p + 6p^2)$$

which is positive if $p > 1$. Similarly consider the determinants

$$C(p,q,r) = \begin{vmatrix} 1-2p & p & & & \cdots & & -1 & & & \\ p & 1-2p & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & 2 & & & & & & \\ \vdots & & & & \ddots & -1 & & & & \\ & & & & -1 & 2 & & & & \\ -1 & & & & & & 2 & -1 & & \\ & & & & & & -1 & 2 & & \\ & & & & & & & & \ddots & -1 \\ & & & & & & & & -1 & 2 \end{vmatrix}$$

Where the first string of 2's is length q and the second of length r. Then $C = C(p, q-1, q-1)$. We claim that $C(p, q, r) > 0$ for any $p > 1$. Similar to before, we see that $C(p, q, r)$ satisfies the recurrences:

$$C(p,q,r) = 2C(p,q-1,r) - Cc(p,q-2,r), \; q > 1$$

$$C(p,q,r) = 2C(p,q,r-1) - C(p,q,r-2), \; r > 1$$

Which lead to the solution:

$$\begin{aligned} C(p,q,r) = {} & rq(C(p,1,1) - C(p,0,1) - C(p,1,0) + C(p,0,0)) \\ & + r(C(p,0,1) - C(p,0,0)) + q(C(p,1,0) - C(p,0,0)) + C(p,0,0) \end{aligned}$$

Hence

$$C(p,q,r) = rq(3p^2) + r(-2p + 3p^2) + q(-2p + 3p^2) + (1 - 4p + 3p^2)$$

which is clearly positive if $p > 1$.

Chapter 5

Knot concordance and correction terms

5.1 Introduction

Given a closed oriented three-manifold with torsion Spin^c structure, the associated Heegaard Floer homology groups come with absolute $\mathbb{Q}$–gradings; see Ozsváth–Szabó [OS06]. This allows one to define numerical invariants of Spin^c three-manifolds, the so-called "correction terms" or "d–invariants". Specifically, suppose $(Y, \mathfrak{s})$ is a Spin^c rational homology three-sphere. Then Ozsváth and Szabó define $d(Y, \mathfrak{s})$ (the correction term) to be the minimal degree of any non-torsion class in $HF^+(Y, \mathfrak{s})$ coming from $HF^\infty(Y, \mathfrak{s})$[1]. This invariant is analogous to the monopole Floer homology h–invariant introduced by Frøyshov [Frø96]. If Y only has a single Spin^c structure $\mathfrak{s}_0$ (ie if Y is an integer homology sphere), then we denote $d(Y, \mathfrak{s}_0)$ by just $d(Y)$. The d–invariants satisfy some useful properties, according to the following theorem of Ozsváth and Szabó:

Theorem 5.1.1 (Ozsváth–Szabó [OS04a])**.** *Let Y be an oriented rational homology three-sphere. Its correction terms satisfy:*

[1] There are correction terms for three-manifolds with positive first Betti number, but we do not discuss them at the moment.

1. *Conjugation invariance*
$$d(Y,\mathfrak{s}) = d(Y,\bar{\mathfrak{s}}).$$

2. *If Y is an integral homology three-sphere and is the oriented boundary of a negative-definite four-manifold W then $d(Y) \geq 0$.*

In fact, item 2 follows from a more general statement, Proposition 5.3.2, and the following theorem of Elkies.

Theorem 5.1.2 (Elkies [Elk95])**.** *Let $Q\colon V \otimes V \to \mathbb{Z}$ be a negative-definite unimodular bilinear form over $\mathbb{Z}$. Denote by $\Xi(Q)$ the set of* characteristic vectors for Q*, ie the set of vectors $\xi \in V$ satisfying*
$$Q(\xi, v) \equiv Q(v,v) \mod 2$$
for all $v \in V$. Then,
$$0 \leq \max_{\xi\in\Xi(Q)} Q(\xi,\xi) + \dim(V),$$
with equality if and only if the bilinear form Q is diagonalizable over $\mathbb{Z}$.

Also, Y can be a disjoint union of rational homology three-spheres, in which case Theorem 5.1.1 (together with Theorem 5.1.2) implies:

Corollary 5.1.3 (Ozsváth–Szabó [OS04a])**.** *Let Y_1 and Y_2 be oriented rational homology three-spheres. Then*

1. *Let $-Y_1$ denote the manifold Y_1 with opposite orientation, then*
$$d(Y_1,\mathfrak{s}) = -d(-Y_1,\mathfrak{s})$$

2. *If $(Y_1,\mathfrak{s}_1)$ is* Spin^c *rational homology cobordant to $(Y_2,\mathfrak{s}_2)$, then*
$$d(Y_1,\mathfrak{s}_1) = d(Y_2,\mathfrak{s}_2).$$

3. *If Y_1 and Y_2 are integral homology three-spheres and W is a negative-definite* Spin^c *cobordism from $(Y_1,\mathfrak{s}_1)$ to $(Y_2,\mathfrak{s}_2)$, then*
$$d(Y_2,\mathfrak{s}_2) \geq d(Y_1,\mathfrak{s}_1).$$

4. If $(Y_1, \mathfrak{s}_1)$ bounds a rational homology four-ball, then $d(Y, \mathfrak{s}) = 0$.

Heegaard Floer homology d–invariants have been used to give restrictions on intersection forms of four-manifolds which can bound a given three-manifold (for instance, Ozsváth and Szabó reproved Donaldson's diagonalization theorem using correction terms). They have also been used to define concordance invariants of knots[2]. For instance, Manolescu and Owens [MO07] used the d–invariants of the branched double cover of a knot to produce concordance invariants (see also Grigsby, Ruberman, and Strle [GRS08], Jabuka [Jab], and Jabuka and Naik [JN04]). In this chapter, given a knot $K \subset S^3$ in the three-sphere, we show that $d(S^3_{+1}(K))$ is a concordance invariant of K and examine some of its properties. Occasionally we denote this invariant by dS^3_1. Note that one could also study $d(S^3_{-1}(K))$, but these invariants are determined by the dS^3_1 since $d(S^3_{-1}(K)) = d(-S^3_1(mK)) = -d(S^3_1(mK))$ where mK denotes the mirror of K. We also establish a "skein inequality" reminiscent of a property of the knot signature. Specifically,

Theorem 5.1.4. *Given a diagram for a knot with distinguished crossing c, let D_+ and D_- be the result of switching c to positive and negative crossings, respectively, as in Figure 5.2. Then*

$$d(S^3_1(D_-); \mathbb{F}) - 2 \leq d(S^3_1(D_+); \mathbb{F}) \leq d(S^3_1(D_-); \mathbb{F})$$

for any field $\mathbb{F}$. Here, $d(Y; \mathbb{F})$ denotes the correction term of Y computed from Floer homology with coefficients in $\mathbb{F}$.

The invariants $d(S^3_{\pm 1}(K))$ also give rise to four-ball genus bounds. Specifically, we have the following:

Theorem 5.1.5. *Let K be a knot in the three-sphere. Then*

$$0 \leq -d(S^3_1(K); \mathbb{Z}_2) \leq 2g_4(K)$$

where $g_4(K)$ denotes the smooth four-ball genus of K.

This should be compared to the following theorem of Frøyshov:

[2] Heegaard Floer theory has led to other concordance invariants, most notably the τ invariant (see Ozsváth–Szabó [OS03b] and Rasmussen [Ras03]).

Theorem 5.1.6 (Frøyshov [Frø04])**.** *Let Y be an oriented homology three-sphere and γ a knot in Y of "slice genus" $\widetilde{g}$. If $Y_{\gamma,-1}$ is the result of -1–surgery on γ then*

$$0 \leq h(Y_{\gamma,-1}) - h(Y) \leq \lceil \widetilde{g}/2 \rceil.$$

Here $h(Y)$ is Frøyshov's instanton Floer homology h–invariant and the "slice genus" is defined to be the smallest non-negative integer $\widetilde{g}$ for which there exists a smooth rational homology cobordism W from Y to some rational homology sphere Y' and a genus $\widetilde{g}$ surface $\Sigma \subset W$ such that $\partial W = \gamma$. It is not clear to the author whether this definition agrees with the usual one for $Y = S^3$. In light of the conjectural relationship $h(Y) = d(Y)/2$ and Theorem 5.1.6, we suspect that the inequality in Theorem 5.1.5 is in general weaker than the h–invariant inequality.

Finally, using the theory of Ozsváth–Szabó [OS04c] and Rasmussen [Ras03], we observe how one can algorithmically compute $d(S^3_{\pm 1}(K))$ if one knows the filtered chain homotopy type of the knot complex $CFK^\infty(K)$. A computer implementation of this algorithm is discussed.

5.1.1 Further questions

What is the relationship between the correction terms of ± 1–surgeries on a knot and the Ozsváth–Szabó, Rasmussen τ invariant? From the discussion in Section 5.5, it seems likely that $|d(S^3_{\pm 1}(K)| \leq 2|\tau(K)|$, but as of the time of this writing a proof remains elusive. Of course if this were the case, then the genus bound, Theorem 5.1.5, would follow immediately from the inequality $|\tau(K)| \leq g_4(K)$ (see Ozsváth and Szabó [OS03b] for a discussion).

5.1.2 Organization

This chapter is organized as follows. In Section 5.2, we discuss basic properties of $d(S^3_1(K))$, including its invariance under concordance. In Section 5.3 we give a proof of the skein inequality, Theorem 5.1.4. In Section 5.4 we prove Theorem 5.1.5. Finally in Section 5.5 we discuss an algorithm to compute $d(S^3_1(K))$ given the knot complex $CFK^\infty(K)$ as well as a computer implementation of this algorithm.

