On Hamilton's Ricci Flow and Bartnik's Construction of Metrics of Prescribed Scalar Curvature

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Abstract

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It is known by work of R. Hamilton and B. Chow that the evolution under Ricci flow of an arbitrary initial metric g on S^2, suitably normalized, exists for all time and converges to a round metric. I construct metrics of prescribed scalar curvature using solutions to the Ricci flow. The problem is converted into a semilinear parabolic equation similar to the quasispherical construction of Bartnik. In this work, I discuss existence results for this equation and applications of such metrics.

Contents

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To my brother.

1 Introduction

Einstein's famous field equation of a space-time (V, γ) is

$$R^V_{ab} - \frac{1}{2}R^V\gamma_{ab} = 8\pi T_{ab}, \quad a, b = 0, 1, 2, 3,$$

where T_{ab} is the space-time energy momentum tensor. The equation admits Cauchy data formulation and initial data cannot be chosen arbitrarily. Let (V, γ) be a solution and consider a spacelike hypersurface (N, g). From the Gauss and Codazzi equations the scalar curvature R_N and the second fundamental form k of (N, g) will satisfy the following constraint equations [Wal84]

$$\begin{aligned} R_N + (tr_g k)^2 - |k|^2_g &= 16\pi T_{00} &(1.0.1)\\ \nabla_j(k_{ij} - trkg_{ij}) &= 8\pi T_{0i}, \quad i, j = 1, 2, 3, &(1.0.2)\end{aligned}$$

where e_0 is the future timelike unit normal vector of the hypersurface N. When $T = 0$, these equations are called the vacuum constraint equations.

There are various ways to construct solutions of the constraint equations. One of the most widely used methods is the conformal method. It reformulates the constraint equations as a coupled quasilinear elliptic PDE system [CBY80]. The other approach is the quasi-spherical ansatz and the parabolic method introduced by Bartnik [Bar93] in 1993. A 3-manifold is called quasi-spherical if it can be foliated by round spheres. Under the quasi-spherical ansatz, Bartnik introduced a new construction of 3-metrics of prescribed scalar curvature. One important observation is that the scalar curvature equation can be rewritten as a semilinear parabolic equation. In 2005 Sharples [Sha05] proves local existence results for the constraint equation in a special case when the spacelike 3-manifold is assumed to satisfy the quasispherical ansatz.

Another important consequnce is that this construction allows insight into the extension problem, where one hopes to extend a bounded Riemannian 3-manifold to an asymptotically flat 3-manifold satisfying weak conditions of the positive mass theorem. This problem arises naturally from Bartniks definition of quasi-local mass [Bar89], where the mass of a bounded Riemannian 3-manifold (Ω, g_0) with connected boundary is defined by

$$m_B(\Omega) := \inf\{m_{ADM}(M) : (M, g) \text{ is an asymptotically flat time-symmetric initial data set satisfying weak-energy conditions such that } (M, g) \text{ contains } (\Omega, g_0) \text{ isometrically M has no horizon.}\}$$

If we consider the extension problem as a problem of matching $M \setminus \Omega$ with Ω along the boundary, then thecondition that the scalar curvature can be defined distributionally and bounded across $\partial(M \setminus \Omega)$ leads to the geometric boundary conditions

$$g|_{\partial(M\setminus\Omega)} = g_0|_{\partial\Omega}, \quad H_{\partial(M\setminus\Omega),g} = H_{\partial\Omega,g_0}$$

Note that this extension is not possible with the traditional conformal method [CBY80, SY79]. Specifying both the boundary metric and the mean curvature leads to simultaneous Dirichlet and Neumann boundary conditions, which are ill-posed for the elliptic equation of the conformal factor.

The evolution under the modified Ricci flow of an arbitrary initial metric g_1 on Σ_1, diffeomorphic to S^2, exists for all time and converges exponentially to the round metric [Ham88, Cho91]. The solutions $g(t)$ therefore give a canonical foliation on $[1, \infty) \times \Sigma_1$. This suggests a construction of 3-metrics with prescribed scalar curvature based on Ricci flow foliation.

1.1 Weighted Hölder spaces

We consider weighted Hölder spaces adapted to the scaling invariant properties for the parabolic equation. For any interval $I \subset \mathbb{R}^+$, let $A_I = I \times \Sigma_1$. For any nonnegative integer k and $0 < \alpha < 1$, define

$$\begin{aligned}
||f||_{0,I} &= \sup\{f(t,x) : (t,x) \in A_I\}, \\
||f||_{\alpha,I} &= ||f||_{0,I} + \langle f \rangle_{\alpha,I}, \\
||f||_{k,I} &= \Sigma_{|i|+2j\le k} ||\nabla^i (t\partial_t)^j f||_{0,I}, \quad \text{and} \\
||f||_{k+\alpha,I} &= ||f||_{k,I} + \Sigma_{|i|+2j=k} \langle \nabla^i (t\partial_t)^j f \rangle_{\alpha,I},
\end{aligned}$$

where

$$\langle f \rangle_{\alpha,I} = \sup \left\{ \begin{array}{l} t_1^{\alpha/2} \dfrac{|f(t_2,x_2) - f(t_1,x_1)|}{|t_2 - t_1|^{\alpha/2}} + \dfrac{|f(t_2,x_2) - f(t_1,x_1)|}{|x_2 - x_1|^{\alpha}} \\ \text{for all } (t_1,x_1),(t_2,x_2) \in A_I \text{ such that } \dfrac{1}{2}t_2 < t_1 < 2t_2 \\ \text{and } (t_1,x_1) \neq (t_2,x_2) \end{array} \right\}.$$

For compact intervals $I \subset \mathbb{R}^+$, the weighted parabolic Hölder space $C^{k+\alpha}(A_I)$ is the Banach space of continuos functions on A_I with finite $||\cdot||_{k+\alpha,I}$ norm. For I noncompact, $C^{k+\alpha}(A_I)$ is defined as the space of continuous functions with bounded $C^{k+\alpha}$ norm on compact subsets of I. $C^{k,\alpha}(\Sigma_1)$ denotes the usual Hölder space on Σ_1 with norm $||\cdot||_{k,\alpha}$.

Remark 1.1. *The weighted Hölder norm $||\cdot||_{k+\alpha,I}$ is invariant under the following dilation. For $\lambda > 0$, and $f \in C^{k+\alpha}(A_I)$, denote $\lambda^{-1}I = \{t \in \mathbb{R}^+ : \lambda t \in I\}$, and f_λ be defined on $\lambda^{-1}I$ as*

$$f_\lambda(t,x) = f(\lambda t, x),$$

then

$$||f_\lambda||_{(k+\alpha),\lambda^{-1}I} = ||f||_{(k+\alpha),I}.$$

A quasi-linear equation in divergence form is an equation of the form

$$\mathscr{L}u \equiv u_t - \frac{d}{dx^i} a^i(x,t,u,u_x) + a(x,t,u,u_x) = 0. \tag{1.1.3}$$

Let Ω be a domain and Q_T be the cylinder $\Omega \times (0,T)$. Let S_T denote the lateral surface of Q_T, i.e., $S_T = \{(x,t) : (x,t) \in \partial\Omega \times [0,T]\}$. The parabolic boundary Γ_T is given by $\Gamma_T = S_T \cup \{(x,t) : x \in \Omega, t = 0\}$.

We define the parabolic distance d by the following. First, we define

$$d_0(x,t) = \min_{(y,s)\in\Gamma_T, s\le t} |(x,t) - (y,s)|,$$

and then

$$d(x,t) = \begin{cases} d_0(x,t) & \text{if } (x,t) \in Q_T \\ -d_0(x,t) & \text{if } (x,t) \notin Q_T. \end{cases}$$

Suppose the functions $a^i(x,t,u,p)$ and $a(x,t,u,p)$ in equation (1.1.3) are defined for $(t,x) \in \bar{A}_I$, continuous in u and $p = (p_1, \cdots, p_n)$, and satisfy the conditions

$$a^i(x,t,u,p)p_i \ge \nu(|u|)p^2 - \varphi_0(x,t), \tag{1.1.4}$$

$$|a^i(x,t,u,p)| \le \mu_1(|u|)|p| + \varphi_1(x,t), \tag{1.1.5}$$

$$|a(x,t,u,p)| \le \mu_2(|u|)p^2 + \varphi_2(x,t), \tag{1.1.6}$$

in which $\nu(\xi)$ and $\mu_i(\xi)$ are positive continuous functions of $\xi \ge 0$, with $\nu(\xi)$ monotonically decreasing, μ_i monotonically increasing, the functions $\varphi_i(x,t)$ nonnegative and having finite $L^{q,r}$ norms

$$||\varphi_0, \varphi_2||_{q,r,A_I}, \quad ||\varphi_1||_{2q,2r,A_I} \le \mu \tag{1.1.7}$$

where q amd r are arbitrary positive numbers satisfying the condition

$$\frac{1}{r} + \frac{n}{q} = 1 - \mathscr{N}$$

with

$$\begin{cases} q \in \left[\frac{n}{2-\mathscr{N}}, \infty\right], & r \in \left[\frac{1}{1-\mathscr{N}}, \infty\right], \quad 0 < \mathscr{N} < 1, \quad \text{for} \quad n \ge 2, \\ q \in [1,\infty], & 2 \in \left[\frac{1}{1-\mathscr{N}}, \frac{2}{1-2\mathscr{N}}\right], \quad 0 < \mathscr{N} < \frac{1}{2}, \quad \text{for} \quad n = 1. \end{cases} \tag{1.1.8}$$

Theorem 1.2. *[LSU68, Theorem V.1.1] Suppose the functions $a_i(x,t,u,p)$ and $a(x,t,u,p)$ possess properties (1.1.4)-(1.1.8), while $u(x,t)$ is a bounded generalized* *solution with ess* $\sup_{A_I}|u| = M$*. Then* $u(x,t) \in C^\alpha(A_I)$ *with some positive* α *depending only on* $n, \nu, \mu, q, r,$ *and* $M\mu_1/\nu$*. The quantity* $||u||_{\alpha,I'}$ *for any* $A_{I'} \subset A_I$ *separated from the parabolic boundary* Γ_T *by a positive distance* d*, is estimated from above by a constant depending only on* $n, M, \nu, \mu, \mu_1, \mu_2, q, r$ *and the distance* d*.*

1.2 Ricci flow

Definition 1.3. *Let* (Σ, g_0) *be a compact Riemannian surface. The normalized Ricci flow is the evolution equation*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = (r-R)g_{ij} \\ g(0,\cdot) = g_0(\cdot) \end{cases}$$

where r *is the mean scalar curvature* $r = \dfrac{\int R d\mu}{\int 1 d\mu}$*.*

Remark 1.4. *Under the normalized Ricci flow, the volume is preserved and the mean scalar curvature* r *is constant.*

If $\mu = \sqrt{\det g_{ij}}$ is the area element, then $\dfrac{\partial}{\partial t}\mu = (r-R)\mu$ and

$$\frac{d}{dt}\int d\mu = \int (r-R)d\mu = 0.$$

The integral of R over the surface M gives the Euler characteristic $\chi(M)$ of M by the Gauss-Bonnet formula

$$\int R d\mu = 4\pi\chi(\Sigma).$$

As a consequence $r = \dfrac{4\pi\chi(\Sigma)}{A(\Sigma)}$ is constant.

