II Kähler / projective
The Kodaira problem.

Characterization Thm :
(Kodaira) A compact
complex mfd X is
projective iff
$\exists$ Kähler form ω on X
with $[\omega] \in H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R})$

$[\omega]$ = coh. class of the
closed 2-form ω.

2

Given X = Kähler cpct

then set

$\{[\omega], \omega$ a Kähler form$\}$

of Kähler classes

is an open

convex cone

(the Kähler cone)

$$K(X) \overset{\text{open}}{\subset} H^{1,1}(X)_{\mathbb{R}} \underset{\text{sub v.space}}{\subset} H^2(X, \mathbb{R})$$

Here $H^{1,1}(X)_{\mathbb{R}} := H^{1,1}(X) \cap H^2(X, \mathbb{R})$

= set of classes representable by a closed real (1,1)-form.

The set of rational coh. classes

$H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R})$

is dense, but

$K(X) \overset{\text{open}}{\subset} H^{1,1}(X)_{\mathbb{R}} \subsetneq H^2(X, \mathbb{R})$ (in general)

and the v. space $H^{1,1}(X)_{\mathbb{R}}$ may well not contain any rational cohomology class.

On the other hand, when $X = X_0$ deforms, say

$(X_t)_{t \in B}$ = family of cx structures on fixed diff. mfd X,

$$H^{1,1}(X_t)_{\mathbb{R}} \subset H^2(X_t, \mathbb{R}) = H^2(X, \mathbb{R})$$

deforms differentiably with $t \in B$.
(as long as X_t remains Kähler)

Assume it deforms enough

so that

$$\bigcup_{t \in B} K(X_t) \quad \text{contains}$$

an open set in $H^2(X, \mathbb{R})$

Then this open set contains

a rational coh. class.

$\Rightarrow$ Some deformation

X_t is projective.

Examples:

. X = complex torus

.. X = hyperkähler mfd

$\Rightarrow$ projective deformations X_t of X are dense in the local universal family of defs of X.

In fact, there is the following infinitesimal criterion for density of proj. defs:

Prop.: Assume X is unobstruc[ted] and for some Kähler class $[\omega] \in H^1(\Omega_X)$,

(†

The map:

$$\lrcorner[\omega] : H^1(T_X) \to H^2(\mathcal{O}_X)$$

is surjective.

(Here $\lrcorner$ is contraction + $\wedge$product

Then proj. def'ns of X are dense in the local univ. family of def'ns of X.

Pf: Use Griffiths' description of inf. var. of Hodge structure to see that assumption $\Leftrightarrow$

$$\bigcup_t H^{1,1}(X_t)_{\mathbb{R}} \overset{\text{open}}{\supset} \bigcup_{t \in B} K(X_t) \to H^2(X, \mathbb{R})$$

is submersive at $(0, [\omega])$.

~~X~~

thi

$$H^{1,1}(X_t)_{\mathbb{R}} = H^2(X_t, \mathbb{R}) \cap H^{1,1}(X_t)$$
$$= H^2(X_t, \mathbb{R}) \cap F^1 H^2(X_t),$$

where $F^1 H^2(X_t) := H^{2,0}(X_t) \oplus H^{1,1}(X_t)$

Thus

$$\bigcup_{t \in B} H^{1,1}(X_t)_{\mathbb{R}} \longrightarrow H^2(X_t, \mathbb{R})$$

is submersive

iff

$$\bigcup_{t \in B} F^1 H^2(X_t) \longrightarrow H^2(X, \mathbb{C})$$

is submersive.

Griffiths computes the diff'l of this last map.

(8

Here $0 \in B \rightsquigarrow X_0 = X$

$T_{B,0} = H^1(X_0, T_{X_0})$

$(X_t)_{t \in B}$ universal.