5.2 The invariant

Proposition 5.2.1. *$d(S^3_1(K))$ is a concordance invariant.*

Proof. It is simple to see that $d(S^3_1(K)) = 0$ if K is smoothly slice: $S^3_1(K)$ bounds the four-manifold obtained by attaching a $+1$–framed two-handle along K to the four-ball. This four-manifold has second homology generated by a sphere of square $+1$. By blowing this down, we see that $S^3_1(K)$ bounds a rational homology four-ball. By item 4 of Corollary 5.1.3, it follows that $d(S^3_1(K)) = 0$. It is just slightly more work to see that if K_1 and K_2 are smoothly concordant, then $d(S^3_1(K_1)) = d(S^3_1(K_2))$: the concordance gives us a smoothly embedded annulus $A \subset S^3 \times I$ (here $I = [0,1]$) with $\partial A = K_1 \cup K_2$ and $K_1 \subset S^3 \times \{0\}$, $K_2 \subset S^3 \times \{1\}$. Attach a two-handle to $S^3 \times \{1\}$ with framing $+1$ along K_2 to give a four-manifold W (see Figure 5.1). Consider a small regular neighborhood of the core disk of this two-handle union a regular neighborhood of the annulus A. This gives cobordisms $W_0 : S^3 \to S^3_1(K_1)$ and $W_1 : S^3_1(K_1) \to S^3_1(K_2)$ such that $W = W_0 \cup W_1$. Notice that W_0 is just $S^3 \times [0,\varepsilon] \cup h^2$, a $+1$–framed two-handle attached along K_1 to a thickened S^3. It follows that $b_2(W_0) = b_2(W) = 1$ and $b_2(W_1) = 0$ (this last fact can be seen from the Mayer–Vietoris sequence applied to the decomposition $W = W_0 \cup W_1$: $0 = H_2(W_0 \cap W_1;\mathbb{Z}) \to H_2(W_0;\mathbb{Z}) \oplus H_2(W_1;\mathbb{Z}) \cong \mathbb{Z} \oplus H_2(W_1;\mathbb{Z}) \to H_2(W;\mathbb{Z}) \cong \mathbb{Z} \to H_1(W_0 \cap W_1;\mathbb{Z}) = 0$). Applying item 2 of Corollary 5.1.3 to W_1 shows that $d(S^3_1(K_1)) = d(S^3_1(K_2))$. □

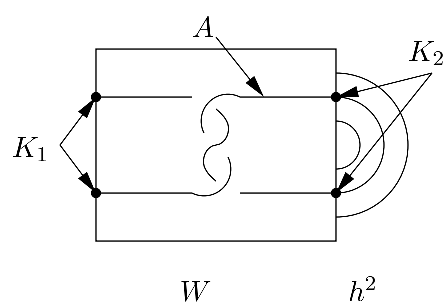

Figure 5.1: The cobordism W.

In general, calculating d–invariants is quite challenging. However, in certain cases explicit formulae exist. For instance, let K be an alternating knot. Then in [OS03a], Ozsváth and Szabó prove that

$$d(S^3_{+1}(K)) = 2\min\left(0, -\lceil\frac{-\sigma(K)}{4}\rceil\right) \tag{5.1}$$

where $\lceil x\rceil$ is the ceiling function and $\sigma(K)$ denotes the knot signature (see also Rasmussen [Ras02]). This formula shows that the concordance invariants $d(S^3_{\pm1}(K))$ do not give group homomorphisms from the smooth concordance group to $\mathbb{Z}$: take the knot $RHT \# LHT$ where RHT denotes the right-handed trefoil and LHT denotes the left-handed trefoil. This knot is slice and hence has vanishing dS^3_1 but $d(S^3_{+1}(RHT)) = -2$ and $d(S^3_{+1}(LHT)) = 0$. Explicit formulae for d–invariants also exist in the case of certain plumbed three-manifolds; see Ozsváth–Szabó [OS03c]. In another direction, since torus knots admit lens space surgeries, one may use Ozsváth–Szabó [OS05b, Theorem 1.2] to calculate dS^3_1 for torus knots.

It may be worth noting that Equation 5.1 does not hold for all knots. For instance, the $(3,4)$–torus knot has signature -6 and $dS^3_1 = -2$.

The non-additivity of dS^3_{+1} can be used to detect relations or establish linear independence in the smooth concordance group, $\mathcal{C}$. For example, recall that $\sigma(LHT) = 2$, $\tau(LHT) = -1$, and $s(LHT) = 2$ (here, $s(K)$ denotes the Rasmussen s concordance invariant of [Ras]). It is also the case that $\sigma(T_{3,4}) = -6$, $\tau(T_{3,4}) = 3$, $s(T_{3,4}) = -6$ where here $T_{3,4}$ denotes the $(3,4)$–torus knot. It follows that $f(LHT \# LHT \# LHT \# T_{3,4}) = 0$ for any f among s, τ, or σ. However, this knot is not slice, since $d(S^3_{-1}(LHT \# LHT \# LHT \# T_{3,4})) = 2$, a fact which can be verified with our program `dCalc`.

Finally, note that $d(S^3_1(K))$ is always even. This follows immediately from the long exact sequence

$$\cdots \to HF^+(S^3) \to HF^+(S^3_0(K)) \to HF^+(S^3_1(K)) \to \cdots \tag{5.2}$$

of Ozsváth and Szabó [OS04d], and the fact that $HF^\infty(S^3_0(K))$ is standard (see, for instance, Section 5.3 for a discussion).

5.3 Skein relations

Recall the axiomatic characterization of the knot signature σ found by Giller (see also Murasugi [Mur96]).

Theorem 5.3.1 (Giller [Gil89]). *Suppose that K is a knot (but not a link) and D is a diagram for K. Then $\sigma(K)$ can be determined from the following three axioms:*

1. *If K is the unknot then $\sigma(K) = 0$.*

2. *If D_+ and D_- are as in Figure 5.2, then*

$$\sigma(D_-) - 2 \leq \sigma(D_+) \leq \sigma(D_-)$$

   (recall that σ is always even).

3. *If $\Delta_K(t)$ is the Conway-normalized Alexander polynomial of K, then*

$$\mathrm{sign}(\Delta_K(-1)) = (-1)^{\sigma(K)/2}.$$

Figure 5.2: Positive and negative crossings, respectfully.

These axioms of course cannot hold for the invariant dS^3_1, but Theorem 5.1.4 does give us an analogue of Theorem 5.3.1, item (2).

In light of the the axiomatic description of σ, it is an interesting question to calculate $dS^3_1/2$ modulo 2. If one could achieve this, it might then be possible to give a completely algorithmic description of dS^3_1.

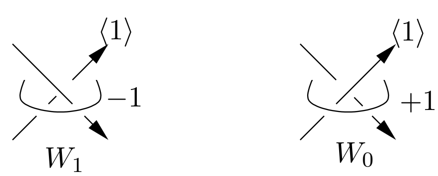

Figure 5.3: A pair of relative handlebodies, representing the cobordisms W_0 and W_1.

We now return to the proof of Theorem 5.1.4.

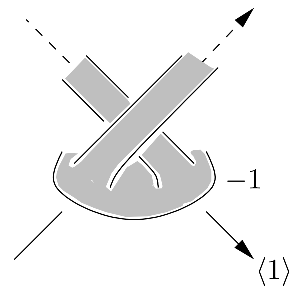

Figure 5.4: The torus T is represented by the shaded region, which is then capped off by the core of the -1–framed two-handle.

Proof of Theorem 5.1.4. **Step 1:** $d(S^3_1(D_-)) \geq d(S^3_1(D_+))$**:**

Given a knot K with diagram $D(K)$ and a distinguished crossing, we have cobordisms $W_0\colon S^3_1(D_-) \to S^3_1(D_+)$ and $W_1\colon S^3_1(D_+) \to S^3_1(D_-)$ given by the Kirby diagrams in Figure 5.3. We claim that $b_2(W_i) = 1$ for $i = 0, 1$. We argue this for W_1, the argument for W_0 being analogous. W_1 fits into a four-manifold $W = X \cup W_1$ where X is obtained by attaching a $+1$–framed two-handle along $K \subset S^3 = \partial B^4$ to the four-ball. Clearly $b_2(W) = 2$ and $b_2(X) = 1$. Consider the Mayer–Vietoris sequence applied to the decomposition $W = X \cup W_1$. In this case, $X \cap W_1 \cong S^3_1(D_+)$, an integral homology three-sphere. So we have $0 = H_2(X \cap W_1; \mathbb{Z}) \to H_2(X; \mathbb{Z}) \oplus H_2(W_1; \mathbb{Z}) \cong \mathbb{Z} \oplus H_2(W_1; \mathbb{Z}) \to H_2(W) \cong \mathbb{Z}^2 \to H_1(X \cap W_1; \mathbb{Z}) = 0$, showing that $H_2(W_1; \mathbb{Z}) \cong \mathbb{Z}$. We may even find a torus T in W_1 which generates $H_2(W_1; \mathbb{Z}) \cong \mathbb{Z}$ as in Figure 5.4. We claim that $[T]^2 = -1$ (likewise, for W_0 we have can find a torus of square $+1$ generating $H_2(W_1; \mathbb{Z})$). T of course sits inside the larger cobordism W. Let $\{\alpha, \beta\}$ be an ordered basis of $H_2(W; \mathbb{Z})$ coming from the two two-handles (more specifically, α is the homology class of the core disk of the two-handle attached to D_+ capped off by a Seifert surface, and β is the homology class of the core disk of the two-handle attached to the -1–framed knot in Figure 5.4 capped off with a Seifert surface pushed slightly into the four-ball). With respect to this basis, we see that the intersection form of W is given by the matrix

$$Q_W = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By Figure 5.4, it is clear that $[T] \cdot \alpha = 0$ and $[T] \cdot \beta = -1$. Therefore $[T] = \beta$ and $[T]^2 = -1$.