Theorem 1.5. *[Ham88] The scalar curvature* R *satisfies the evolution equation*

$$\frac{\partial R}{\partial t} = \Delta R + R^2 - rR.$$

If $R \geq 0$ *at the start, it remains so for all time.*

Definition 1.6. *The potential f is the solution of the equation*

$$\Delta f = R - r \tag{1.2.9}$$

with mean value zero.

Remark 1.7. *Since $R - r$ has mean value zero, the equation (1.2.9) is solvable and the solution is unique up to a constant. So we can make f has mean value zero. The equation for the potential f is given by*

$$\frac{\partial f}{\partial t} = \Delta f + rf - b,$$

where $b = \dfrac{\int |Df|^2 d\mu}{\int 1 d\mu}$.

Definition 1.8. *The modified Ricci flow is the following evolution equation*

$$\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij} + 2D_i D_j f = 2M_{ij},$$

where f is the Ricci potential and M_{ij} is the trace-free part of $\mathrm{Hess}(f)$.

From a straightforward calculation, $|M_{ij}|^2$ as a two tensor satisfies

$$\frac{\partial}{\partial t} |M_{ij}|^2 = \Delta |M_{ij}|^2 - 2|D_k M_{ij}|^2 - 2R|M_{ij}|^2.$$

Applying the maximum principle to the equation, we have

Corollary 1.9. *[Ham88] If $R \geq c > 0$ then*

$$|M_{ij}| \leq Ce^{-ct}$$

for some constant C. Hence $|M_{ij}| \to 0$ exponentially.

Theorem 1.10. *[Ham88] If M is diffeomorphic to S^2, then for any metric g with positive Gauss curvature the solution of the Ricci flow exists for all time, and g converges to a metric of constant curvature.*

Theorem 1.11. *[Cho91] If g is any metric on S^2, then under Hamilton's Ricci flow, the Gauss curvature becomes positive in finite time.*

Combining the theorems above yields:

Corollary 1.12. *If g is any metric on S^2, then under Hamilton's Ricci flow $|M_{ij}| \to 0$ and g converges to a metric of constant curvature.*

2 The Main Result and Previous Research

2.1 The main result

Given any metric g_1 on a topological sphere Σ_1 with area 4π, we obtain a family of 2-metrics $g(t,x)$ by solving the modified Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = (r-R)g_{ij} + 2D_iD_jf = 2M_{ij},$$

with initial condition $g(1) = g_1$.

Let $N = [1,\infty)\times\Sigma_1$, R_N be a given function on N, and $g_N = u^2dt^2 + t^2 g_{ij}(t,x)dx^idx^j$ a metric with scalar curvature R_N. Then u satisfies the parabolic equation

$$t\frac{\partial u}{\partial t} = \frac{1}{2}u^2\Delta u + \frac{t^2}{4}|M|^2u + \frac{1}{2}u - \frac{R}{4}u^3 + \frac{t^2R_N}{4}u^3. \tag{2.1.10}$$

For any $\varphi \in C^0(N)$, we define function $\varphi_*(t)$ and $\varphi^*(t)$ by

$$\varphi_*(t) = \inf\{\varphi(t,x) : x \in \Sigma_1\}, \quad \varphi^*(t) = \sup\{\varphi(t,x) : x \in \Sigma_1\}.$$

Theorem 2.1. *Assume that $R_N \in C^\alpha(N)$ and the constant K is defined by*

$$K = \sup_{1\le t<\infty}\left\{-\int_1^t \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)_* \exp\left(\int_1^{t'} \frac{s|M|^{*2}}{2}ds\right)dt'\right\} < \infty. \tag{2.1.11}$$

Suppose that there is a constant $C > 0$ such that for all $t \geq 1$ and $I_t = [t, 4t]$,

$$||R_Nt^2||_{\alpha,I_t} \le \frac{C}{t}, \quad \text{and} \quad \int_1^\infty |R_N|^* t^2 dt < \infty.$$

Then for any function $\varphi \in C^{2,\alpha}(\Sigma_1)$ *such that* $0 < \varphi < \dfrac{1}{\sqrt{K}}$*, there is a unique positive solution* $u \in C^{2+\alpha}(N)$ *of (2.1.10) with initial condition* $u(1,\cdot) = \varphi(\cdot)$*. such that the metric* $g_N = u^2dt^2 + t^2g(t)$ *on* N *satisfies the asymptotically flat condition*

$$|g^N_{ab} - \delta_{ab}| + t|\partial_a g^N_{bc}| < \frac{C}{t}, \quad a,b,c = 1,2,3 \tag{2.1.12}$$

with finite ADM mass and $m_{ADM} = \lim_{t\to\infty} \dfrac{1}{4\pi} \oint_{\Sigma_t} \dfrac{t}{2}(1-u^{-2})d\sigma$*. Moreover, the Riemannian curvature* Rm_N *of the 3-metric* g_N *on* N *is Hölder continuous and decays as* $|Rm_N| < \dfrac{C}{t^3}$*.*

Theorem 2.2 (Black hole initial data)**.** *Let* $R_N \in C^\alpha(N)$*. Further suppose that* $R_N t^2 < R$ *for* $1 \le t < \infty$*. Then there exists* $u^{-1} \in C^{2+\alpha}(N)$ *such that the metric* g_N *constructed in the previous theorem has curvature uniformly bounded with outermost totally geodesic boundary* Σ_1*.*

Let $0 < \eta < 1$ *be such that*

$$0 \le [\frac{R}{2} - \frac{t^2}{2}R_N]_{t=1} < (1-\eta)^{-1}. \tag{2.1.13}$$

Then there is t_0' *such that for* $1 < t < t_0'$*,*

$$0 < u^{-2}(t) \le \frac{t-1}{t}(1-\eta)^{-1}.$$

2.2 Previous research

It is shown in [Bar93] that a quasi-spherical metric g can always be expressed as

$$g = u^2dr^2 + (\beta_1 dr + r\sigma_1)^2 + (\beta_2 dr + r\sigma_2)^2$$

where $d\sigma^2 = \sigma_1^2 + \sigma_2^2$ is the standard metric on the unit 2-sphere. The scalar curvature is given by

$$\begin{aligned} \frac{1}{2}r^2u^2R(g) \quad &= \quad (2 - div\beta)u^{-1}(r\frac{\partial u}{\partial r} - \beta_i u_{;i}) - u\Delta u + u^2 - 1 + r\frac{\partial}{\partial r}div\beta \\ &\quad -\beta_{;i}div\beta_{;i} + 2div\beta - \frac{1}{2}(div\beta)^2 - \frac{1}{4}|\beta_{i;j} + \beta_{j;i}|^2 \end{aligned} \tag{2.2.14}$$

where $\cdot_{;i}$ is the covariant derivative of $d\sigma^2$. Consider u is an unknown function. Prescribing scalar curvature could be viewed as solving a semilinear parabolic equation for u. If $R(g)$ is determined by the Hamiltonian constraint equation (1.0.1) and β_i is regarded as prescribed, then solving (2.2.14) leads to a solution to the Hamiltonian constraint. Bartink has studied global existence theorems for (2.2.14). With suitable decay conditions on the prescribed fields β_i, these results give asymptotically flat solutions of the Hamiltonian constraint with either black hole $H = 0$ or regular center $r = 0$. The main theorems generalize Bartnik's work on constructing metrics of prescribed scalar curvature. Instead of assuming quasi-spherical foliation, we study the metric with prescribed scalar curvature constructed by a foliation from solutions of Hamilton's Ricci flow.

2.3 Outline of the proof

Given a Riemannian surface (Σ_1, g_1), Hamilton's Ricci flow gives a smooth family of metrics $g(t)$. We consider the warp product metric $g_N = u^2dt^2 + t^2g(t)$. By viewing $R(g_N) = R_N$ as prescribed scalar curvature and u as an unknown function. Let $w = u^{-2}$ and $m = \frac{t}{2}(1 - u^{-2})$. In Section 3.1 we derive the semi-linear parabolic equation for u and the equivalent equations for w and m which will be useful in the proof of a priori estimates and decay estimates. Section 3.2 is devoted to proving the existence and uniqueness of the solution. Since the equation is semilinear, following from Schauder theory and implicit function theorem, we have the short-time existence of solutions. Long-time existence of solutions follows from a priori estimates obtained by the maximum principle and the Schauder estimates for u and m. The asymptotic behavior of the solution is not controlled. In Section 3.3 and 3.4 we describe conditions on R_N which ensures existence of solutions satisfying the boundary behavior.

3 Proof of the Main Result

We consider the metric

$$g_N = u^2 dt^2 + t^2 g_{ij}(t,x) dx^i dx^j$$

and look for u so that the metric g_N has the prescribed scalar curvature R_N, where $g_{ij}(t)$ is the solution to the modified Ricci flow and $t^2 g_{ij}$ is the induced metric on Σ_t, which is the level surface t =constant. By straightforward computation, we obtain the parabolic equation (2.1.10) for u, where R is the scalar curvature of $g(t)$ on each leaf Σ_t, $|M|$ is the length of M_{ij} with respect to the metric $g(t)$, and R_N the prescribed scalar curvature.

3.1 The equation with prescribed scalar curvature

In this section we derive the equation of foliation with prescribed scalar curvature.

Lemma 3.1. *The metric $g_N = u^2 dt^2 + t^2 g(t,x)$ has the scalar curvature R_N if and only if u satisfies*

$$t\frac{\partial u}{\partial t} = \frac{1}{2}u^2 \Delta u + \frac{t^2}{4}|M|^2 u + \frac{1}{2}u - \frac{R}{4}u^3 + \frac{t^2 R_N}{4}u^3,$$

and the second fundamental forms h_{ij}, $i,j = 1,2$ of Σ_t with respect to the normal $\nu = \frac{1}{u}\frac{\partial}{\partial t}$ are given by

$$h_{ij} = \frac{1}{u}\left(\frac{1}{t}g_{N\,ij} + t^2 M_{ij}\right).$$

In particular, the mean curvature, H, and the norm squared of the second fundamental form $|A|$, are given by

$$H = \frac{2}{tu}, \quad \textit{and} \quad |A|^2 = \frac{2}{t^2u^2} + \frac{|M_{ij}|^2}{u^2} \tag{3.1.15}$$

respectively.