Now as B is smooth submersive $\Rightarrow$ open and the previous criterion applies. $\boxed{\text{U}}$

~~The criterion is ...~~

Buchdahl uses this criterion to reprove the following Kodaira's Theorem in case of unobstructed surfaces.

<u>Thm (Kodaira)</u> A compact Kähler surface admits arbitrarily small deformations which are projective

Kodaira's proof is by classification.

Buchdahl's proof is for unobstructed surfaces:

He proves that if S is such that (ω) Kähler and $\exists\, \eta \in H^0(K_S)$ $\Big($ $\eta(\omega) = 0$ in $H^1(\Omega_S(K_S))$ $\overset{\shortparallel}{H^1(T_S)^*}$ $\Big)$

Then S is projective.

[by duality this $\Leftrightarrow$ to $[\omega]\lrcorner$ non surj]

* The Kodaira problem:

What about higher dimension?

Q: Let X be a compact Kähler manifold. Does X deform to a projective mfld Y?

If we have such a deformation
$$\begin{array}{c}\mathcal{X} \\ \pi \downarrow \\ B\end{array}$$

π = smooth proper

B = connected analytic space

$X = \mathcal{X}_0$, $Y = \mathcal{X}_b$, for some b,

Then $X \underset{\text{diffeo}}{\simeq} Y$

because B is path connected,

and over paths the family $\mathcal{X}$ is differentiably trivial.

Thus, can ask weaker questions:

Qn: Is a compact Kähler manifold diffeomorphic
homeomorphic
homotopy equivalent

to a projective complex manifold?

The last questions is very natural ~~make sense~~ as the classically known topological restrictions on projective mflds come from Hodge theory, hence are satisfied as well by Kähler cpct mflds:

In fact they can be used to distinguish topologically the class of cpt Kähler mflds from that of symplectic ones.

simplest ones

Topological restrictions on Kähler manifolds coming from Hodge θ:

a) Hodge decomposition

$\Rightarrow$ b_{2i+1} is even

b) Hodge decomposition is compatible with

$\cup : H^k(X) \otimes H^l(X) \to H^{k+l}(X)$

in the sense that

$H^{p,q}(X) \otimes H^{r,s}(X) \to H^{p+r,q+s}(X)$

$\Rightarrow$ restrictions on cohomology ring

c) Hard Lefschetz theorem $\Rightarrow$

$b_{2i}(X)$, $2i \leq n$

are increasing with i

(L is injective on

$H^*(X)$, $* \leq n-1$)

Similarly, $b_{2i+1}(X)$, $2i+1 \leq n$

are increasing with i.

d) Hodge index -

Case of surfaces :

$\langle , \rangle$ on $H^2(S, \mathbb{R})$

$\overset{\shortparallel}{q_L}$ positive on $\langle [\omega] \rangle + (H^{2,0}(S) \oplus H^{0,2}$
even dimens.

negative on $H^{1,1}(S)_{\mathbb{R},prim}$

$\Rightarrow$ b_2^+ is odd.

→ Q p 11

~~[illegible], despite of this~~ Qp!!

Theorem: (V. 03) In any dimension $n \geq 4$, $\exists$ compact Kähler mlds, which do not have the homotopy type [the cohomology ring] of a projective mld.

In dimension $n \geq 6$, $\exists$ simply connected such examples.

Another version of Kodaira problem:

The examples are very simple geometrically

- Start from a complex torus + blow it up along Cx submanifolds.

- (Simply connected case) Start from K = Kummer variety of a complex torus + blow-up complex submanifolds in $K \times K$.

NB: the complex torus is very special

These examples are bimeromorphically equivalent to either a torus, or a self-product $K \times K$, K = Kummer mfd, which satisfy themselves the property of deforming to a projective mfd.

Leads to:

Qn (Campana, Buchdahl, Yau)

Does a compact Kähler mfd

admit a smooth bimeromorphic model, which deforms to a projective mfd?

[= A birational version of Kodaira's problem].