By item 3 of Corollary 5.1.3 it follows that

$$d(S^3_1(D_-)) \geq d(S^3_1(D_+)).$$

Step 2: $d(S^3_1(D_+)) \geq d(S^3_1(D_-)) - 2$**:**
By a similar argument to the previous, we see that the cobordism $W_0\colon S^3_1(D_-) \to S^3_1(D_+)$ has second homology generated by a torus T of square $+1$. Taking an internal connected sum of $S^3_1(D_-)$ with a regular neighborhood of T, $\nu(T)$, we get cobordisms $V_0\colon S^3_1(D_-) \to S^3_1(D_-) \# \partial\nu(T)$ and $V_1\colon S^3_1(D_-) \# \partial\nu(T) \to S^3_1(D_+)$ such that $W_0 = V_0 \cup V_1$. Here $\partial\nu(T)$ denotes the boundary of a regular neighborhood of the surface T in W_0. This is of course a circle bundle over the two-torus with Euler number $+1$ (it is also a torus bundle over the circle with reducible monodromy). It may be realized as $(0,0,1)$–surgery on the Borromean rings, which we denote by $M\{0,0,1\}$. Clearly $b_2^+(V_1) = 0$ for otherwise we would have a surface F with positive square in W_0 which does not intersect the generating torus $[T] \in H_2(W_0;\mathbb{Z})$. Similarly, we have that $b_2^-(V_1) = 0$. Notice that V_0 deformation retracts onto the wedge $S^3_1(D_-) \vee T^2$ and hence has euler characteristic $\chi(V_0) = -1$. Since $\chi(W_1) = \chi(V_0) + \chi(V_1) - \chi(V_0 \cap V_1)$ and $\chi(V_0 \cap V_1) = 0$ ($V_0 \cap V_1$ is a three-manifold) we see that $\chi(V_1) = 2$. Therefore the cobordism $V_1\colon S^3_1(D_-) \# \partial\nu(T) \to S^3_1(D_+)$ has $\chi = 2$, $\sigma = 0$ (here σ denotes the signature of the intersection form of V_1), and $c_1(\mathfrak{s})^2 = 0$ for all Spin^c structures $\mathfrak{s} \in \mathrm{Spin}^c\, V_1$. By the formula for grading shifts in Heegaard Floer homology (see Ozsváth and Szabó [OS06]), it follows that the maps on Floer homology associated with this cobordism have grading shift

$$\frac{c_1(\mathfrak{s})^2 - 2\chi(V_1) - 3\sigma(V_1)}{4} = -1.$$

Before continuing with d–invariant calculations, we pause to recall some constructions in Heegaard Floer theory for manifolds with $b_1 > 0$. In this case, there is a natural action of the exterior algebra $\Lambda^* H_1(Y;\mathbb{Z})/\mathrm{Tors}$ on all versions of Floer homology HF°. Under this action, elements of $H_1(Y;\mathbb{Z})$ drop relative gradings by one. As an example, let $\mathbb{Z}_{(k)}$ denote the graded abelian group $\mathbb{Z}$ supported in grading k. Under the graded isomorphism $\widehat{HF}(S^2 \times S^1) \cong \mathbb{Z}_{(1/2)}\langle a\rangle \oplus \mathbb{Z}_{(-1/2)}\langle b\rangle$, the action of the circle factor $\gamma := [* \times S^1] \in H_1(S^2 \times S^1;\mathbb{Z})$ is given by $\gamma \cdot a = b$ and $\gamma \cdot b = 0$.

A Spin^c three-manifold $(Y, \mathfrak{s})$ with torsion Spin^c structure $\mathfrak{s}$ is said to have *standard* HF^∞ if there is a graded isomorphism of $\Lambda^* H_1(Y;\mathbb{Z})/\mathrm{Tors} \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$–modules

$$HF^\infty(Y, \mathfrak{s}) \cong \Lambda^* H^1(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}] \tag{5.3}$$

where the action of $H_1(Y;\mathbb{Z})/\mathrm{Tors}$ on the right hand side is given by contraction on $\Lambda^* H^1(Y;\mathbb{Z})$. Here $\Lambda^* H^1(Y;\mathbb{Z})$ is graded by the requirement that $\mathrm{gr}\left(\Lambda^{b_1 Y} H^1(Y;\mathbb{Z})\right) = b_1(Y)/2$ and the fact that $H_1(Y;\mathbb{Z})$ drops gradings by 1. For example, $\#^k S^2 \times S^1$ has standard HF^∞ for any k as does any three-manifold with $b_1 < 3$ by a theorem of Ozsváth and Szabó [OS04d, Theorem 10.1]. For Spin^c three-manifolds $(Y, \mathfrak{s})$ with standard HF^∞ there is a "bottom-most" correction term, denoted $d_b(Y, \mathfrak{s})$, which is defined to be the smallest grading of any non-torsion element $x \in HF^+(Y, \mathfrak{s})$ coming from an element $x' \in HF^\infty(Y, \mathfrak{s})$ which lies in the kernel of the action by $H_1(Y;\mathbb{Z})/\mathrm{Tors}$. Notice that, in contrast to "ordinary" correction terms, it is not true in general that $d_b(Y, \mathfrak{s}) = -d_b(-Y, \mathfrak{s})$ (for instance, take $Y = S^2 \times S^1$). The correction terms d_b give restrictions on intersection forms of negative semi-definite four-manifolds bounding a given three-manifold according to:

Proposition 5.3.2 (Ozsváth–Szabó [OS04a])**.** *Let Y be a closed oriented three-manifold (not necessarily connected) with torsion* Spin^c *structure $\mathfrak{t}$ and standard HF^∞. Then for each negative semi-definite four-manifold W which bounds Y so that the restriction map $H^1(W;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, we have the inequality*

$$c_1(\mathfrak{s})^2 + b_2^-(W) \leq 4 d_b(Y, \mathfrak{t}) + 2 b_1(Y)$$

for all Spin^c *structures $\mathfrak{s}$ over W whose restriction to Y is $\mathfrak{t}$.*

Returning to the proof of Theorem 5.1.4, recall that we have a cobordism

$$V_1 \colon\ S^3_1(D_-) \# M\{0,0,1\} \to S^3_1(D_+)$$

with $b_2^\pm(V_1) = 0$. Therefore $c_1(\mathfrak{s})^2 = 0$ for all Spin^c structures $\mathfrak{s}$ on V_1. Note also that $H^1(V_1;\mathbb{Z}) = 0$: consider the Mayer–Vietoris sequence applied to $W_0 = V_0 \cup V_1$:

$$H_1(V_0 \cap V_1;\mathbb{Z}) \cong \mathbb{Z}^2 \to H_1(V_0;\mathbb{Z}) \oplus H_1(V_1;\mathbb{Z}) \cong \mathbb{Z}^2 \oplus H_1(V_1;\mathbb{Z}) \to H_1(W_0;\mathbb{Z}) = 0$$

(recall that $V_0 \cap V_1 = S^3_1(D_-) \# M\{0,0,1\}$ and that $V_0 \simeq S^3_1(D_-) \vee T^2$; $\simeq$ denoting homotopy equivalence). Applying Proposition 5.3.2 to V_1 we see that:

$$0 \leq 4d(S^3_1(D_+)) + 4d_b(-\left(S^3_1(D_-) \# M\{0,0,1\}\right)) + 2 \cdot 2 \tag{5.4}$$

Claim 5.3.3. $d_b(-\left(S^3_1(D_-) \# M\{0,0,1\}\right)) = -d(S^3_1(D_-)) + 1$.

Notice that this would imply Theorem 5.1.4. To prove the claim, recall that in [OS04a], Ozsváth and Szabó calculated that $\widehat{HF}(M\{0,0,1\}) = \mathbb{Z}^2_{(0)} \oplus \mathbb{Z}^2_{(-1)}$, supported completely in the unique torsion Spin^c structure. This implies that $\widehat{HF}(-M\{0,0,1\}) = \mathbb{Z}^2_{(0)} \oplus \mathbb{Z}^2_{(1)}$, and by the long exact sequence

$$\cdots \longrightarrow \widehat{HF}_i(Y,\mathfrak{s}) \longrightarrow HF^+_i(Y,\mathfrak{s}) \xrightarrow{U} HF^+_{i-2}(Y,\mathfrak{s}) \longrightarrow \cdots$$

it follows that $HF^+(-M\{0,0,1\}) = (\mathcal{T}^+_{(0)})^2 \oplus (\mathcal{T}^+_{(1)})^2$ where $\mathcal{T}_{(k)}$ denotes the graded $\mathbb{Z}[U]$-module $\mathbb{Z}[U,U^{-1}]/U \cdot \mathbb{Z}[U]$ graded so that multiplication by U is degree -2 and $1 \in \mathcal{T}_{(k)}$ lies in grading k. Writing $HF^+(S^3_+(D_-)) = \mathcal{T}^+_{\left(d(S^3_1(D_-))\right)} \oplus Q$ for some torsion $\mathbb{Z}[U]$ module Q (Q is called the *reduced* Floer homology of Y, and is also written $HF^+_{\mathrm{red}}(Y)$), we get that $HF^+(-S^3_1(D_-)) = \mathcal{T}^+_{\left(-d(S^3_1(D_-))\right)} \oplus Q'$, by item 1 of Corollary 5.1.3. By the long exact sequence

$$\cdots \longrightarrow HF^-_i(Y,\mathfrak{s}) \longrightarrow HF^\infty_i(Y,\mathfrak{s}) \longrightarrow HF^+_i(Y,\mathfrak{s}) \longrightarrow HF^-_{i-1}(Y,\mathfrak{s}) \longrightarrow \cdots$$

we get that $HF^-(-S^3_1(D_-)) = \mathcal{T}^-_{\left(-d(S^3_1(D_-))-2\right)} \oplus Q''$ for some torsion $\mathbb{Z}[U]$-module Q'' where here $\mathcal{T}^-_{(k)}$ denotes the graded $\mathbb{Z}[U]$–module $U \cdot \mathbb{Z}[U]$ graded so that multiplication by U is degree -2 and U lies in grading k. Using the formula

$$HF^-(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong H_*\left(CF^-(Y_1,\mathfrak{s}_1) \otimes_{\mathbb{Z}[U]} CF^-(Y_2,\mathfrak{s}_2)\right)$$

from Ozsváth–Szabó [OS04d], if we use Floer homology with field coefficients $\mathbb{F}$, we have:

$$\begin{aligned} HF^-(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong{}& HF^-(Y_1,\mathfrak{s}_1) \otimes_{\mathbb{F}[U]} HF^-(Y_2,\mathfrak{s}_2) \\ & \bigoplus \mathrm{Tor}^{\mathbb{F}[U]}\left(HF^-(Y_1,\mathfrak{s}_1), HF^-(Y_2,\mathfrak{s}_2)\right) \end{aligned}$$