Proof. The second fundamental forms measure the change of the metric along the normal direction. For all $i, j = 1, 2$,

$$\begin{aligned} h_{ij} &= g_N \left(\nabla_\nu \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \frac{1}{2} \frac{\partial}{\partial \nu} g_{N\,ij} \\ &= \frac{1}{2u} \partial_3 g_{N\,ij} = \frac{1}{u} \left(\frac{1}{t} g_{N\,ij} + t^2 M_{ij} \right). \end{aligned}$$

Direct computations show that

$$H = \frac{2}{tu}, \quad \text{and} \quad |A|^2 = \frac{2}{t^2 u^2} + \frac{|M_{ij}|^2}{u^2}.$$

Also for $j, l, p = 1, 2$,

$$\begin{aligned} \Gamma_N{}^3_{jl} &= \frac{1}{2} g_N{}^{33} \left(\partial_j g_{N\,3l} + \partial_l g_{N\,3j} - \partial_3 g_{N\,jl} \right) = -\frac{1}{u} h_{jl}, \\ \Gamma_N{}^3_{3l} &= \frac{1}{2} g_N{}^{33} \left(\partial_3 g_{N\,3l} + \partial_l g_{N\,33} - \partial_3 g_{N\,l3} \right) = \frac{1}{u} \frac{\partial u}{\partial x^l}, \\ \Gamma_N{}^3_{33} &= \frac{1}{2} g^{33} (\partial_3 g_{N\,33}) = \frac{1}{u} \partial_3 u, \\ \Gamma_N{}^p_{3l} &= \frac{1}{2} g_N{}^{pq} \left(\partial_3 g_{N\,ql} + \partial_l g_{N\,3q} - \partial_q g_{N\,3l} \right) = u g_N{}^{pq} h_{lq}, \\ \Gamma_N{}^p_{jl} &= \frac{1}{2} g_N{}^{pq} \left(\partial_j g_{N\,ql} + \partial_l g_{N\,qj} - \partial_q g_{N\,jl} \right) = \Gamma^p_{jl}. \end{aligned}$$

From above, the curvature tensor $R_N{}^3_{3jl}$ is

$$\begin{aligned} R_N{}^3_{3jl} &= \partial_3 \Gamma_N{}^3_{jl} - \partial_j \Gamma_N{}^3_{3l} + \Gamma_N{}^3_{3p} \Gamma_N{}^p_{jl} + \Gamma_N{}^3_{33} \Gamma_N{}^3_{jl} - \Gamma_N{}^3_{jp} \Gamma_N{}^p_{3l} - \Gamma_N{}^3_{j3} \Gamma_N{}^3_{l3} \\ &= -\frac{1}{u} \frac{\partial h_{jl}}{\partial t} - \frac{1}{u} \left(\partial^2_{jl} u - \partial_p u \Gamma_N{}^p_{jl} \right) + g_N^{pq} h_{ql} h_{jp}. \end{aligned}$$

Thus the Ricci curvature $R_N{}^3_3$ is

$$\begin{aligned} g_N^{jl} R_N{}^3_{3jl} &= -\frac{1}{u} g^{jl} \partial_t h_{jl} - \frac{1}{t^2 u} \Delta u + |A|^2 \\ &= -\frac{1}{u} \frac{\partial}{\partial t} H - \frac{1}{t^2 u} \Delta u - |A|^2. \end{aligned} \tag{3.1.16}$$

The Gauss equation to Σ_t shows that for any $i, j, k, l = 1, 2$

$$g_N^{ik} g_N^{jl} R_{N\,ijkl} = \frac{R}{t^2} - H^2 + |A|^2, \tag{3.1.17}$$

where R is the intrinsic scalar curvature of Σ_t with respect to the metric $g(t,x)$. Combining the above two equations, we have the scalar curvature R_N of g_N,

$$\begin{aligned} R_N &= 2g_N^{jl} R_N{}^0_{0jl} + g_N^{ik} g_N^{jl} R_{Nijkl} \\ &= -2\frac{1}{u}\frac{\partial}{\partial t}H - 2\frac{1}{t^2 u}\Delta u - 2|A|^2 + \frac{R}{t^2} - H^2 + |A|^2 \\ &= -2\frac{1}{u}\frac{\partial}{\partial t}\left(\frac{2}{tu}\right) - 2\frac{1}{t^2 u}\Delta u + \frac{R}{t^2} - \left(\frac{2}{tu}\right)^2 - \frac{2}{t^2 u^2} - \frac{|M|^2}{u^2} \\ &= \frac{4}{tu^3}\frac{\partial u}{\partial t} - \frac{2}{t^2 u}\Delta u + \frac{R}{t^2} - \frac{2}{t^2 u^2} - \frac{|M|^2}{u^2}. \end{aligned} \tag{3.1.18}$$

We rewrite the above equation as

$$t\frac{\partial u}{\partial t} = \frac{1}{2}u^2 \Delta u + \frac{t^2}{4}|M|^2 u + \frac{1}{2}u - \frac{R}{4}u^3 + \frac{t^2 R_N}{4}u^3.$$

□

In order to eliminate the nonlinear term u^3, we will look at $w = u^{-2}$ to obtain the C^0 a priori bounds. We also introduce $m = \frac{t}{2}(1 - u^{-2})$, which will be important for obtaining decay estimates of u.

Lemma 3.2. *The equations for w and m are*

$$t\partial_t w = \frac{1}{2w}\Delta w - \frac{3}{4w^2}|Dw|^2 - \left(\frac{t^2}{2}|M|^2 + 1\right) w + \frac{R}{2} - \frac{t^2}{2}R_N, \tag{3.1.19}$$

$$\begin{aligned} t\partial_t m &= \frac{1}{2}u^2 \Delta m + \frac{3u^2}{2t}|\nabla m|^2 - \frac{t^2}{2}|M|^2 m + \frac{t^3}{4}|M|^2 + \frac{t}{2} - \frac{tR}{4} \\ &\quad + \frac{t^3}{4}R_N. \end{aligned} \tag{3.1.20}$$

Proof. Straightforward computations give that

$$\frac{\partial w}{\partial t} = -\frac{2}{u^3}\frac{\partial u}{\partial t}, \quad \nabla w = -\frac{2}{u^3}\nabla u, \quad \text{and} \quad \Delta w = -\frac{2}{u^3}\Delta u + \frac{6}{u^4}|\nabla u|^2.$$

These identities together with equation (2.1.10), we have

$$t\frac{\partial w}{\partial t} = \frac{1}{2w}\Delta w - \frac{3}{4w^2}|Dw|^2 - \left(\frac{t^2}{2}|M|^2 + 1\right) w + \frac{R}{2} - \frac{t^2}{2}R_N.$$

Similarly, for $m = \frac{t}{2}(1-u^{-2})$

$$\frac{\partial m}{\partial t} = t\frac{1}{u^3}\frac{\partial u}{\partial t} + \frac{1}{2}(1-u^{-2}),$$
$$\nabla m = \frac{t}{u^3}\nabla u, \quad \text{and} \quad \Delta m = \frac{t}{u^3}\Delta u - 3\frac{t}{u^4}|\nabla u|^2.$$

Therefore,

$$\begin{aligned} t\frac{\partial m}{\partial t} &= \frac{t}{2u}\Delta u - \frac{t^2}{2}|M|^2 m + \frac{t^3}{4}|M|^2 + \frac{t}{2} - \frac{tR}{4} + \frac{t^3}{4}R_N \\ &= \frac{1}{2}u^2\Delta m + \frac{3}{2}u\nabla u\cdot\nabla m - \frac{t^2}{2}|M|^2 m + \frac{t^3}{4}|M|^2 + \left(\frac{t}{2} - \frac{tR}{4}\right) + \frac{t^3}{4}R_N. \end{aligned}$$

□

3.2 Existence for prescribed scalar curvature

By specifying the initial condition $u(1,\cdot) = \varphi(\cdot)$ and solving the parabolic equation (2.1.10) for u, there exists a unique solution u such that the metric $g_N = u^2dt^2 + t^2g(t)$ has the desired scalar curvature R_N. In this section, we describe conditions on R_N under which the existence and uniqueness of the solution u can be established.

Suppose u is a solution of (2.1.10). Applying the maximum principle and Schauder theory, we have the following a priori estimates.

Proposition 3.3. *Suppose $u \in C^{2+\alpha}(A_{[t_0,t_1]})$, $1 < t_0 < t_1$ is a positive solution. Then for $t_0 \le t \le t_1$ we have*

$$\begin{aligned} u^{-2}(t,x) &\ge \frac{t_0}{t}(u^*(t_0))^{-2}\exp\left(-\int_{t_0}^{t}\frac{s|M|^{*2}}{2}ds\right) \\ &\quad + \frac{1}{t}\int_{t_0}^{t}\left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)_*\exp\left(-\int_{t'}^{t}\frac{s(|M|^*)^2}{2}ds\right)dt' \\ u^{-2}(t,x) &\le \frac{t_0}{t}(u_*(t_0))^{-2}\exp\left(-\int_{t_0}^{t}\frac{s|M|_*{}^2}{2}ds\right) \\ &\quad + \frac{1}{t}\int_{t_0}^{t}\left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)^*\exp\left(-\int_{t'}^{t}\frac{s|M|_*^2}{2}ds\right)dt'. \end{aligned}$$

If we further assume that R_N is defined on $A_{[1,\infty)}$ such that the functions

$$\delta_*(t) = \frac{1}{t}\int_1^t \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)_* (t') \exp\left(-\int_{t'}^t \frac{s|M|^{*2}}{2} ds\right) dt' \qquad (3.2.21)$$

$$\delta^*(t) = \frac{1}{t}\int_1^t \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)^* (t') \exp\left(-\int_{t'}^t \frac{s|M|_*^2}{2} ds\right) dt' \qquad (3.2.22)$$

are defined and finite for all $t \in [t_0, \infty)$, then the estimates may be rewritten as

$$u^{-2}(t,x) \geq \delta_*(t) + \frac{t_0}{t}\left(u^*(t_0)^{-2} - \delta_*(t_0)\right) \exp\left(-\int_{t_0}^t \frac{s|M|^{*2}}{2} ds\right)$$

$$u^{-2}(t,x) \leq \delta^*(t) + \frac{t_0}{t}\left(u_*(t_0)^{-2} - \delta^*(t_0)\right) \exp\left(-\int_{t_0}^t \frac{s|M|_*{}^2}{2} ds\right).$$

Proof. From Lemma 3.2 the parabolic equation for $w = u^{-2}$ is

$$t\partial_t w = \frac{1}{2w}\Delta w - \frac{3}{4w^2}|Dw|^2 - \frac{t^2}{2}|M|^2 w - w + \frac{R}{2} - \frac{t^2}{2}R_N.$$

Consider the associated ordinary differential equation,

$$t\frac{df}{dt} = -\left(\frac{t^2}{2}|M|^2 + 1\right)^* f + \left(\frac{R}{2} - \frac{t^2}{2}R_N\right)_*$$

with initial condition $f(t_0) = t_0 w_*(t_0)$.