Answer is again NO, for topological reasons

Thm (V. 04) In dimension ≥ 10, $\exists$ compact Kähler mfds, no smooth birational model of which has the homotopy type of a projective mfd

18 bis

Outline of the topological obstruction for X = compact Kähler to admit a projective complex structure:

The ring

$$H^*(X, \mathbb{Z}) \text{ is}$$

given.

If X admits a projective

(•) If the ring structure on $H^*(X)$ is rich enough, compatibility with $\cup$ may force the existence of deg2 Hodge classes. (integral deg 2 classes of type (1,1))

(••) They in turn may force the existence of endo's of Hodge structures on H^1 or H^2

(•••) Certain endo's prevent the existence of polarizatio complex structure

existence is not a pr. as X is Kähler

$\exists$ Hodge structures of weight k on each $H^k(X, \mathbb{Z})$ compatible with cup-product,

AND

a integral polarization on the Hodge structure on H^1 or H^2

The starting point of all these constructions is: The existence of certain endomorphisms φ_T acting on a complex torus T may prevent T to be projective. (= illustration of point (100))

<u>Construction</u>: Γ = rk $2n$ lattice, $\varphi: \Gamma \to \Gamma$ endomorphism.

(20

Assume:
The eigenvalues of φ are all distinct, + none is real.

Choose $\lambda_1, \dots, \lambda_n$, n eigenvalues pairwise not conjugate.

Consider $\Gamma^{1,0} :=$ eigenspace of $\varphi_{\mathbb{C}}$ associated to $\lambda_1, \dots, \lambda_n$. Thus

$$\Gamma^{1,0} \subset \Gamma_{\mathbb{C}} \quad \text{and}$$

$$\Gamma^{1,0} \oplus \underbrace{\overline{\Gamma^{1,0}}}_{\substack{\shortparallel \\ \text{eigenspace assoc. to } \overline{\lambda}_1, \dots, \overline{\lambda}_n}} = \Gamma_{\mathbb{C}}$$

$\Rightarrow$ Complex torus

$$T = \Gamma_{\mathbb{C}} / \Gamma^{1,0} \oplus \Gamma \ , \ \varphi$$

induces $\varphi_T : T \to T$.

Prop: Assume $n \geq 2$

and: $\mathrm{Gal}(\mathbb{Q}[\lambda_1, \ldots, \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n] : \mathbb{Q})$ acts as $\mathfrak{S}_{2n}$ on eigenvalues of φ.

Then T is not projective.

Pf: One proves the stronger statement $NS(T) = 0$, where

$NS(T) = \{ c_1(\mathcal{L}),\ \mathcal{L} = $ holo line bundle on $T \}$

$= \{ H^2(T, \mathbb{Z}) \cap H^{1,1}(T) \}$.

Lefschetz thm on (1,1)-classes

In fact : $H^2(T, \mathbb{Q}) = \Lambda^2 \Gamma^*_{\mathbb{Q}}$

and $\varphi_T^* = \Lambda^2 {}^t\varphi \curvearrowright H^2(T, \mathbb{Q})$

Assumption on $\mathrm{Gal}(\mathbb{Q}(\lambda_i, \bar{\lambda}_i) : \mathbb{Q}) \Rightarrow$

The action of $\Lambda^2 {}^t\varphi$ is irreducible.

But $\Lambda^2 {}^t\varphi = \varphi_T^*$ leaves $NS(T) \otimes \mathbb{Q}$ stable.

Thus either $NS(T)_{\mathbb{Q}} = 0$ (2

or $NS(T)_{\mathbb{Q}} = H^2(T, \mathbb{Q})$.

as $H^{2,0}(T) \neq 0$ because

$n \geq 2$, and

$NS(T)_{\mathbb{Q}} \subset H^{1,1}$,

the second case is impossible.

$\Rightarrow NS(T)_{\mathbb{Q}} = 0$

$\boxed{S}$