(since $\mathbb{F}[U]$ is a principal ideal domain). It follows that

$$HF^-\left((-S^3_1(D_-)) \# (-M\{0,0,1\})\right) \cong \left(\mathcal{T}^-_{\left(-d(S^3_1(D_-))-2\right)}\right)^2 \oplus \left(\mathcal{T}^-_{\left(-d(S^3_1(D_-))-1\right)}\right)^2 \oplus Q'''$$

for some torsion $\mathbb{F}[U]$ module Q'''. Therefore

$$HF^+\left((-S^3_1(D_-))\,\#\,(-M\{0,0,1\})\right) \cong \left(\mathcal{T}^+_{\left(-d(S^3_1(D_-))\right)}\right)^2 \oplus \left(\mathcal{T}^+_{\left(-d(S^3_1(D_-))+1\right)}\right)^2 \oplus Q''''$$

and we have shown that

$$d_b\left(-(S^3_1(D_-)\,\#\,M\{0,0,1\})\right) = d_b\left((-S^3_1(D_-))\,\#\,(-M\{0,0,1\})\right) = -d(S^3_1(D_-)) + 1$$

proving Claim 5.3.3 and hence Theorem 5.1.4 □

5.4 Genus bounds

Proof of Theorem 5.1.5. **Step 1:** $d(S^3_1(K)) \leq 0$:

Let $g = g_4(K)$, the smooth four-ball genus of K, ie the minimum genus of any smooth surface smoothly embedded in the four-ball with boundary K. Now attach a -1–framed two-handle to the four-ball along the mirror of K, denoted mK. Now delete a small ball from the four-ball. This gives a negative definite cobordism $S^3 \to S^3_{-1}(mK)$ whose second homology is generated by a surface of genus g and square -1. By item 3 of Corollary 5.1.3 and the fact that $d(S^3) = 0$ we get that $d(S^3_{-1}(mK)) \geq 0$. Since $d(S^3_1(K)) = -d(-S^3_{-1}(mK))$, we are done.

Step 2: $-d(S^3_1(K)) \leq 2g$:

Similar to the previous paragraph, by removing a small ball from the four-ball and then attaching a $+1$–framed two-handle to the boundary three-sphere along K, we obtain a cobordism

$$W\colon\ S^3 \to S^3_1(K)$$

which contains a genus g surface of square $+1$, Σ_g. Let $Y_g(\pm 1)$ denote an euler number ± 1 circle bundle over a surface of genus g. In the notation of the previous section, we have $Y_1(\pm 1) = M\{0,0,\pm 1\}$. $Y_g(+1)$ is of course homeomorphic to the boundary of a regular neighborhood of $\Sigma_g \subset W$. Similar to previous discussions, by taking an internal connected sum we get a pair of cobordisms

$$W_1\colon\ S^3 \to Y_g(+1)$$

and

$$W_2\colon\ Y_g(+1) \to S^3_1(K)$$

with $b_2^{\pm}(W_2) = b_1(W_2) = 0$. Notice that the oriented manifold $-Y_g(+1) = Y_g(-1)$ has standard HF^∞ by Ozsváth–Szabó [OS04a, Propositions 9.3 and 9.4] since we may connect it via a negative-definite cobordism to $\#^{2g}S^2 \times S^1$, which has standard HF^∞. Applying Proposition 5.3.2, we see that

$$0 \leq 4d\left(S^3_1(K)\right) + 4d_b\left(-Y_g(+1)\right) + 2 \cdot 2g. \tag{5.5}$$

Theorem 5.1.5 would follow if we could show that $d_b\left(-Y_g(+1)\right) = g$. Indeed, we show this in Lemma 5.4.2. The calculation of $d_b\left(-Y_g(+1)\right)$ follows quickly from the machinery of [OS08], which we recall in Section 5.4.1. □

5.4.1 Review of the integer surgery formula

In this section we review the essential details needed to state Ozsváth and Szabó's "integer surgery formula," referring the reader to [OS08] for more details. Suppose $(Y, \mathfrak{t})$ is a Spin^c three-manifold with $\mathfrak{t}$ torsion and suppose that $K \subset Y$ is a null-homologous knot. Fixing a Seifert surface $F \subset Y$ for K, we can assign to K its knot Floer homology $C := CFK^\infty(Y, K, F, \mathfrak{t})$, a $\mathbb{Z} \oplus \mathbb{Z}$–bifiltered chain complex well-defined up to filtered chain homotopy type as described in Section 2.1.2 and Ozsváth–Szabó [OS04c]. This is an abelian group generated by tuples $[\mathbf{x}, i, j]$ for integers i, j and intersection points $\mathbf{x}$ coming from a particular Heegaard diagram for K (see Ozsváth–Szabó [OS04c] for a proper discussion). This group comes with an absolute $\mathbb{Q}$–grading as well as an action by $\Lambda^* H_1(Y;\mathbb{Z})/\mathrm{Tors} \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$. There is an identification of Spin^c structures over $Y_n(K)$ which are Spin^c–cobordant to $\mathfrak{t}$ over a certain cobordism $W_n(K)$ with $\mathbb{Z}/n\mathbb{Z}$. For $i \in \mathbb{Z}/n\mathbb{Z}$, let $CF^+(Y_n(K), i, \mathfrak{t})$ denote the corresponding summand of $CF^+(Y_n(K))$.

Let $A^+_{s,\mathfrak{t}} = C\{i \geq 0 \text{ or } j \geq s\}$ and $B^+_{s,\mathfrak{t}} = C\{i \geq 0\}$, the latter being identified with $CF^+(Y, \mathfrak{t})$. There are maps

$$v^+_{s,\mathfrak{t}}\colon\ A^+_{s,\mathfrak{t}} \to B^+_{s,\mathfrak{t}}$$

and

$$h^+_{n,s,\mathfrak{t}}\colon\ A^+_{s,\mathfrak{t}} \to B^+_{s+n,\mathfrak{t}}$$

defined as follows: $v_{s,\mathfrak{t}}$ is just the projection $C\{i \geq 0 \text{ or } j \geq s\} \to C\{i \geq 0\}$ while $h^+_{s,\mathfrak{t}}$ is the projection $C\{i \geq 0 \text{ or } j \geq s\} \to C\{j \geq s\}$ followed by an identification $C\{j \geq s\} \cong C\{j \geq 0\}$ (induced by multiplication by U^s) followed by a "natural" homotopy equivalence $h\colon C\{j \geq 0\} \to C\{i \geq 0\}$. The map h is obtained by the handleslide invariance of Heegaard Floer homology and is natural in the sense that the induced map on homology is independent (up to a sign) of a chosen sequence of handleslides. Set

$$\mathbb{A}^+_{i,\mathfrak{t}} = \bigoplus_{\{s \in \mathbb{Z} \,|\, s \equiv i \mod n\}} A^+_{s,\mathfrak{t}}$$

and:

$$\mathbb{B}^+_{i,\mathfrak{t}} = \bigoplus_{\{s \in \mathbb{Z} \,|\, s \equiv i \mod n\}} B^+_{s,\mathfrak{t}}$$

Define

$$\mathcal{D}^+_{i,\mathfrak{t},n}\colon \mathbb{A}^+_{i,\mathfrak{t}} \to \mathbb{B}^+_{i,\mathfrak{t}}$$

by

$$\mathcal{D}^+_{i,\mathfrak{t},n}\{a_s\}_{s\in\mathbb{Z}} = \{v^+_{s,\mathfrak{t}}(a_s) + h^+_{s-n,\mathfrak{t}}(a_{s-n})\}_{s\in\mathbb{Z}}.$$

Assign gradings to $\mathbb{A}^+_{i,\mathfrak{t}}$, $\mathbb{B}^+_{i,\mathfrak{t}}$ as follows. Under the identification $B^+_{s,\mathfrak{t}} \cong CF^+(Y,\mathfrak{t})$, we map homogeneous elements of degree d in $CF^+(Y,\mathfrak{t})$ to homogeneous elements of $B^+_{s,\mathfrak{t}}$ of degree

$$d + 2\sigma l + n\ell(\ell-1) - 1, \text{ where } 0 \leq \sigma < n \text{ and } s = \sigma + n\ell \text{ if } n > 0, \tag{5.6}$$

or

$$d - 2\sigma\ell + n\ell(\ell-1) \text{ where } 0 \leq \sigma < -n \text{ and } s = -(\sigma - n\ell) \text{ if } n < 0. \tag{5.7}$$

It is then possible to assign gradings to the $A^+_{s,\mathfrak{t}}$ which are consistent with their natural relative $\mathbb{Z}$–gradings in such a way that the maps $v^+_{s,\mathfrak{t}}$ and $h^+_{s,\mathfrak{t}}$ are homogeneous of degree -1. With this all in place, we may now state the "integer surgery formula" of [OS04a]:

Theorem 5.4.1 (Ozsváth–Szabó [OS04a])**.** *Fix a* Spin^c *structure* $\mathfrak{t}$ *over* Y *whose first Chern class is torsion,* $K \subset Y$ *a null-homologous knot, and* n *a non-zero integer. For each* $i \in \mathbb{Z}/n\mathbb{Z}$*, the mapping cone* $\mathbb{X}^+_{i,\mathfrak{t}}(n)$ *of*

$$\mathcal{D}^+_{n,i,\mathfrak{t}}\colon \mathbb{A}^+_{i,\mathfrak{t}} \to \mathbb{B}^+_{i,\mathfrak{t}}$$

is isomorphic, as a relatively graded $\Lambda^* H_1(Y;\mathbb{Z})/\mathrm{Tors}\otimes_{\mathbb{Z}}\mathbb{Z}[U]$*–module, to* $CF^+(Y_n(K), i, \mathfrak{t})$*. In fact, this isomorphism* $\mathbb{X}^+_{i,\mathfrak{t}(n)} \to CF^+(Y_n(K), i, \mathfrak{t})$ *is homogeneous of degree* $d(n,i)$ *where*

$$d(n,i) = -\max_{\{s\in\mathbb{Z}\,|\,s\equiv i \mod n\}} \frac{1}{4}\left(1-\left(\frac{n+2s}{n}\right)^2\right)$$

for $n>0$ *and* $d(n,i) = -d(-n,i)$ *for* $n<0$*.*

Recall that the mapping cone of the map $\mathcal{D}^+_{n,i,\mathfrak{t}}\colon \mathbb{A}^+_{i,\mathfrak{t}} \to \mathbb{B}^+_{i,\mathfrak{t}}$ has underlying group $\mathbb{X}^+_{i,\mathfrak{t}}(n) = \mathbb{A}^+_{i,\mathfrak{t}} \oplus \mathbb{B}^+_{i,\mathfrak{t}}$ and differential

$$\partial_{\mathbb{X}^+_{i,\mathfrak{t}}(n)} = \begin{pmatrix} \partial_{\mathbb{A}^+_{i,\mathfrak{t}}} & 0 \\ \mathcal{D}^+_{n,i,\mathfrak{t}} & \partial_{\mathbb{B}^+_{i,\mathfrak{t}}} \end{pmatrix}.$$

Also, when $n=\pm1$, $\mathrm{Spin}^c(Y_n(K)) \cong \mathrm{Spin}^c(Y)$ and there is no additional choice of $i\in\mathbb{Z}/n\mathbb{Z}$. When this is satisfied, we write simply $\mathbb{X}^+_{\mathfrak{t}}(n)$ instead of $\mathbb{X}^+_{i,\mathfrak{t}}(n)$.