Using the integrating factor method, we have

$$\frac{d}{dt}\left(t \exp\left(\int_{t_0}^t \frac{s|M|^{*2}}{2} ds\right) f\right) = \left(\frac{R}{2} - \frac{t^2}{2}R_N\right)_* \exp\left(\int_{t_0}^t \frac{s|M|^{*2}}{2} ds\right)$$

and

$$\begin{aligned} f(t) &= \frac{1}{t}\exp\left(-\int_{t_0}^t \frac{s|M|^{*2}}{2} ds\right) \\ &\quad \times \left(\int_{t_0}^t \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)_* \exp(\int_{t_0}^{t'} \frac{s|M|^{*2}}{2} ds) dt' + f(t_0)\right) \\ &= \frac{1}{t}\int_{t_0}^t \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)_* \exp(-\int_{t'}^t \frac{s|M|^{*2}}{2} ds) dt' \\ &\quad + \frac{1}{t} f(t_0) \exp\left(-\int_{t_0}^t \frac{s|M|^{*2}}{2} ds\right) \\ &= \delta_*(t) + \frac{t_0}{t}\left(w_*(t_0) - \delta_*(t_0)\right) \exp\left(-\int_{t_0}^t \frac{s|M|^{*2}}{2} ds\right). \end{aligned}$$

Applying the parabolic maximum principle to the equation,

$$\begin{aligned} t\frac{dw_*}{dt} &\geq -\left(\frac{t^2}{2}|M|^2+1\right)w_* + \left(\frac{R}{2}-\frac{t^2}{2}R_N\right) \\ &\geq -\left(\frac{t^2}{2}|M|^2+1\right)^* w_* + \left(\frac{R}{2}-\frac{t^2}{2}R_N\right)_*, \end{aligned}$$

we have $w_* \geq f$. Hence

$$u^{-2}(t,x) \geq \delta_*(t) + \frac{t_0}{t}\left(w_*(t_0)-\delta_*(t_0)\right)\exp\left(-\int_{t_0}^{t}\frac{s|M|^{*2}}{2}ds\right).$$

Similarly, applying the maximum principle to w^* we get the upper bound of u^{-2}.

□

We also need the following interior Schauder estimates.

Proposition 3.4. *Let $I=[1,t_1]$ and $I'=[t_0,t_1]$ with $1<t_0<t_1$, and suppose $u\in C^{2+\alpha}(A_I)$ is a solution in A_I with a source function $R_N\in C^{\alpha}(A_I)$, and*

$$0<\delta_0\leq u^{-2}(t,x)\leq \delta_0^{-1}, \quad \textit{for all} \quad (t,x)\in A_I,$$

for some constant δ_0. Then with $m=\frac{1}{2}(1-u^{-2})$ as above, there is a constant C, depending on t_0, t_1, δ_0, $||m||_{0,I}$, $||M||_{\alpha,I}$, $||R||_{\alpha,I}$ and $||R_N||_{\alpha,I}$ such that

$$||m||_{2+\alpha,I'}\leq C. \tag{3.2.23}$$

If $R_N\in C^{k+\alpha}(A_I)$, $k\in\mathbb{N}$ then there is a constant C depending on t_0, t_1, δ_0, $||M||_{k+\alpha,I}$, $||R||_{k+\alpha,I}$, and $||R_N||_{k+\alpha,I}$ such that

$$||m||_{k+2+\alpha,I'}\leq C. \tag{3.2.24}$$

Proof. In local coordinates (x_1,x_2) on Σ_t, the Laplace-Beltrami operator is given by $\Delta = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial}{\partial x^j}\right)$. Hence

$$\begin{aligned} u^2\Delta u &= \frac{u^2}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial u}{\partial x^j}\right) \\ &= \frac{\partial}{\partial x^i}\left(u^2 g^{ij}\frac{\partial u}{\partial x^j}\right) - \frac{\partial}{\partial x^i}\left(\frac{u^2}{\sqrt{\det g}}\right)\left(\sqrt{\det g}\,g^{ij}\frac{\partial u}{\partial x^j}\right). \end{aligned}$$

Let $t = e^s$, and so $\dfrac{\partial}{\partial s} = t\dfrac{\partial}{\partial t}$.

$$\begin{aligned}
\frac{\partial u}{\partial s} &= t\frac{\partial u}{\partial t} = \frac{1}{2}u^2\Delta u + \frac{t^2}{4}|M|^2 u + \frac{1}{2}u - \frac{R}{4}u^3 + \frac{t^2 R_N}{4}u^3 \\
&= \frac{\partial}{\partial x^i}\left(\frac{1}{2}u^2 g^{ij}\frac{\partial u}{\partial x^j}\right) - \frac{\partial}{\partial x^i}\left(\frac{u^2}{2\sqrt{\det g}}\right)\left(\sqrt{\det g}g^{ij}\frac{\partial u}{\partial x^j}\right) \\
&\quad + \frac{t^2}{4}|M|^2 u + \frac{1}{2}u - \frac{R}{4}u^3 + \frac{t^2 R_N}{4}u^3 \\
&= \frac{\partial}{\partial x^i}a^i(x,t,u,\partial u) - a(x,t,u,\partial u).
\end{aligned}$$

where

$$a^i(x,t,u,\vec{p}) = \frac{1}{2}u^2 g^{ij}p_j$$

and

$$\begin{aligned}
a(x,t,u,\vec{p}) &= \left(\sqrt{\det g}g^{ij}p_j u^2\right)\frac{\partial}{\partial x^i}\left(\frac{1}{2\sqrt{\det g}}\right) + u g^{ij}p_i p_j \\
&\quad - \frac{t}{4}|M|^2 u - \frac{1}{2t}u + \frac{R}{4t}u^3 - \frac{tR_N}{4}u^3.
\end{aligned}$$

By the assumption, $a^i p_i \geq C|p|^2$, $|a_i| \leq C'|p|$, and $|a| \leq C''(1+|p|^2)$ for some positive constants C, C', and C''. Applying Theorem V.1.1 in [LSU68], we have

$$||u||_{\alpha',I''} \leq C_1$$

for some $0 < \alpha' < 1$, $\alpha' = \alpha'(\delta_0)$ and $C_1 = C_1(t_0, t_1, \delta_0, ||R||_{0,I}, ||M||_{0,I}, ||R_N||_{0,I})$, where $I' \subset I'' \subset I$.

Without loss of generality we may assume $\alpha' \leq \alpha$. The usual Schauder interior estimates, Theorem IV.10.1, in [LSU68] give

$$||u||_{2+\alpha',I'} \leq C_2(C_1, ||R||_{\alpha,I}, ||M||_{\alpha,I}, ||R_N||_{\alpha,I})$$

Apply the Schauder estimates again to (3.1.20)

$$t\partial_t m = \frac{1}{2}u^2\Delta m + \frac{3u^2}{2t}|\nabla m|^2 - \frac{t^2}{2}|M|^2 m + \frac{t^3}{4}|M|^2 + \frac{t}{2} - \frac{tR}{4} + \frac{t^3}{4}R_N.$$

The desired estimate (3.2.23) follows, and (3.2.24) follows by the usual bootstrap argument.

$\square$

The short time existence of solutions to (2.1.10) follows from the linear Schauder theory and a standard implicit function theorem argument.

Proposition 3.5. *Let* $I = [t_0, t_1]$, $1 \le t_0 < t_1 < \infty$, *and* $R_N \in C^\alpha(A_I)$. *Then for any initial condition*

$$u(t_0, x) = \varphi(x), \quad x \in \Sigma_1,$$

where $\varphi \in C^{2,\alpha}(\Sigma)$ *satisfies*

$$0 < \delta_0 \le \varphi^{-2}(x) \le \delta_0^{-1}, \quad x \in \Sigma_1,$$

for some constant δ_0, *the initial value problem has a solution* $u \in C^{2+\alpha}(A_{[t_0,t_0+T]})$ *for some* $T > 0$, *where* T *depends on* t_0, δ_0, $||M||_{\alpha,I}$, $||R||_{\alpha,I}$, $||R_N||_{\alpha,I}$, *and* $||\varphi||_{2,\alpha}$.

Theorem 3.6. *Assume that* $R_N \in C^\alpha(N)$ *and the constant* K *defined by (2.1.11). Then for every* $\varphi \in C^{2,\alpha}(\Sigma_1)$ *such that*

$$0 < \varphi(x) < \frac{1}{\sqrt{K}}, \quad \text{for all } x \in \Sigma_1, \tag{3.2.25}$$

there is a unique positive solution $u \in C^{2+\alpha}(A_{[1,\infty)})$ *of (2.1.10) with initial condition*

$$u(1, \cdot) = \varphi(\cdot). \tag{3.2.26}$$

Proof. The upper bound of (3.2.25) implies

$$\delta_*(t) + \frac{1}{t}(u^*(1))^{-2} \exp\left(-\int_{t_0}^{t} \frac{s|M|^{*2}}{2} ds\right) > 0 \text{ for all } t \ge 1. \tag{3.2.27}$$

Since the equation is parabolic, by Proposition 3.5 and 3.4, there is $\epsilon > 0$ and $u \in C^{2+\alpha}(A_{[1,1+\epsilon]})$ satisfying the initial condition $u(1) = \varphi$ on $[1, 1+\epsilon]$ for some $\epsilon > 0$. By Proposition 3.3 and (3.2.27), there are functions $0 < \delta_1(t) \le \delta_2(t) < \infty$, $1 \le t$, independent of ϵ, such that

$$0 < \delta_1(t) \le u^{-2}(t, x) \le \delta_2(t) \quad \text{for all} \quad t \in [1, 1+\epsilon].$$

Let $U = \{t \in \mathbb{R}^+ : \exists u \in C^{2+\alpha}(A_{[1,1+t]}) \text{ satisfying (2.1.10) and (3.2.26)}\}$. The local existence Proposition 3.5 guarantees U is open in $\mathbb{R}^+$. Since $[1, 1+t]$ is compact, there is δ_0 such that $0 < \delta_0 \leq u^{-2}(t,x) \leq \delta_0^{-1}$ for all $(t,x) \in A_{[1,1+t]}$. From the interior estimate (3.2.23) of Proposition 3.4, we have an a priori estimate for $||u(1+t,\cdot)||_{2,\alpha}$. By Proposition 3.5 the solution can be extended to $A_{[1,1+t+T]}$ for some T independent of u, which shows that U is closed. Hence u extends to a global solution $u \in C^{2+\alpha}(A_{[1,\infty]})$.

□

An immediate corollary of Theorem 3.6 is the existence of extension metrics having boundary Σ_1 with prescribed positive mean curvature.

Corollary 3.7. *Let R_N and K be as given in Theorem 3.6. Suppose $H \in C^{2,\alpha}(\Sigma_1)$ satisfies*

$$H(x) > 2\sqrt{K} \quad \textit{for all } x \in \Sigma_1.$$

Then there is a metric g_N with scalar curvature R_N having boundary Σ_1 with mean curvature H.