5.4.2 A useful computation

Lemma 5.4.2. *For* $Y_g(\pm1)$ *as before, we have*

$$d_b(Y_g(\pm1);\mathbb{Z}_2) = \mp g$$

where $d(Y,\mathfrak{s};\mathbb{Z}_2)$ *denotes the d–invariant of* $(Y,\mathfrak{s})$ *as computed from Floer homology with coefficients in* $\mathbb{Z}_2$*.*

Proof. The oriented manifold $-Y_g(+1) = Y_g(-1)$ may be obtained as -1–surgery on the knot K (the "Borromean knot") shown in Figure 5.5.

We start with the calculation for $Y_g(-1)$. Since the second homology of $Y_g(-1)$ is generated by embedded tori, the adjunction inequality (Ozsváth–Szabó [OS04d, Theorem 7.1]) implies that $HF^+(Y_g(+1),\mathfrak{t})$ is non-zero only in the unique torsion Spin^c structure $\mathfrak{t}_0 \in \mathrm{Spin}^c(Y_g(-1))$. The knot Floer complex for the Borromean knot K is calculated in Ozsváth–Szabó [OS04c] to be

$$C := CFK^\infty(\#^{2g} S^2\times S^1, K) \cong \Lambda^* H^1(\Sigma_g;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[U,U^{-1}]$$

with $\mathbb{Z}\oplus\mathbb{Z}$–bifiltration given by:

$$C\{i,j\} = \Lambda^{g-i+j} H^1(\Sigma_g;\mathbb{Z}) \otimes_{\mathbb{Z}} U^{-i} \tag{5.8}$$

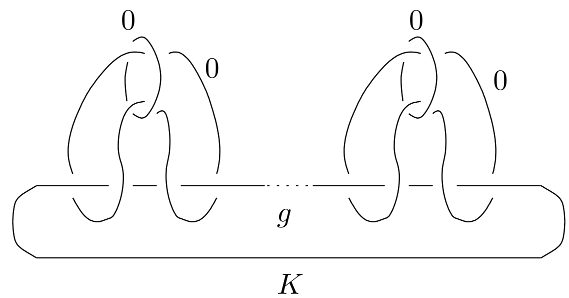

Figure 5.5: The Borromean knot K.

Furthermore, the group $C\{i, j\}$ is supported in grading $i + j$ and all differentials vanish (including all "higher" differentials coming from the spectral sequence $HFK^\infty \Rightarrow HF^\infty$).

Under the above identification, and the identification $H_1(\Sigma_g; \mathbb{Z}) \cong H_1(\#^{2g} S^2 \times S^1; \mathbb{Z})$, the action of $\gamma \in H_1(\Sigma_g; \mathbb{Z})$ on $C \cong \Lambda^* H^1(\Sigma_g; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$ is given explicitly by

$$\gamma \cdot (\omega \otimes U^j) = \iota_\gamma \omega \otimes U^j + \mathrm{PD}(\gamma) \wedge \omega \otimes U^{j+1} \tag{5.9}$$

where ι_γ denotes contraction. Since $HF^\infty(\#^{2g} S^2 \times S^1) \cong C$, this action may be viewed as a reflection of the fact that $HF^\infty(\#^{2g} S^2 \times S^1)$ is standard.

The only presumably non-combinatorial ingredient in the integer surgery formula (once the complex $C = CFK^\infty(Y, K, i)$ is at hand) is the necessary explicit identification of the natural homotopy equivalence $h \colon C\{j \geq 0\} \to C\{i \geq 0\}$. The homotopy h takes a particularly simple form for Floer homology with coefficients in $\mathbb{Z}_2$, with which we work for the remainder of this section. The description of h is as follows:

Proposition 5.4.3 (Ozsváth–Szabó [OS04a])**.** *For the Borromean knot K, the natural homotopy equivalence $h \colon C\{j \geq 0\} \to C\{i \geq 0\}$ sends $C\{j, i\}$ to $C\{i, j\}$.*

Interestingly, the above proposition does not hold for Floer homology with coefficients in $\mathbb{Z}$ (see Jabuka–Mark [JM08b] for a description).

We picture the complex $\mathbb{X}^+(-1)$ as below:

$$\begin{tikzcd}
\cdots & A^+_{-1} \arrow[d, "v^+_{-1}"'] & A^+_0 \arrow[dl, "h^+_0"'] \arrow[d, "v^+_0"'] & A^+_1 \arrow[dl, "h^+_1"'] \arrow[d, "v^+_1"'] & \cdots \\
& B^+_{-1} & B^+_0 & B^+_1 &
\end{tikzcd}$$

For simplicity of discussion, we currently restrict to the case of $g = 1$. In this case, a piece of $\mathbb{X}^+(-1)$ looks like Figure 5.6.

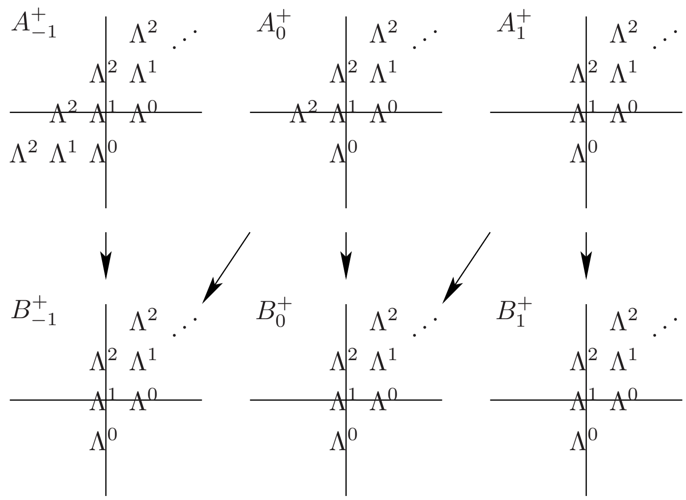

Figure 5.6: A portion of the complex $\mathbb{X}(-1)$. We suppress the U's from the notation, since they can be determined from the position in the plane, according to Equation 5.8.

We claim that the correction terms of $Y_1(-1)$ can be read off from Figure 5.6. Indeed, writing $H_1(\Sigma_1;\mathbb{Z}) = \mathbb{Z}\langle a, b\rangle$ and $\Lambda^* H_1(\Sigma_1;\mathbb{Z}) = \mathbb{Z}\langle 1, a, b, a\wedge b\rangle$, consider the element $1 \otimes U^{-1} \in B^+_0\{1, 0\}$. Since

$$\partial_{\mathbb{X}^+(-1)} = \begin{pmatrix} \partial_{\mathbb{A}^+} & 0 \\ \mathcal{D}^+ & \partial_{\mathbb{B}^+} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathcal{D}^+ & 0 \end{pmatrix},$$

it follows that $1 \otimes U^{-1} \in B^+_0$ is a cycle. We claim that $1 \otimes U^{-1} \in B^+_0$ is also not a boundary. Indeed, suppose that $\partial_{\mathbb{X}^+(-1)}(x) = 1 \otimes U^{-1} \in B^+_0$ for some $x \in \mathbb{X}^+(-1)$. Then x would necessarily have either a non-zero component in $A^+_0\{1, 0\}$ or a non-zero component in $A^+_1\{1, 2\}$. In either case, a simple diagram chase shows that x cannot be extended to a cycle in $\mathbb{X}^+(-1)$ (ride the zig-zag and notice that x would have infinitely many non-zero components in $\mathbb{A}^+$).

It is, however, the case that $U \cdot (1 \otimes U^{-1}) = 1 \otimes U^0 \in B_0^+\{0,-1\}$ is a boundary: the element $1 \otimes U^0 \in A_0^+\{0,-1\}$ maps to it under $\partial_{\mathbb{X}^+(-1)}$.

Using similar reasoning, one can show that the elements

$$1 \otimes U^{-1} \in B_0^+\{1,0\},\ a \otimes U^0 \in B_0^+\{0,0\},\ b \otimes U^0 \in B_0^+\{0,0\},\ \text{ and } (a \wedge b) \otimes U^0 \in B_0^+\{0,1\}$$

all represent generators for the four "towers" of $HF^+(Y_1(-1))$. According to the grading formula, Equation 5.7, these elements have grading 1,0,0, and 1, respectfully. It follows that:

$$HF^+(Y_1(-1)) = \left(\mathcal{T}^+_{(0)}\right)^2 \oplus \left(\mathcal{T}^+_{(+1)}\right)^2 \oplus HF^+_{\mathrm{red}}(Y_1(-1))$$

where here we are using a slight abuse of notation: $\mathcal{T}^+$ now denotes $\mathcal{T}^+ \otimes \mathbb{Z}_2$ (for the previous definition of $\mathcal{T}^+$) and Floer homology with $\mathbb{Z}_2$–coefficients is understood. Further, according to the action, Equation 5.9, it follows that $d_b(Y_1(-1);\mathbb{Z}_2) = 1$ (alternatively this follows since $Y_1(-1)$ has standard HF^∞ via Equation 5.3). Although we already knew how to compute this, the advantage of this calculation is that the reasoning generalizes to arbitrary g. Indeed, for general g one can check that the intersection

$$B_0^+\{(i \geq 0 \text{ and } j = 0) \ \text{ or } \ (i = 0 \text{ and } j \geq 0)\}$$

gives representatives for generators of the 2^{2g} "towers" of $HF^+(Y_g(-1))$. Using the grading formula, Equation 5.7, it follows that:

$$\begin{aligned} HF^+(Y_g(-1)) \cong \left(\mathcal{T}^+_{(0)}\right)^{\binom{2g}{g}} \oplus \left(\mathcal{T}^+_{(1)}\right)^{2\binom{2g}{g-1}} \oplus \left(\mathcal{T}^+_{(2)}\right)^{2\binom{2g}{g-2}} \oplus \cdots \\ \cdots \oplus \left(\mathcal{T}^+_{(g-1)}\right)^{2\binom{2g}{1}} \oplus \left(\mathcal{T}^+_{(g)}\right)^{2} \oplus HF^+_{\mathrm{red}}(Y_g(-1)) \end{aligned}$$

By the action formula, Equation 5.9, (or the fact that $HF^\infty(Y_g(-1))$ is standard) it follows that

$$d_b(Y_g(-1);\mathbb{Z}_2) = g.$$

Calculating $d_b(Y_g(-1);\mathbb{Z}_2)$ is similar. In that case, the generators of the intersection

$$A_0^+\{(i \leq 0 \text{ and } j = 0) \ \text{ or } \ (i = 0 \text{ and } j \leq 0)\}$$

may be extended (in one step) to representative cycles for the homology $HF^+(Y_g(+1))$. Using Equation 5.6 one calculates that:

$$HF^+(Y_g(+1)) \cong \left(\mathcal{T}^+_{(0)}\right)^{\binom{2g}{g}} \oplus \left(\mathcal{T}^+_{(-1)}\right)^{2\binom{2g}{g-1}} \oplus \left(\mathcal{T}^+_{(-2)}\right)^{2\binom{2g}{g-2}} \oplus \cdots$$
$$\cdots \oplus \left(\mathcal{T}^+_{(-g+1)}\right)^{2\binom{2g}{1}} \oplus \left(\mathcal{T}^+_{(-g)}\right)^{2} \oplus HF^+_{\mathrm{red}}(Y_g(+1))$$

The action formula, Equation 5.9, gives

$$d_b(Y_g(+1);\mathbb{Z}_2) = -g \tag{5.10}$$

proving Lemma 5.4.2 □

An alternative approach to the calculation of the correction terms of ±1–surgery on the Borromean knot of genus g is the integer surgery exact sequence, together with Jabuka-Mark's calculation of the Floer homology of $S^1 \times \Sigma_g$ [JM08c]. It is interesting to note that $HF^+_{\mathrm{red}}(Y_g(-1))$ has been calculated in Ozsváth–Szabó [OS04c].

In fact, the methods in this section give the following:

Theorem 5.4.4. *Let X be a smooth, simply-connected, compact, oriented four-manifold with a homology sphere Y as boundary with $b_2^+(X) = 1$ and $b_2^-(X) = 0$. Let $\Sigma \subset X$ be a closed surface of genus g and self-intersection $\Sigma \cdot \Sigma = 1$. Then $0 \le d(Y;\mathbb{Z}/2) + 2g$.*

5.5 Computations

In this section we discuss an algorithm to compute the invariants $d(S^3_{\pm1}(K))$ assuming we know the filtered chain homotopy type of the knot complex $CFK^\infty(K)$ of Ozsváth–Szabó [OS04c]. We also discuss a computer implementation of this algorithm. The algorithm we use is based on the theory of Ozsváth–Szabó [OS04c; OS04a], and Rasmussen [Ras03] and has three steps:

1. Use the theory of Ozsváth–Szabó [OS04c] or Rasmussen [Ras03] to compute the graded $\mathbb{F}[U]$-module

   $$HF^+(S^3_{-p}(K), \mathfrak{s}_0)$$

   where $-p$ is "sufficiently negative" and $\mathfrak{s}_0$ is a suitable Spin^c structure on $S^3_{-p}(K)$ (and similarly for $S^3_p(K)$ for p "sufficiently positive").

2. Use exact sequences to compute the correction terms of $S^3_0(K)$.

3. Use a simple relation between the correction terms of $S^3_0(K)$ and the correction terms of $S^3_{\pm 1}(K)$.

We describe the steps in reverse order. Recall from [OS04a] that for a closed oriented three-manifold Y_0 with $H_1(Y_0;\mathbb{Z}) \cong \mathbb{Z}$, there are two correction terms $d_{\pm 1/2}(Y_0)$, where $d_{\pm 1/2}(Y_0)$ is the minimal grading of any non-torsion element in the image of $HF^\infty(Y_0, \mathfrak{t}_0)$ in $HF^\pm(Y_0, \mathfrak{t}_0)$ with grading $\pm 1/2$ modulo 2. Then step 3 follows from Ozsváth–Szabó [OS04a, Proposition 4.12] which states that

$$d_{1/2}(S^3_0(K)) - 1/2 = d(S^3_1(K)) \tag{5.11}$$

and

$$d_{-1/2}(S^3_0(K)) + 1/2 = d(S^3_{-1}(K)). \tag{5.12}$$

This proposition is an easy consequence of the fact that $HF^\infty(S^3_0(K))$ is standard and the exact sequence, Equation 5.2.

For step 2, recall the integral surgeries long exact sequence (see Ozsváth–Szabó [OS04d, Theorem 9.19]; [OS04a] for the graded version). Let $K \subset Y$ be a knot in an integral homology three-sphere and p a positive integer. Then we get a map

$$Q\colon \operatorname{Spin}^c(Y_0) \to \operatorname{Spin}^c(Y_p)$$

and a long exact sequence of the form

$$\cdots \xrightarrow{F_1} HF^+(Y_0, [\mathfrak{t}]) \xrightarrow{F_2} HF^+(Y_p, \mathfrak{t}) \xrightarrow{F_3} HF^+(Y) \longrightarrow \cdots \tag{5.13}$$

where

$$HF^+(Y_0, [\mathfrak{t}]) = \bigoplus_{\mathfrak{t}' \in Q^{-1}(\mathfrak{t})} HF^+(Y_0, \mathfrak{t}').$$

Moreover, the component of F_1 in the above exact sequence which takes $HF^+(Y)$ into the $\mathfrak{t}_0$-component of $HF^+(Y_0, [Q(\mathfrak{t}_0)])$ has degree $-1/2$, while the restriction of F_2 to the $HF^+(Y_0, \mathfrak{t}_0)$–summand of $HF^+(Y_0, [Q(\mathfrak{t}_0)])$ has degree $\binom{\frac{p-3}{4}}{}$. It follows from this exact sequence that $d_{1/2}(S^3_0(K)) + \left(\frac{p-3}{4}\right) = d(S^3_{+p}(K), \mathfrak{s}_0)$. To get $d_{-1/2}(S^3_0(K))$, we may use a similar exact sequence for negative surgeries.

$CFK^\infty(K)$ is finitely generated as a complex over $\mathbb{Z}[U, U^{-1}]$. It comes with an absolute $\mathbb{Z}$–grading and a $\mathbb{Z} \oplus \mathbb{Z}$–bifiltration. Generators are written $[\mathbf{x}, i, j]$ for integers i, j and intersection points $\mathbf{x}$ in a Heegaard diagram for K. The U–action is given by $U \cdot [\mathbf{x}, i, j] = [\mathbf{x}, i-1, j-1]$. We picture these complexes as graphs in the plane, as in Figure 5.7 (1). Dots represent generating $\mathbb{Z}$'s while arrows represent differentials. The absolute grading is (basically) pinned down by the fact that if we consider the "y–slice" quotient complex $CFK^\infty\{i = 0\}$, then its homology (which is guaranteed to be a single copy of $\mathbb{Z}$—the generator of $\widehat{HF}(S^3) \cong \mathbb{Z}_{(0)}$) is supported in grading 0, the fact that the U–action drops absolute grading by 2, and the fact that differentials drop grading by 1[3].

In order to accomplish step 1, we use the following theorem of Ozsváth–Szabó, Rasmussen (see, for instance, Ozsváth–Szabó [OS04c, Corollary 4.3])

Proposition 5.5.1 (Ozsváth–Szabó [OS04c], Rasmussen [Ras03])**.** *Let K be a knot in the three-sphere. Then there exists a positive integer N with the property that for all $p \geq N$ we have that*

$$HF^+_\ell(S^3_{-p}(K), [0]) \cong H_k(CFK^\infty(K)\{i \geq 0 \text{ and } j \geq 0\}$$

where

$$\ell = k + \left(\frac{1-p}{4}\right).$$

Similarly,

$$HF^+_\ell(S^3_p(K), [0]) \cong H_k(CFK^\infty(K)\{i \geq 0 \text{ or } j \geq 0\})$$

where

$$\ell = k + \left(\frac{p-1}{4}\right).$$

In fact, we can take $N = 2g - 1$ where $g = g(K)$ is the knot genus.

We can pick off the correction terms $d(S^3_{\pm p}(K), 0)$ from this theorem: we know that

$$H_*(CFK^\infty(K)) \cong \mathbb{Z}[U, U^{-1}]$$

[3] This is actually not quite true: not all vertices appearing in the complex CFK^∞ are related to the generator of $\widehat{HF}(S^3)$ by a sequence of U–maps and differentials. To grade these remaining vertices, we have to go back to the Heegaard diagram. However, for the purpose of computing d–invariants, we do not need to look at these at all.

so choose a class $a \in CFK^\infty(K)$ which generates this homology as a $\mathbb{Z}[U, U^{-1}]$–module. Look at a sufficiently negative U–power of this generator. This generates the "tower" of $H_*(CFK^\infty(K)\{i \geq 0 \text{ or } j \geq 0\}$. All one has to do is start taking U–powers of $[a] \in H_k(CFK^\infty(K)\{i \geq 0 \text{ or } j \geq 0\}$ and see when they vanish in homology. The grading of the last-surviving U–power of $[a]$ is $d(S^3_{+p}(K), [0])$ (after the grading shift). A similar story allows one to compute $d(S^3_{-p}(K), 0)$.