Proof. Let $\varphi(x) = \dfrac{2}{H(x)}$. Then the assumption

$$H(x) = \frac{2}{\varphi(x)} > 2\sqrt{K} \quad \text{for all } x \in \Sigma_1. \tag{3.2.28}$$

is equivalent to $\varphi^*(x) < \dfrac{1}{\sqrt{K}}$. Theorem 3.6 shows that there exists a unique solution $u \in C^{2+\alpha}(N)$ to the initial value problem, and the resulting metric has boundary Σ_1 with mean curvature H by (3.1.15).

□

Remark 3.8. *Blowup is possible for the initial value problem, and the condition (3.2.25) is nearly optimal. If the initial condition $\varphi(x) > \dfrac{1}{\sqrt{K}}$ for all $x \in \Sigma_1$, then the solution u will blow up within finite time.*

Proof. The lower bound from Proposition 3.3 shows that if $u(t,x)$ is a positive solution on A_I then,

$$u^{-2}(t,x) \le \frac{1}{t}\int_1^t \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)^* \exp(-\int_{t'}^t \frac{s|M|_*^{\ 2}}{2}ds)dt' + \frac{1}{t}\varphi(1)_*^{-2}\exp\left(-\int_1^t \frac{s|M|_*^{\ 2}}{2}ds\right)$$

for all $(t,x) \in A_I$.

Since $\varphi(x) > \dfrac{1}{\sqrt{K}}$, there exists $t_0 > 1$ such that

$$\frac{1}{t}\int_1^{t_0} \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)^* \exp(-\int_{t'}^t \frac{s|M|_*^{\ 2}}{2}ds)dt' + \frac{1}{t}\varphi(1)_*^{-2}\exp\left(-\int_1^{t_0} \frac{s|M|_*^{\ 2}}{2}ds\right) < 0.$$

This shows that there can be no solution on $A_{[1,t_0]}$ and there is a maximal T, $1 < T < t_0$ so that the solution exists on $A_{[1,T]}$ and $\lim_{t\to T}\sup u^*(t) = \infty$.

$\square$

3.3 Asymptotic metric behavior

Under suitable decay conditions on the prescribed scalar curvature, we can further obtain the asymptotic flatness of the resulting 3-metric g_N so that ADM mass is well-defined [Bar86].

Lemma 3.9. *If R_N is given such that*

$$\int_1^\infty |R_N|^* t^2 dt < \infty,$$

then there is a constant C such that for all $t \ge 1$

$$1 - \frac{C}{t} \le \delta_*(t) \le \delta^*(t) \le 1 + \frac{C}{t}, \tag{3.3.29}$$

where $\delta_*(t)$ and $\delta^*(t)$ are defined by (3.2.21) and (3.4.38).

Proof. Using

$$|e^\eta - 1| \le 2|\eta| \text{ for } |\eta| \le 1,$$

we see that

$$
\begin{aligned}
&\left|\int_1^t \exp\left(-\int_{t'}^t \frac{s|M|^{*2}}{2}ds\right)dt' - t\right| \\
&\leq \left|\int_1^t \exp\left(-\int_{t'}^t \frac{s|M|^{*2}}{2}ds\right) - 1dt' - 1\right| \\
&\leq C + 2\int_1^t \int_{t'}^t \frac{s|M|^{*2}}{2}dsdt' \\
&\leq C.
\end{aligned}
$$

Since R converges to 2 exponentially, we have the integral

$$\int_1^\infty \left(\frac{R-2}{2}\right)^* dt < \infty.$$

This together with the curvature assumption implies that for $t \geq 1$,

$$
\begin{aligned}
\delta^*(t) &= \frac{1}{t}\int_1^t \left(\frac{R}{2} - \frac{t'^2}{2}R_N\right)^*(t')\exp\left(-\int_{t'}^t \frac{s|M|_*^2}{2}ds\right)dt' \\
&= \frac{1}{t}\int_1^t \exp\left(-\int_{t'}^t \frac{s|M|^{*2}}{2}ds\right)dt' \\
&\quad + \frac{1}{t}\int_1^t \left(\frac{R-2}{2} - \frac{t'^2}{2}R_N\right)^*(t')\exp\left(-\int_{t'}^t \frac{s|M|_*^2}{2}ds\right)dt' \\
&\leq 1 + \frac{C}{t}
\end{aligned}
$$

for some constant C.

Similarly, $\delta_*(t) \geq 1 - \frac{C}{t}$. Thus

$$1 - \frac{C}{t} \leq \delta_*(t) \leq \delta^*(t) \leq 1 + \frac{C}{t}$$

for $t \geq 1$.

□

Theorem 3.10 (Asymptotic flatness)**.** *Let $u \in C^{2+\alpha}(N)$ be a solution of (2.1.10). Suppose that there is a constant $C > 0$ such that for all $t \geq 1$ and $I_t = [t, 4t]$, $||R_N t^2||_{\alpha, I_t} \leq \frac{C}{t}$, and $\int_1^\infty |R_N|^* t^2 dt < \infty$. Then g_N satisfies the asymptotically flat* *condition (2.1.12) for $t > t_0$, where t_0 and C are some fixed constants. Moreover, the Riemannian curvature Rm_N of the 3-metric g_N on N is Hölder continuous and decays as $|Rm_N| < \dfrac{C}{t^3}$, and ADM mass can be expressed as*

$$m_{ADM} = \lim_{t\to\infty} \frac{1}{4\pi} \oint_{\Sigma_t} \frac{t}{2}(1-u^{-2})d\sigma. \tag{3.3.30}$$

Proof. We compare g_N with the flat metric $g_E = dt^2 + t^2 g_{S^2}$, and get

$$g_N - g_E = (u^2-1)dt^2 + t^2(g(t) - g_{S^2}).$$

Since $g(t)$ converges to the round metric g_{S^2} exponentially, to prove the asymptotic flatness (2.1.12) of the metric g_N, it suffices to show that

$$|u^2-1| + |t\partial_t u| + |\nabla_i u| \le \frac{C}{t}, \quad i = 1, 2.$$

Define $\tilde{u}(t,x) = u(\tau t, x)$. Observe that $u \in C^{2+\alpha}(A_{[\tau,4\tau]})$ satisfies (2.1.10) if and only if $\tilde{u}$ on the interval $[1,4]$ satisfies

$$t\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2}\tilde{u}^2\Delta\tilde{u} + \frac{t^2}{4}|\tilde{M}|^2\tilde{u} + \frac{1}{2}\tilde{u} - \frac{\tilde{R}}{4}\tilde{u}^3 + \frac{t^2\tilde{R}_N}{4}\tilde{u}^3, \tag{3.3.31}$$

where $\tilde{M}(t,x) = \tau M(\tau t, x)$, $\tilde{R}(t,x) = R(\tau t, x)$, and $\tilde{R}_N(t,x) = \tau^2 R_N(\tau t, x)$.

Applying Proposition 3.4 to $\tilde{u}$ on the interval $[1,4]$, and then rescaling back, we obtain $||u||_{2+\alpha, I'_\tau} \le C$, and

$$||m||_{2+\alpha,I'_\tau} \le C||m||_{0,I_\tau} + C\tau \left(||\tau M||_{\alpha,I_\tau} + ||R-2||_{\alpha,I_\tau} + ||\tau^2 R_N||_{\alpha,I_\tau}\right), \tag{3.3.32}$$

where $I'_\tau = [2\tau, 4\tau]$, and C is independent of u and τ. The bound (3.3.29) controls $||m||_{0,I_\tau}$. M converges to 0 and R converges to 2 exponentially fast. This, together with the decay assumption $||R_N t^2||_{\alpha,I_t} \le C/t$, controls the second term of (3.3.32). Hence there is a uniform bound

$$||m||_{2+\alpha,I'_\tau} \le C, \quad \text{for all } \tau \ge 1.$$

Expressing this in terms of u and derivatives of u gives

$$||1-u^{-2}||_{\alpha,I'_\tau} + ||\tau\partial_\tau u||_{\alpha,I'_\tau} + ||\nabla u||_{\alpha,I'_\tau} + ||\nabla^2 u||_{\alpha,I'_\tau} \le \frac{C}{\tau}.$$

The estimates for $1-u^{-2}$ and ∇u show that g_N is asymptotically Euclidean in the sense of (2.1.12). The estimate for $\nabla^2 u$, together with the expression (3.1.16) and (3.1.17), shows that $Ric \in C^{0,\alpha}(A_{[1,\infty)})$ and $|Ric(t,x)| \leq \frac{C}{t^3}$. Since N is of dimension 3, the Ricci curvature determines the full curvature tensor. The Riemannian curvature on N is Hölder continuous and decays as $|Rm_N| < \frac{C}{t^3}$.

Instead of the rectangular coordinates, we choose the coordinates $\{x^a\} = \{x^1, x^2, t\}$ which will be the usual spherical polar coordinates at $t=\infty$. Following from [Bar86] that ADM mass of N is uniquely defined by

$$m_{ADM} = \frac{1}{16\pi} \lim_{t\to\infty} \oint_{\Sigma_t} \left(g_E^{ab} \nabla_a^E g_{N\,bc} - \nabla_c^E (g_E^{ab} g_{N\,ab}) \right) d\sigma^c,$$

where

$$\begin{aligned}
g_E^{ab}\nabla_a^E g_{N\,bc} &= g_E^{33}\nabla_3^E g_{N\,33} + g_E^{12}\nabla_1^E g_{N\,23} + g_E^{21}\nabla_2^E g_{N\,13} + g_E^{11}\nabla_1 g_{N\,13} + g_E^{22}\nabla_2^E g_{N\,23} \\
&= g_E^{33}(\partial_3 g_{N\,33} - 2\Gamma_{E\,33}^{\;3} g_{N\,33}) + g_E^{12}(\partial_1 g_{N\,23} - \Gamma_{E\,12}^{\;a} g_{N\,a3} - \Gamma_{E\,13}^{\;a} g_{N\,2a}) \\
&\quad + g_E^{12}(\partial_2 g_{N\,13} - \Gamma_{E\,12}^{\;a} g_{N\,a3} - \Gamma_{E\,23}^{\;a} g_{N\,1a}) \\
&\quad + g_E^{11}(\partial_1 g_{N\,13} - \Gamma_{E\,11}^{\;a} g_{N\,a3} - \Gamma_{E\,13}^{\;a} g_{N\,1a}) \\
&\quad + g_E^{22}(\partial_2 g_{N\,23} - \Gamma_{E\,22}^{\;a} g_{N\,a3} - \Gamma_{E\,23}^{\;a} g_{N\,2a}) \\
\nabla_3^E(\mathrm{tr}_{g_E} g_N) &= \partial_3({g_E}^{33} g_{N\,33} + 2{g_E}^{12} g_{N\,12} + {g_E}^{11} g_{N\,11} + {g_E}^{22} g_{N\,22}).
\end{aligned}$$

For $i,j = 1,2$, the connection of the Euclidean metric g_E is given by

$$\begin{aligned}
\Gamma_{E\,33}^{\;3} &= \frac{1}{2} g_E^{33} \partial_3 g_{33}^E = 0, \\
\Gamma_{E\,ij}^{\;3} &= \frac{1}{2} g_E^{33} \left(\partial_i g_{j3}^E + \partial_j g_{i3}^E - \partial_3 g_{ij}^E \right) \\
&= -\frac{1}{2}\partial_3 g_{ij}^E, \\
\Gamma_{E\,j3}^{\;i} &= \frac{1}{2} \sum_{a=1}^{3} g_E^{ia} \left(\partial_j g_{3a}^E + \partial_3 g_{ja}^E - \partial_a g_{j3}^E \right) \\
&= \frac{1}{2} g_E^{i1} \partial_3 g_{j1}^E + \frac{1}{2} {g_E}^{i2} \partial_3 g_{j2}^E,
\end{aligned}$$

$$\begin{aligned}
\Gamma_{E}{}^{k}_{ij} &= \frac{1}{2}\sum_{a=1}^{3} g_E^{ka}\left(\partial_i g^E_{ja} + \partial_j g^E_{ia} - \partial_a g^E_{ij}\right)\\
&= \frac{1}{2}\sum_{l=1}^{2} g_E^{kl}\left(\partial_i g^E_{jl} + \partial_j g^E_{il} - \partial_l g^E_{ij}\right).
\end{aligned}$$

Since $g(t)$ converges to the round metric g_{S^2} exponentially, g_{N12} and $|M_{ij}|$ converge to zero exponentially, and g_N converges to g_E. Thus by the definition of ADM mass

$$\begin{aligned}
16\pi m_{ADM} &= \lim_{t\to\infty}\oint_{\Sigma_t}\left(g_E^{ab}\nabla^E_a g_{N bc} - \nabla^E_c(g_E^{ab}g_{N ab})\right)d\sigma^c\\
&= \lim_{t\to\infty}\oint_{\Sigma_t} g_E^{12}(-\Gamma_{E}{}^{3}_{12}g_{N33} - \Gamma_{E}{}^{2}_{13}g_{N22}) + g_E^{12}(-\Gamma_{E}{}^{3}_{12}g_{N33} - \Gamma_{E}{}^{1}_{23}g_{N11})\\
&\quad + g_E^{11}(-\Gamma_{E}{}^{3}_{11}g_{N33} - \Gamma_{E}{}^{1}_{13}g_{N11}) + g_E^{22}(-\Gamma_{E}{}^{3}_{22}g_{N33} - \Gamma_{E}{}^{2}_{23}g_{N22})\\
&\quad \left. - \partial_3(2{g_E}^{12}g_{N12} + {g_E}^{11}g_{N11} + {g_E}^{22}g_{N22})\right\} u^{-2}t^2 d\sigma\\
&= \lim_{t\to\infty}\oint_{\Sigma_t}\left\{ g_E^{11}\left(\frac{1}{2}\partial_3 g^E_{11}g_{N33} - \frac{1}{2}g_E^{11}\partial_3 g^E_{11}g_{N11}\right)\right.\\
&\quad + {g_E}^{22}\left(\frac{1}{2}\partial_3 g^E_{22}g_{N33} - \frac{1}{2}g_E^{22}\partial_3 g^E_{22}g_{N22}\right) + g_E^{11}\partial_3 g^E_{11}g_E^{11}g_{N11}\\
&\quad \left. + g_E^{22}\partial_3 g^E_{22}g_E^{22}g_{N22} - g_E^{11}\frac{2}{t}g_{N11} - g_E^{22}\frac{2}{t}g_{N22}\right\} u^{-2}t^2 d\sigma\\
&= \lim_{t\to\infty}\oint_{\Sigma_t} 2t(1-u^{-2})d\sigma.
\end{aligned}$$

Hence

$$m_{ADM} = \lim_{t\to\infty}\frac{1}{4\pi}\oint_{\Sigma_t}\frac{t}{2}(1-u^{-2})d\sigma.$$

Observe that from equation (3.1.20)

$$\begin{aligned}
\frac{d}{dt}\oint_{\Sigma_t} m d\sigma &= \oint \frac{1}{2t}u^2\Delta m + \frac{3u^2}{2t^2}|\nabla m|^2 - \frac{t}{2}|M|^2 m d\sigma\\
&\qquad + \oint \frac{t^2}{4}|M|^2 + \frac{1}{2} - \frac{R}{4} + \frac{t^2}{4}R_N d\sigma\\
&= -\oint \frac{1}{t^2}u^3|\nabla m|^2 + \frac{3u^2}{2t^2}|\nabla m|^2 - \frac{t}{2}|M|^2 m d\sigma\\
&\qquad + \oint \frac{t^2}{4}|M|^2 + \left(\frac{1}{2} - \frac{R}{4}\right) + \frac{t^2}{4}R_N d\sigma.
\end{aligned}$$

Since u, m, amd ∇m are bounded, and $|M|$ decays to zero exponentially, the first term is integrable on $(1, \infty)$. The decay condition $\int_1^\infty |R_N|^* t^2 dt < \infty$ ensures the second term is also integrable on $(1, \infty)$. Therefore the limit in (3.3.30) exists. $\square$

Collecting the above theorems, we summarize as follows.

Theorem 3.11. *Assume that $R_N \in C^\alpha(N)$ and the constant K defined by (2.1.11). Suppose that there is a constant $C > 0$ such that for all $t \geq 1$ and $I_t = [t, 4t]$, $||R_N t^2||_{\alpha, I_t} \leq \frac{C}{t}$, and $\int_1^\infty |R_N|^* t^2 dt < \infty$.*

Then for every $\varphi \in C^{2,\alpha}(\Sigma_1)$ such that

$$0 < \varphi(x) < \frac{1}{\sqrt{K}}, \quad \text{for all } x \in \Sigma_1,$$

there is a unique positive solution $u \in C^{2+\alpha}(N)$ of (2.1.10) with initial condition

$$u(1, \cdot) = \varphi(\cdot)$$

such that the metric $g_N = u^2 dt^2 + t^2 g(t)$ satisfies the asymptotically flat condition (2.1.12) with finite ADM mass (3.3.30). Moreover, the Riemannian curvature Rm_N of the 3-metric g_N on N is Hölder continuous and decays as $|Rm_N| < \frac{C}{t^3}$.

Let $(\Omega, g) \hookrightarrow M$ be a compact three manifold with smooth boundary Σ. The Hawking quasi local mass $m_H(\Sigma)$ is defined by (see [Haw68])

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int H^2 d\sigma\right).$$

Remark 3.12. *Although Σ_t are not round spheres, the Hawking mass $m_H(\Sigma_t)$ approaches ADM mass of the asymptotically flat 3-manifold N as $t \to \infty$.*

Proof. Since the area of Σ_1 is normalized to 4π and the Ricci flow preserves the area, we have the area $A(\Sigma_t) = 4\pi t^2$, the mean curvature $H = \frac{2}{tu}$ for each leaf Σ_t.

$$t\partial_t w = -\frac{1}{u}\Delta u - \frac{t^2}{2}|M|^2 w - w + \frac{R}{2} - \frac{t^2}{2} R_N.$$

Therefore the Hawking mass $m_H(\Sigma_t)$ is given by

$$\begin{aligned} m_H(\Sigma_t) &= \sqrt{\frac{A(\Sigma_t)}{16\pi}}\left(1-\frac{1}{16\pi}\int H_t^2 d\sigma_t\right) \\ &= \frac{t}{2}\left(1-\frac{1}{16\pi}\int\left(\frac{4}{t^2u^2}\right)t^2 d\sigma\right) \\ &= \frac{t}{2}\left(1-\frac{1}{4\pi}\int u^{-2}d\sigma\right) \\ &= \frac{1}{4\pi}\int\frac{t}{2}(1-u^{-2})d\sigma. \end{aligned}$$

□

Remark 3.13. *The Hawking mass $m_H(\Sigma_t)$ is nondecreasing in t. In particular, if we impose the initial condition $u(1)^{-1}=0$, i.e., minimal boundary surface, and assume that $R_N \geq 0$, then ADM mass is bounded below by*

$$m_{ADM}(N) \geq \frac{1}{2}\sqrt{\frac{A(\Sigma_1)}{4\pi}} = \frac{1}{2}.$$

Proof. From the calculation above, the Hawking mass of Σ_t can be expressed as

$$\begin{aligned} m_H(\Sigma_t) &= \frac{t}{2}\left(1-\frac{1}{4\pi}\int u^{-2}d\sigma\right) \\ &= \frac{t}{2}\left(1-\frac{1}{4\pi}\int w d\sigma\right). \end{aligned}$$

The equation for w in Lemma 3.2 is

$$t\partial_t w = \frac{t}{2u}\Delta u - \frac{t^2}{2}|M|^2 m + \frac{t^3}{4}|M|^2 + \frac{t}{2} - \frac{tR}{4} + \frac{t^3}{4}R_N.$$

By direct computation, we see

$$\begin{aligned} \frac{d}{dt}m_H(\Sigma_t) &= \frac{1}{2}\left(1-\frac{1}{4\pi}\int w d\sigma\right) - \frac{1}{8\pi}\int t\frac{\partial}{\partial t}w d\sigma \\ &= \frac{1}{2} - \frac{1}{8\pi}\int w d\sigma - \frac{1}{8\pi}\int t\frac{\partial}{\partial t}w d\sigma \\ &= \frac{1}{2} - \frac{1}{8\pi}\int w d\sigma - \frac{1}{8\pi}\int\left(\frac{1}{u}\Delta u - \frac{t^2}{2}|M|^2 w - w + \frac{R}{2} - \frac{t^2}{2}R_N\right)d\sigma \\ &= \frac{1}{8\pi}\int\frac{1}{u}\Delta u + \frac{t^2}{2}|M|^2 w + \frac{t^2}{2}R_N d\sigma \\ &= \frac{1}{8\pi}\int u^{-2}|\nabla u|^2 + \frac{t^2}{2}|M|^2 w + \frac{t^2}{2}R_N d\sigma \geq 0. \end{aligned}$$

□

3.4 Minimal surface interior boundary

Another advantage of having the foliation by the modified Ricci flow solutions is that the scalar curvature R on each Σ_t evolves by

$$\frac{\partial R}{\partial t} = \Delta_{g(t)} R + R^2 - 2R.$$

Using the maximum principle we can show that if $R \geq 0$ is preserved under the flow. Moreover R becomes strictly positive immediately. This allows us to construct asymptotically flat time symmetric initial data on $N = [1, \infty) \times \Sigma_1$ with totally geodesic boundary Σ_1 with non-negative Gaussian curvature on (Σ_1, g_1).