Write $\widetilde{d}(S^3_p(K), [0]) = d(S^3_p(K), [0]) - \frac{p-1}{4}$, the "unshifted" correction term of the group $H_k(CFK^\infty(K)\{i \geq 0 \text{ or } j \geq 0\})$. Putting together the previous discussion, by the exact sequence, Equation 5.13 we have:

$$d_{\frac{1}{2}}(S^3_0(K)) + \frac{p-3}{4} = d(S^3_p(K), [0]) = \widetilde{d}(S^3_p(K), 0) + \frac{p-1}{4}$$

Using Equation 5.11 we get:

$$\frac{1}{2} + d(S^3_1(K)) + \frac{p-3}{4} = \widetilde{d}(S^3_p(K), 0) + \frac{p-1}{4}$$

ie

$$d(S^3_1(K)) = \widetilde{d}(S^3_p(K), [0]). \tag{5.14}$$

We now discuss how to teach a computer to do step 1. In fact, we implemented this in C++ in a program called `dCalc` (`beta`). The source code is available at

`http://www.math.columbia.edu/~tpeters`.

As previously mentioned, $CFK^\infty(K)$ is finitely generated as a complex over $\mathbb{Z}[U, U^{-1}]$. By a symmetry property of the knot Floer homology, we may assume that the corresponding graph is symmetric about the line $i = j$. With such a graph at hand, we choose a generating set which is

1. Minimal: no smaller subset of it generates $CFK^\infty(Y)$.

2. In the first quadrant, $i \geq 0$ and $j \geq 0$.

3. As close to the origin as possible.

(See Figure 5.7 for an example). We started with a digraph data structure to represent the complexes. Vertices were marked with $\mathbb{Z} \oplus \mathbb{Z}$–bifiltration levels and could be marked with gradings. Vertices are also marked to keep track of bases. The first step was to fill in the gradings. For this we needed to compute the "y–slice" described before. This is determined by a finite number of U–translates of our chosen generating set. Once we find the generator of $\widehat{HF}(S^3)$, it is a problem in graph traversal to fill in (most of—see footnote 3) the other gradings. Our chosen generating set will necessarily have 1–dimensional homology (over $\mathbb{Z}_2$) and its generator x will have a grading computed from the graph traversal. This generator maps to a generator of $HF^+(S^3) \cong \mathcal{T}^+_{(0)}$. To compute $d(S^3_{+1}(K))$, we start by taking a finite piece of the complex $CFK^\infty(K)\{i \geq 0 \text{ or } j \geq 0\}$ (more specifically, take our chosen generating set and start hitting it by U—at some point it will disappear out of the "hook" region $\{i \geq 0 \text{ or } j \geq 0\}$. Take only those images which appear in the hook). In the graph implementation, this just involves shifting filtration levels and throwing away some vertices as they exit the hook. Now take the generator x and start pushing it down by U until it dies in homology. Its grading just before it dies is $d(S^3_{+1}(K))$, by Equation 5.14. To compute $d(S^3_{-1}(K))$, one runs a similar story, but instead of using the "hook region" $\{i \geq 0 \text{ or } j \geq 0\}$, one uses the first quadrant $\{i \geq 0 \text{ and } j \geq 0\}$.

Since one knows how knot complexes behave under connected sum of knots (tensor product over $\mathbb{Z}[U, U^{-1}]$; see Ozsváth–Szabó [OS04c, Theorem 7.1] for the precise formulation), we implemented this as well, allowing users to compute correction terms of surgeries on connected sums of knots.

5.5.1 A few examples

In this section, Floer homology with mod-2 coefficients is understood. Figure 5.7 shows an example of the algorithm described in Section 5.5. Here, we are given the knot complex $CFK^\infty(T_{3,4})$ for the $(3, 4)$–torus knot, which was computed in Ozsváth–Szabó [OS04a, Section 5.1] or by [OS05b, Theorem 1.2].

As another example, consider the right-handed trefoil RHT. This knot has knot Floer

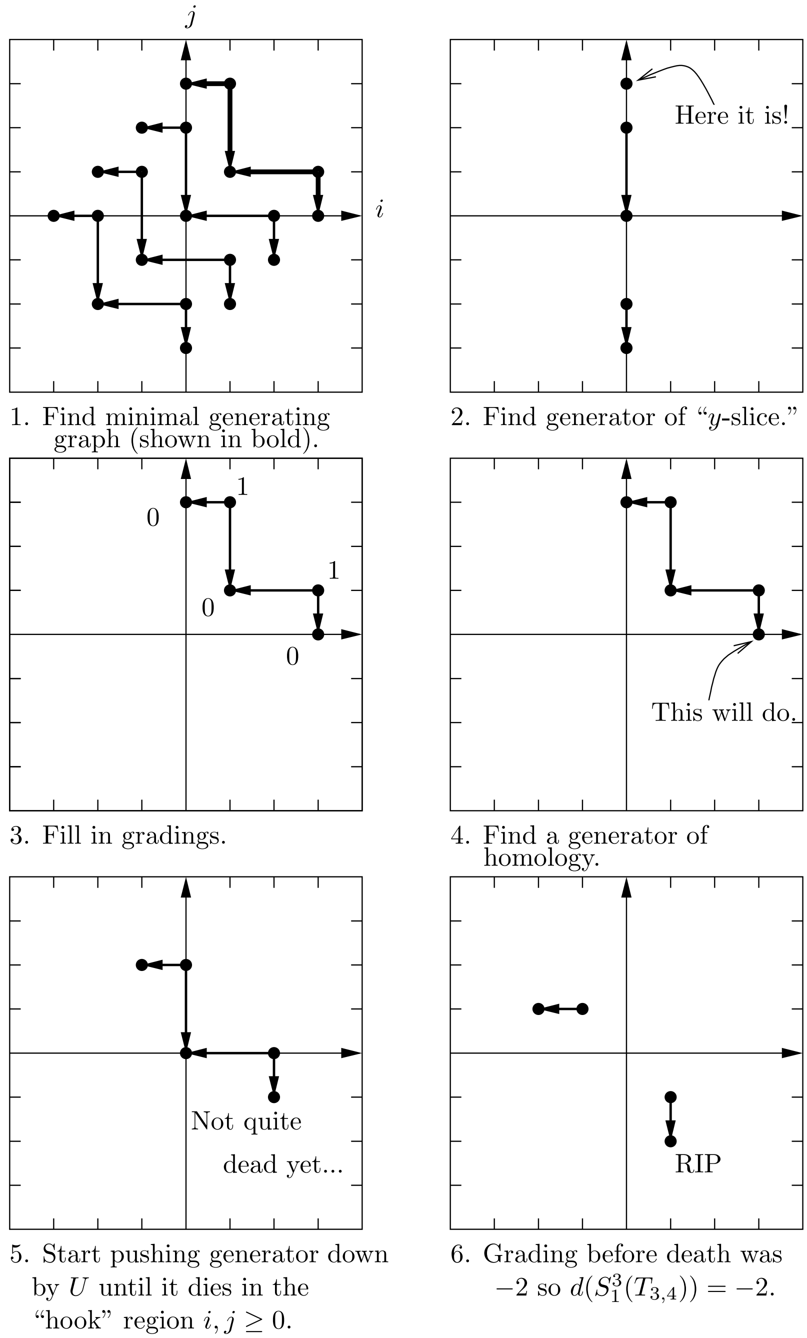

Figure 5.7: An example of the algorithm used, applied to the $(3, 4)$–torus knot, $T_{3,4}$.

homology given by

$$\widehat{HFK}_j(RHT, i) = \begin{cases} \mathbb{Z}_2 \text{ if } (i, j) = (1, 0) \\ \mathbb{Z}_2 \text{ if } (i, j) = (0, -1) \\ \mathbb{Z}_2 \text{ if } (i, j) = (-1, -2) \\ 0 \text{ otherwise} \end{cases}$$

Here, i, j denote the Alexander and Maslov gradings, respectfully. Since we know there is a spectral sequence, induced by the Alexander filtration on $\widehat{CF}(S^3)$, converging to $\widehat{HF}(S^3) \cong \mathbb{Z}$ (supported in grading 0), it follows that the E^1 page of this spectral sequence is given in Figure 5.8.

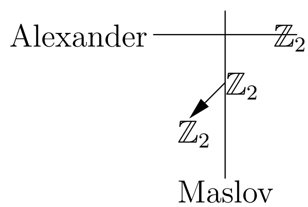

Figure 5.8: The E^1 page of the spectral sequence $\widehat{HFK}(RHT) \Rightarrow \widehat{HF}(S^3)$.

By the symmetry of CFK^∞, it follows that $CFK^\infty(RHT)$ is generated as a $\mathbb{Z}[U, U^{-1}]$–module by the complex in Figure 5.9.

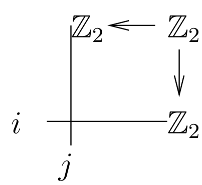

Figure 5.9: A generating complex for the knot complex of the right-handed trefoil.

which shows that $d(S^3_1(RHT)) = -2$, a fact which more readily follows from Equation 5.1.

Next, consider the figure eight knot, 4_1. This knot has knot Floer homology

$$\widehat{HFK}_j(4_1, i) = \begin{cases} \mathbb{Z}_2 \text{ if } (i,j) = (1,1) \\ \mathbb{Z}_2^3 \text{ if } (i,j) = (0,0) \\ \mathbb{Z}_2 \text{ if } (i,j) = (-1,-1) \\ 0 \text{ otherwise} \end{cases}$$

Again, by considering the spectral sequence $\widehat{HFK}(4_1) \Rightarrow \widehat{HF}(S^3)$, it follows that the E^1 page of this spectral sequence is given Figure 5.10.

It follows that $CFK^\infty(4_1)$ is generated as a $\mathbb{Z}[U, U^{-1}]$–module by the complex shown in Figure 5.11.