Theorem 3.14 (Minimal surface interior boundary)**.** *Let $R_N \in C^\alpha(N)$. Further suppose that $R_N t^2 < R$ for $1 \leq t < \infty$. Then there is $u^{-1} \in C^{2+\alpha}(N)$ such that the constructed metric on N has curvature uniformly bounded on $A_{[1,2]}$ with totally geodesic boundary Σ_1.*

Let $0 < \eta < 1$ be such that

$$1 - \eta \leq R - R_N|_{t=1} < (1 - \eta)^{-1}. \tag{3.4.33}$$

Then there is t_0' such that for $1 < t < t_0'$,

$$\frac{t-1}{t}(1 - \eta) \leq u^{-2}(t) \leq \frac{t-1}{t}(1 - \eta)^{-1}.$$

To prove Theorem 3.14, we need the following three Lemmas. First, we introduce the rescaling transformation

$$\tilde{u}(t) = \sqrt{\frac{t}{t+1}} u(t+1) \quad \text{where } t \in (0, \infty),$$

and obtain a parabolic equation for $\tilde{u}$ (Lemma 3.15). Secondly, the equation of $\tilde{u}$ has the same form as the u-equation. We show that for the solution $\tilde{u}$ exists for all time $[t_0, \infty)$, Lemma 3.16. In Lemma 3.17, by applying Arzela-Ascoli theorem and taking a convergent subsequence of solutions $\tilde{u}_\epsilon$ on interval (ϵ, ∞), we show that there is a global solution u on $(0, \infty)$. Note that the global solution here is not necessarily unique.

Lemma 3.15. *Define* $\tilde{u}(t) = \sqrt{\frac{t}{t+1}}u(t+1)$ *for* $t > 0$*.* u *satisfies (2.1.10) on* $(1, \infty)$ *if and only if* $\tilde{u}$ *on* $(0, \infty)$ *satisfies*

$$t\frac{\partial \tilde{u}}{\partial t}(t) = \frac{1}{2}\tilde{u}^2\Delta\tilde{u} + \frac{t^2}{4}|\tilde{M}|^2\tilde{u} + \frac{1}{2}\tilde{u} - \frac{\tilde{R}}{4}\tilde{u}^3 + \frac{t^2\tilde{R}_N}{4}\tilde{u}^3, \tag{3.4.34}$$

where the fields $\tilde{M}$*,* $\tilde{R}$*, and* $\tilde{R}_N$ *are defined by*

$$\begin{aligned} \tilde{M}(t) &= \frac{t+1}{t}M(t+1), \\ \tilde{R}(t) &= R(t+1), \text{ and} \\ t^2\tilde{R}_N(t) &= (t+1)^2R_N(t+1). \end{aligned} \tag{3.4.35}$$

Proof. By substituting the parabolic equation (2.1.10) of u, we get the evolution equation of u

$$\begin{aligned} t\frac{\partial \tilde{u}}{\partial t}(t) &= \left(\frac{t}{t+1}\right)^3 (t+1)\frac{\partial u}{\partial t}(t+1) + \frac{1}{2(t+1)}\sqrt{\frac{t}{t+1}}u(t+1) \\ &= \left(\frac{t}{t+1}\right)^{3/2}\left(\frac{1}{2}u^2\Delta u + \frac{(t+1)^2}{4}|M|^2u + \frac{1}{2}u - \frac{R}{4}u^3 + \frac{(t+1)^2R_N}{4}u^3\right) \\ &\quad + \frac{1}{2}\frac{1}{t+1}\sqrt{\frac{t}{t+1}}u(t+1) \\ &= \frac{1}{2}\tilde{u}\Delta\tilde{u} + \frac{t^2}{4}|\tilde{M}|^2\tilde{u} + \frac{1}{2}\tilde{u} - \frac{\tilde{R}}{4}\tilde{u}^3 + \frac{t^2\tilde{R}_N}{4}\tilde{u}^3. \end{aligned}$$

□

Lemma 3.16. *Suppose* $t_0 > 0$*,* $R_N \in C^\alpha(N)$ *and* $R_Nt^2 < R$ *for* $1 \leq t < \infty$*. Then for any* $\varphi \in C^{2,\alpha}(\Sigma_1)$*, there is a positive solution* $\tilde{u} \in C^{2+\alpha}(A_{[t_0,\infty)})$ *of (3.4.34) with initial condition* $\tilde{u}(t_0, \cdot) = \varphi(\cdot)$*.*

Proof. Define $\tilde{K}$ as

$$\tilde{K} = \sup_{t_0 \leq t < \infty}\left\{-\frac{1}{t_0}\int_0^t \left(\frac{\tilde{R}}{2} - \frac{t'^2}{2}\tilde{R}_N\right)_* (t')\exp\left(\int_{t_0}^{t'}\frac{s|\tilde{M}|_*^2}{2}ds\right)dt'\right\}. \tag{3.4.36}$$

Since $R_Nt^2 < R$ for $1 \leq t < \infty$, and $\tilde{R}$ and $\tilde{R}_N$ are defined as

$$\begin{aligned} \tilde{R}(t) &= R(t+1), \text{ and} \\ t^2\tilde{R}_N(t) &= (t+1)^2R_N(t+1), \end{aligned}$$

we have

$$t^2 \tilde{R}_N(t) < \tilde{R} \quad \text{for all} \quad 0 \leq t < \infty,$$

and thus $\tilde{K} = 0$.

Observe that the $\tilde{u}$-equation is of the same form as the u-equation (2.1.10), except that $\tilde{M}$ is defined by (3.4.35). Following by the same argument in Proposition 3.6, the solution $\tilde{u}$ exists for all time $t_0 \leq t < \infty$.

$\square$

The rescaled scalar curvature $\tilde{R}_N$ is defined on $A_{(0,\infty)}$. Define functions $\tilde{\delta}_*(t)$ and $\tilde{\delta}^*(t)$ on $A_{(0,\infty)}$

$$\tilde{\delta}_*(t) = \frac{1}{t}\int_0^t \left(\frac{\tilde{R}}{2} - \frac{t'^2}{2}\tilde{R}_N\right)_* (t') \exp\left(-\int_{t'}^t \frac{s|\tilde{M}|^{*2}}{2} ds\right) dt' \tag{3.4.37}$$

$$\tilde{\delta}^*(t) = \frac{1}{t}\int_0^t \left(\frac{\tilde{R}}{2} - \frac{t'^2}{2}\tilde{R}_N\right)^* (t') \exp\left(-\int_{t'}^t \frac{s|\tilde{M}|_*^2}{2} ds\right) dt' \tag{3.4.38}$$

Applying the maximum principle to the (3.4.34), we obtain a priori estimates for $\tilde{u}$

$$\tilde{u}^{-2}(t,x) \geq \tilde{\delta}_*(t) + \frac{t_0}{t}\left(\tilde{u}^*(t_0)^{-2} - \tilde{\delta}_*(t_0)\right) \exp\left(-\int_{t_0}^t \frac{s|\tilde{M}|^{*2}}{2} ds\right) \tag{3.4.39}$$

$$\tilde{u}^{-2}(t,x) \leq \delta^*(t) + \frac{t_0}{t}\left(\tilde{u}_*(t_0)^{-2} - \tilde{\delta}^*(t_0)\right) \exp\left(-\int_{t_0}^t \frac{s|\tilde{M}|_*^2}{2} ds\right). \tag{3.4.40}$$

Lemma 3.17. *Suppose that $R_N \in C^\alpha(N)$ and $R_N t^2 < R$ for $1 \leq t < \infty$. Then there is a solution $\tilde{u} \in C^{2+\alpha}(A_{(0,\infty)})$ of (3.4.34) such that for all $(t,x) \in \mathbb{R}^+ \times \Sigma_1$*

$$\frac{1}{\sqrt{\tilde{\delta}^*(t)}} \leq \tilde{u}(t,x) \leq \frac{1}{\sqrt{\tilde{\delta}_*(t)}}, \tag{3.4.41}$$

Proof. The curvature assumption $R_N t^2 < R$ for $1 \leq t < \infty$ implies that

$$0 < \tilde{\delta}_*(t) \leq \tilde{\delta}^*(t) < \infty \quad \text{for all} \quad t > 0.$$

Let $\varphi_\epsilon \in C^{2,\alpha}(\Sigma_1)$, $0 < \epsilon < 1$ be any family of functions satisfying

$$\tilde{\delta}_*(\epsilon) \leq \varphi_\epsilon^{-2}(x) \leq \tilde{\delta}^*(\epsilon).$$

Let $\tilde{u}_\epsilon$ be the solution of (3.4.34) on $A_{[\epsilon,\infty)}$ with initial condition φ_ϵ. The existence of the solution to the initial value problem is shown in Lemma 3.16. From the a priori estimates above (3.4.39) and (3.4.40) we have solutions $\tilde{u}_\epsilon$ are bounded and

$$\tilde{\delta}_*(t) \leq \tilde{u}_\epsilon^{-2}(t,x) \leq \tilde{\delta}^*(t), \quad \epsilon \leq t < \infty,$$

for all $0 < \epsilon < 1$.

Now suppose $t_0 > 0$ and $v \in C^{2+\alpha}(A_I)$, $I = [t_0, 4t_0]$, is a solution of (3.4.34) satisfying

$$\tilde{\delta}_*(t) \leq v^{-2}(t,x) \leq \tilde{\delta}^*(t), \quad \text{for all } t \in I,$$

and define $\tilde{v}(t,x) = v(t/t_0, x)$. Spplying Proposition 3.4 to $\tilde{v}$ on the interval $[1,4]$ and rescaling back, from (3.2.23) we obtain an estimate of the form

$$||v||_{2+\alpha,I'} \leq C, \quad I' = [2t_0, 4t_0], \tag{3.4.42}$$

where C is a constant independent of v.

Applying (3.4.42) to $\tilde{u}_\epsilon$, we obtain an uniform bound of $||\tilde{u}_\epsilon||_{2+\alpha,[2\epsilon,\infty)}$. By Arzela-Ascoli theorem, there is a sequence $\epsilon_j \to 0$ such that $\{\tilde{u}_{\epsilon_j}\}$ converges uniformly in $C^{2+\alpha}(A_I)$ for any compact interval $I \subset \mathbb{R}^+$ to the desired solution $\tilde{u} \in C^{2+\alpha}(\mathbb{R}^+ \times \Sigma_1)$. □

Proof of Theorem 3.14. Lemma 3.17 gives a solution $\tilde{u} \in C^{2+\alpha}(A_{(0,\infty)})$ to (3.4.34), bounded by

$$0 < \tilde{\delta}_*(t) \leq \tilde{u}^{-2}(t) \leq \tilde{\delta}^*(t),$$

where $\tilde{\delta}_*(t)$ and $\tilde{\delta}^*(t)$ are defined by (3.2.21) and (3.4.38), using $\tilde{M}$, $\tilde{R}$, and $\tilde{R}_N$. Since $|M| \leq Ce^{-ct}$ for some constants c and C, and (3.4.33), there is a small ϵ such that on $(0,\epsilon)$ $1-\eta \leq R - t^2 R_N < (1-\eta)^{-1}$,

$$\begin{aligned}
\frac{1}{t}\int_0^t \exp\left(-\int_{t'}^t \frac{s|\tilde{M}|^{*2}}{2} ds\right) dt' &< 1, \quad \text{and} \\
\frac{1}{t}\int_0^t \exp\left(-\int_{t'}^t \frac{s|\tilde{M}|_*^{2}}{2} ds\right) dt' &< 1.
\end{aligned}$$

It shows that on $(0,\epsilon)$

$$\begin{aligned} 1-\eta \le \tilde{\delta}_*(t) &\le \tilde{\delta}^*(t) < (1-\eta)^{-1}, \text{and} \\ \frac{-\eta t}{1-\eta} &< 2\tilde{m}(t) < \eta t, \end{aligned}$$

where $\tilde{m}(t) = \frac{t}{2}(1-\tilde{u}^{-2}(t)) = m(1+t) - \frac{1}{2}$. The rescaling estimate (3.3.32) applied to $\tilde{m}$ shows that the covariant derivatives of m decay,

$$|\nabla m(t)| + |\nabla^2 m(t)| \le C(t-1).$$

It follows that the curvature of g_N is bounded on $A_{[1,2]}$.

$\square$

Remark 3.18. *Let $\tilde{u} \in C^{2+\alpha}(A_{(0,t_0)})$ be a solution constructed in Lemma 3.17. Then $\tilde{u}$ is unique in the class of solutions satisfying*

$$|\tilde{u}-1| \le Ct^{\epsilon}$$

for some $\epsilon > 0$, C and all $0 < t < t_0$.

Proof. Suppose that $\tilde{u}_1$ and $\tilde{u}_2$ are two solutions of (3.4.34). Let $v = \tilde{m}_1 - \tilde{m}_2$ where $\tilde{m}_i = \frac{t}{2}(1-\tilde{u}_i^{-2}), i = 1,2$. From Lemma 3.2, the equation for v is given by

$$t\frac{\partial v}{\partial t} = \frac{t}{2}(\tilde{u}_1^{-1}\Delta\tilde{u}_1 - \tilde{u}_2^{-1}\Delta\tilde{u}_2) - \frac{t^2}{2}|\tilde{M}|^2 v.$$

$$\begin{aligned} t\frac{d}{dt}\oint v^2 d\sigma &= -\oint |\tilde{M}|^2 v^2 d\sigma + \oint tv(\tilde{u}_1^{-1}\Delta\tilde{u}_1 - \tilde{u}_2^{-1}\Delta\tilde{u}_2) d\sigma \\ &= -\oint |\tilde{M}|^2 v^2 d\sigma \\ &\quad + t\oint -\nabla v\left(\frac{\nabla\tilde{u}_1}{\tilde{u}_1}\right) + \nabla v\left(\frac{\nabla\tilde{u}_2}{\tilde{u}_2}\right) + v\left(\frac{|\nabla\tilde{u}_1|^2}{\tilde{u}_1^2}\right) - v\left(\frac{|\nabla\tilde{u}_2|^2}{\tilde{u}_2^2}\right) d\sigma \\ &= -\oint |\tilde{M}|^2 v^2 d\sigma \\ &\quad + t\oint \frac{\tilde{u}_1^2\tilde{u}_2^2}{t^2}\Big\{(t-\tilde{m}_1-\tilde{m}_2)\left(|\nabla v|^2 + \frac{\tilde{u}_1^2\tilde{u}_2^2}{t^2}|\nabla v|^2 v^2\right) \\ &\qquad + \nabla(\tilde{m}_1+\tilde{m}_2)\left(\frac{\nabla\tilde{u}_1}{\tilde{u}_1} + \frac{\nabla\tilde{u}_2}{\tilde{u}_2} + \frac{\tilde{u}_1^2\tilde{u}_2^2}{t^2} v\nabla v\right) v^2\Big\}. \end{aligned}$$

There exist some constants C and c such that

$$t\frac{d}{dt}\oint v^2 d\sigma \;\leq\; C\oint \left(e^{-ct} + \left(|\nabla\tilde{m}_1|^2 + |\nabla\tilde{m}_1|^2\right)(t^{-2} + |v|t^{-3})\right) v^2 d\sigma.$$

Since $\tilde{u}_1$ and $\tilde{u}_2$ are given in the class $|\tilde{u}-1| \leq Ct^{\epsilon}$,

$$|\tilde{u}_1 - 1| + |\tilde{u}_2 - 1| \leq Ct^{\epsilon},$$

and then

$$|v| \leq |\tilde{m}_1| + |\tilde{m}_2| \leq Ct^{1+\epsilon}.$$

The rescaling estimates of $\tilde{m}$ give

$$|\nabla m_1| + |\nabla m_2| \leq Ct^{1+\epsilon}.$$

Therefore we find that

$$\frac{d}{dt}\oint v^2 d\sigma \leq Ct^{-1+2\epsilon}\oint v^2 d\sigma.$$

By solving the differential inequality, we have

$$\oint_{\Sigma_{t_2}} v^2 d\sigma \leq C\epsilon \exp\left(t_2^{2\epsilon} - t_1^{2\epsilon}\right)\oint_{\Sigma_{t_1}} v^2 d\sigma$$

for all $0 < t_1 < t_2 < t_0$. Since $v \to 0$, $\oint_{\Sigma_{t_1}} v^2 d\sigma$ goes to 0 as $t_1 \to 0$ and hence $v \equiv 0$, which proves the uniqueness.

□

Corollary 3.19. *If we start with the standard metric (Σ_1, g_{S^2}) and prescribe the scalar curvature $R_N \equiv 0$, then the metric g_N obtained from above is exactly a Schwarzschild metric with ADM mass $m_{ADM} = \frac{1}{2}$.*

Proof. Since the initial metric is the standard round metric, the modified Ricci flow doesn't change the metric and $g(t) = g_{S^2}$, $R \equiv 2$, and $M_{ij} \equiv 0$ for all $t \geq 1$. Since $R_N \equiv 0$, the rescaling fields are

$$\tilde{R} = 2, \quad \tilde{M}_{ij} \equiv 0, \quad \text{and} \quad \tilde{R}_N \equiv 0$$

for all $t \geq 0$. Moreover the a priori bounds $\tilde{\delta}_*(t) = 1$ and $\tilde{\delta}^*(t) = 1$ for all $t \geq 0$. Hence the solution $\tilde{u} \equiv 1$ and the metric

$$g_N = \frac{1}{1 - \frac{1}{t}} dt^2 + t^2 g_{S^2}.$$

□

4 Asymptotically hyperbolic metrics

The same idea works for constructing asymptotic hyperbolic 3-metrics of prescribed scalar curvature. Given any topological 2-surface (Σ_1, g_1) with area $A(\Sigma_1) = 4\pi$. Let $N = [1, \infty)$ and

$$g_N = u^2 dt^2 + \sinh^2 t g_{ij}(t) dx^i dx^j$$

where $g(t)$ is the solution obtained by the modified Ricci flow with initial metric $g(1) = g_1$.

Similar calculations show that the second fundamental forms h_{ij} on each leaf Σ_t are

$$\begin{aligned} h_{ij} &= -g_N\left(\frac{1}{u}\frac{\partial}{\partial t}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) \\ &= -\frac{1}{2u}\left(-\frac{\partial}{\partial t}\bar{g}_{ij}\right) \\ &= \frac{1}{u}\left(\frac{\cosh t}{\sinh t} g_{N ij} + \sinh^2 t M_{ij}\right) \end{aligned}$$

In particular, the mean curvature H and the norm squared of the second fundamental form h are given by

$$\begin{aligned} H &= 2\frac{\cosh t}{\sinh t}\frac{1}{u}, \\ |A|^2 &= 2\frac{\cosh^2 t}{\sinh^2 t}\frac{1}{u^2} + \frac{|M_{ij}|^2}{u^2} \end{aligned}$$

respectively.

Also,

$$\begin{aligned}
g_N^{jl} R_N{}^3_{3jl} &= -\frac{1}{u}\bar{g}^{jl}\partial_t \bar{h}_{jl} - \frac{1}{u\sinh^2 t}\Delta u + |A|^2 \\
&= -\frac{1}{u}\frac{\partial}{\partial t}H - 2|A|^2 - \frac{1}{u\sinh^2 t}\Delta u + |A|^2 \\
&= -\frac{1}{u}\frac{\partial}{\partial t}H - \frac{1}{u\sinh^2 t}\Delta u - |A|^2 \\
&= \frac{2\cosh t}{u^3\sinh t}\frac{\partial u}{\partial t} + \frac{2}{u^2\sinh^2 t} - \frac{1}{u\sinh^2 t}\Delta u - |A|^2
\end{aligned}$$

and

$$g_N^{ik} g_N^{jl} R_{N\,ijkl} = \frac{R}{\sinh^2 t} - H^2 + |A|^2$$

where $i, j, k, l = 1, 2$ and R is the scalar curvature of $g(t)$ on Σ_t.

The scalar curvature R_N of the metric g_N is given by

$$\begin{aligned}
R_N &= 2g_N^{jl} R_N{}^3_{3jl} + g_N^{ik} g_N^{jl} R_{N\,ijkl} \\
&= \frac{4\cosh t}{u^3\sinh t}\frac{\partial u}{\partial t} + \frac{4}{u^2\sinh^2 t} - \frac{2}{u\sinh^2 t}\Delta u - 2|A|^2 + \frac{R}{\sinh^2 t} - H^2 + |A|^2 \\
&= \frac{4\cosh t}{u^3\sinh t}\frac{\partial u}{\partial t} + \frac{4}{u^2\sinh^2 t} - \frac{2}{u\sinh^2 t}\Delta u + \frac{R}{\sinh^2 t} - H^2 - |A|^2 \\
&= \frac{4\cosh t}{u^3\sinh t}\frac{\partial u}{\partial t} - \frac{2}{u\sinh^2 t}\Delta u + \frac{R}{\sinh^2 t} + \frac{4}{u^2\sinh^2 t} - \frac{6\cosh^2 t}{u^2\sinh^2 t} - \frac{|M_{ij}|^2}{u^2}.
\end{aligned}$$

Rewrite the equation. We have the parabolic equation for the prescribed scalar curvature

$$\begin{aligned}
\cosh^2 t\sinh t\frac{\partial u}{\partial t} &= \frac{\cosh t}{2}u^2\Delta u + \left(-\cosh t + \frac{3\cosh^3 t}{2} + \cosh t\sinh^2 t\frac{|M_{ij}|^2}{4}\right)u \\
&\quad -\frac{1}{4}\left(\cosh tR - \cosh t\sinh^2 tR_N\right)u^3,
\end{aligned}$$

and the solution u converges to 1 exponentially.

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