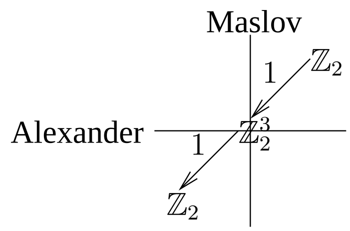

Figure 5.10: The E^1 page of the spectral sequence $\widehat{HFK}(4_1) \Rightarrow \widehat{HF}(S^3)$. The markings on the arrows signify the ranks of the maps.

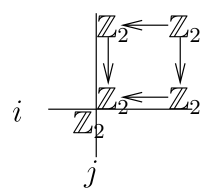

Figure 5.11: A generating complex for the figure eight knot, 4_1. Here we have two $\mathbb{Z}_2$'s at the origin—one is isolated while the other is part of a null-homologous "box".

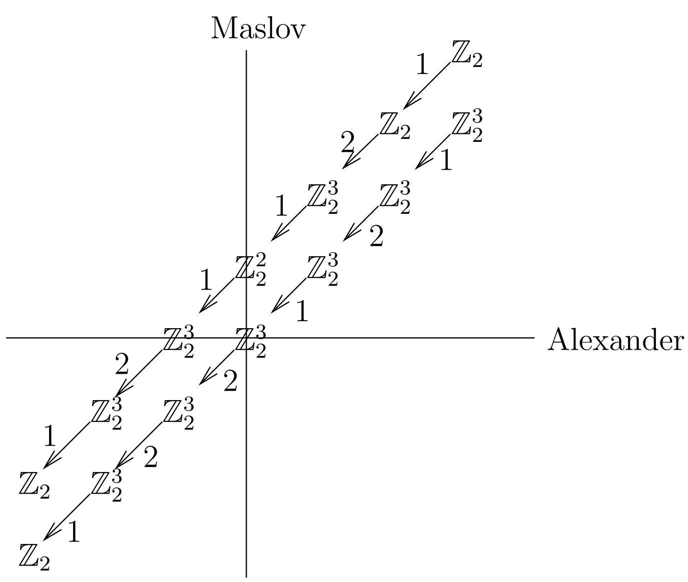

Figure 5.12: The E^1 term of the spectral sequence $\widehat{HFK}(C_{2,1}) \Rightarrow \widehat{HF}(S^3)$.

Here, we see a single isolated $\mathbb{Z}_2$ at the origin plus a null-homologous "box". It is then easy to see that $d(S^3_{\pm 1}(4_1)) = 0$ (again, this follows more quickly from Equation 5.1).

Recall that while the Kinoshita–Terasaka knot $K_{2,1}$ is smoothly slice, it is currently unknown if its Conway mutant $C_{2,1}$ is smoothly slice (though it is *topologically* slice since it has trivial Alexander polynomial, by a result of Freedman [FQ90; Fre83]). Indeed, we currently show that $d(S^3_1(C_{2,1})) = 0$, showing that our invariant gives no information. In [BG], Baldwin and Gillam calculated the knot Floer homology polynomial[4] of $C_{2,1}$ to be:

$$(q^{-3}+q^{-2})t^{-3}+3(q^{-2}+q^{-1})t^{-2}+3(q^{-1}+1)t^{-1}+3+2q+3(q+q^2)t+3(q^2+q^3)t^2+(q^3+q^4)t^3 \tag{5.15}$$

Similar to previous computations, it follows that the E^1 term of the spectral sequence $\widehat{HFK}(C_{2,1}) \Rightarrow \widehat{HF}(S^3)$ is forced to be as in Figure 5.12. From this, it follows that $CFK^\infty(C_{2,1})$ can be computed by a complex generated as a $\mathbb{Z}[U, U^{-1}]$–module with a single isolated $\mathbb{Z}_2$ at the origin plus a collection of null-homologous "boxes". As in the computation for the figure eight, it follows that $d(S^3_{\pm 1}(C_{2,1})) = 0$.

5.5.2 An example session

In this section we show an example session of our program `dCalc`. We first input a generating complex for CFK^∞ of the right-handed trefoil, as in Figure 5.9. We then form the complex for the connect-sum $RHT\#RHT$. Finally we compute the correction terms of $S^3_{\pm 1}(RHT\#RHT)$.

```
*****************************************************************
Welcome to the d invariant calculator!


This program computes the d invariants of +/-1 surgery on a knot


Copyright (C) 2009 Thomas Peters
```

[4]The knot Floer homology polynomial of a knot K is defined to be

$$\Sigma_{i,j} \dim_{\mathbb{Z}_2} \widehat{HFK}_j(S^3, K, i)\, q^j t^i$$

.

```
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******************************************************************
----------------------------------
   Main menu.
(1) Enter a new knot
(2) View current knots
(3) Select a knot
(4) Connect-sum two knots
(0) Quit
-----------------------------------
1
Enter the name of your knot
trefoil
Enter the knot vertex keys (non-neg integers).  input -1 to stop
0
```

```
1
2
-1
entered vertices 0,1,2,
enter the adjacency lists (type a vertex key, press enter,
continue. input -1 to stop)
successors of 0:
1
2
-1
successors of 1:
-1
successors of 2:
-1
enter the bifiltration levels
(type i value, press enter, then type j value, then press enter)
(input -1 to stop)

F_i[0] = 1
F_j[0] = 1

F_i[1] = 0
F_j[1] = 1

F_i[2] = 1
F_j[2] = 0
added knot trefoil with adjacency list
[0]1,2,
[1]
[2]
```

```
and bifiltration levels
F(0) = (1,1)
F(1) = (0,1)
F(2) = (1,0)



---------------------------------
   Main menu.
(1) Enter a new knot
(2) View current knots
(3) Select a knot
(4) Connect-sum two knots
(0) Quit
----------------------------------
4
Current knots are:
(index, name)
------------
(0, trefoil)
Enter the indices of the two knots to add
0
0
Computing tensor product...
Computation took 0min0sec.
Created knot trefoil#trefoil having adjacency list
[12]
[8]
[5]8,12,
[7]
```

```
[4]
[2]4,7,
[3]7,12,
[1]4,8,
[0]2,5,1,3,
and bifiltrations
F(12) = (2,0)
F(8) = (1,1)
F(5) = (2,1)
F(7) = (1,1)
F(4) = (0,2)
F(2) = (1,2)
F(3) = (2,1)
F(1) = (1,2)
F(0) = (2,2)
---------------------------------
   Main menu.
(1) Enter a new knot
(2) View current knots
(3) Select a knot
(4) Connect-sum two knots
(0) Quit
----------------------------------
3
Current knots are:
(index, name)
------------
(0, trefoil)
(1, trefoil#trefoil)
input an index
```

```
1
What would you like to do with your knot complex?
(1) Print its adjacency list
(2) Show its bifiltration levels
(3) Check if it defines a complex
(4) Check if it is filtered
(5) Compute its homology
(6) Compute d invariants!
(7) Nothing--bring me back to the main menu
6
d(S^3_{+1}(K)) = -2
d(S^3_{-1}(K)) = 0
---------------------------------
   Main menu.
(1) Enter a new knot
(2) View current knots
(3) Select a knot
(4) Connect-sum two knots
(0) Quit
----------------------------------
0
Really quit d calculator? (y/n) y
```

5.5.3 Issues with the implementation

`dCalc` does not do any checking on inputted complexes. If one inputs a complex which does not come from a knot, `dCalc` may `return garbage` or have undefined behavior. `dCalc` does, however, come with a few basic functions useful in determining the feasibility of a given complex. For instance, it can check if the user's graph actually represents a complex.

It is also worth mentioning that our implementation was for Floer homology with coefficients in $\mathbb{Z}_2$, so we are really computing correction terms for mod-2 coefficients. It is an interesting question to determine whether or not d–invariants for Floer homology with $\mathbb{Z}_2$ coefficients can ever differ from d–invariants calculated with $\mathbb{Z}$ coefficients.

More seriously, by default `dCalc` uniquely identifies vertices by `int` keys. One is therefore limited by the maximum value of `int`, `INT_MAX` (this is defined in the header file `<limits.h>` and varies from platform to platform, though is guaranteed to be at least 32,767). This is only realistically a problem after taking tensor products, where we rely on an explicit bijection $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ to assign vertex keys for the tensor product. This function is quadratic in its two arguments so it can grow quite quickly. Surpassing `INT_MAX` can result in undefined behavior (including segmentation faults). If one were limited by this feature, one could change the underlying data structure of the vertex keys to a more flexible structure, for instance something like the `tuple` structure found in `python`, or to a larger integer structure, such as a `long unsigned int`. The latter can be done by changing the line "`typedef int KEYTYPE;`" of `vertex.h` to, for instance, "`typedef long KEYTYPE;`" and them recompiling. Of course, such operations increase run time. One way to check if `INT_MAX` has been exceeded (assuming `KEYTYPE` is not `unsigned`) is by printing out the adjacency matrix (or filtration levels) for a particular knot complex. If negatives appear as keys, `INT_MAX` has been surpassed (though, in principle, this need not be a necessary condition).

One place in which this program is inefficient memory-wise is in checking whether or not a given element in a complex is a boundary. We do this by row-reduction. If a complex has n generators, the row-reduction requires a `char` array of roughly size n^2 to be allocated from the heap. Of course, one should not need to create these matrices considering the homology itself can be checked just by performing an algorithm on the graph (see Baldwin and Gillam [BG] for a discussion).

We stress that in order to compute $d(S^3_{\pm 1}(K))$ for a given knot, one must have at hand the filtered chain homotopy type of the $\mathbb{Z}[U, U^{-1}]$–module $CFK^\infty(K)$. Computing these complexes is quite challenging, in general. In the case that K is alternating or is a torus knot, then one may recover $CFK^\infty(K)$ from the usually weaker invariant $\widehat{HFK}(K)$. In the former case we have Equation 5.1 and in the latter we have Ozsváth–Szabó [OS05b, Theorem 1.2], so we do not need to use a computer at all. Depending on one's proficiency in Heegaard Floer homology, it is sometimes possible (though one should not expect in general) to calculate $CFK^\infty(K)$ from $\widehat{HFK}(K)$ (for instance, see the examples in Section 5.5.1).

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