Hilbert's original 14th problem and certain moduli spaces

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$\rho : G \longrightarrow GL(N, \mathbf{C})$, or $G \overset{\rho}{\curvearrowright} V \simeq \mathbf{C}^N$
N-dimensional linear representation of an algebraic group G

$G \curvearrowright \mathbf{C}[x_1, \ldots, x_N] = \mathbf{C}[V] =: S$
induced action (called *linear action* on a polynomial ring.)

$$S^G = \{f(x_1, \ldots, x_N) \mid f^g = f \quad \forall g \in G\}$$

Original 14th problem Is S^G finitely generated (as ring over $\mathbf{C}$)?

RIMS preprint #1343(2001), #1502(2005)
http://www.kurims.kyoto-u.ac.jp

Yes,

if G is finite. (Easy)

if $G = SL(m)$. (Hilbert 1890)

if G is reductive. (Hilbert $+\cdots$)

More generally, let $G \curvearrowright R$ be action on a ring over $\mathbf{C}$.

Theorem R finitely generated, G reductive $\Rightarrow R^G$ finitely generated

By the exact sequence

$$1 \to G^u \to G \to G^{red} \to 1,$$

we have

Corollary R^{G^u} finitely generated $\Rightarrow R^G$ finitely generated

Boiled down 14th problem Is S^G finitely generated for unipotent G?

Yes,

if $G = \mathbf{G}_a$. (thm of Weitzenböck)

(action of $\mathbf{G}_a$
$\Leftrightarrow$ action of $\mathbf{C}$ with polynomial coefficients
$\Leftrightarrow$ locally finite derivation)

No Counterexample by Nagata in 1958

Metaproblem
Find good criteria of finite and non-finite generation of S^G
(for unipotent algebraic group G).

No Counterexample for $\mathbf{G}_a^3$ (M. 2001)

Open problem

Is S^G finitely generated for a linear action of $G = \mathbf{G}_a^2$ on a polynomial ring?

(action of $\mathbf{G}_a^2 \Leftrightarrow$ commutative pair of locally finite derivations)

I will answer two problems affirmatively for Nagata invariant rings.

§1 Nagata action and the main theorem

Consider the standard unipotent action

$$\mathbf{C}^n \curvearrowright \mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] =: S_{2n}$$

$$(t_1, \ldots, t_n) \qquad \begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases} \qquad 1 \leq i \leq n.$$

$G \subset \mathbf{C}^n$ s-dimensional general linear subspace, $r := n - s$ (codimension)

Restriction

$$\mathbf{G}_a^s = \mathbf{C}^s \simeq G \quad \curvearrowright \quad S_{2n}$$

is called a Nagata action.

Nagata'58 studied the case $r = 3$ and showed that S^G is not finitely generated for square numbers $n = m^2 \geq 16$.

Theorem The invariant ring S_{2n}^G, $\dim G = s$, is finitely generated if and only if

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{2} > 1.$$

This condition is equivalent to the finiteness of the Weyl group of $T_{r,s,2}$.

Special cases

(1) $\dim G = 2 \Rightarrow S_{2n}^G$ is f.g. for $\forall n$.

(2) $\dim G = 3$

$(n, r) = (8, 5)$, $\mathbf{C}^3 \curvearrowright S_{16} \Rightarrow$ f.g.

$(n, r) = (9, 6)$, $\mathbf{C}^3 \curvearrowright S_{18} \Rightarrow$ not f.g.

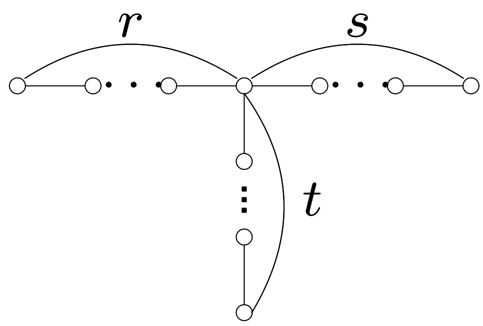
Two proofs for ‘if’ (Nagata) part

(M.) geometry of moduli of vector bundles
advantage: Determines movable cone and chamber structure

(Castravet-Tevelev) algebraic
advantage: Determination of set of generators

Three-legged diagram

$T_{r,s,t}$

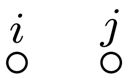


$W(T_{r,s,t})$ Weyl group

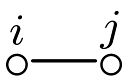
generators $w_1, \ldots, w_n$

$$n = r + s + t - 2 = (\#\text{ of vertices})$$

relations $w_1^2 = \cdots = w_n^2 = 1$

$w_i w_j = w_j w_i$ if 

(not joined)

$(w_i w_j)^3 = 1$ if 

(joined)

finite group $\Leftrightarrow$ $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} > 1$

$\Leftrightarrow A_n,\ D_n$ or $E_{6,7,8}$

§2 Geometrization

$G \subset \mathbf{C}^n$:general linear subspace of codim r

X_G = Blow-up of $\mathbf{P}^{r-1}$, the projectivization of $\mathbf{C}^n/G$, at n points $p_1, \ldots, p_n$ which are the images of standard basis of $\mathbf{C}^n$

Theorem $(r \geq 3)$

$$S_{2n}^G \simeq \bigoplus_{a,b_1,\ldots,b_n \in \mathbb{Z}} H^0(X_G, \mathcal{O}_X(ah - \sum_i b_i e_i))$$

$$\simeq \bigoplus_{L \in \mathrm{Pic} X} H^0(X, L) =: TC(X_G), \text{ or } Cox(X_G)$$

$\mathcal{O}_X(h) := \pi^* \mathcal{O}_{\mathbf{P}}(1)$

$$\begin{array}{cccc} & X_G & \supset e_1, \ldots, e_n \\ \pi & \downarrow & \downarrow \\ & \mathbf{P}^{r-1} & \ni p_1, \ldots, p_n \end{array}$$

$e_i :=$ exceptional divisor over p_i

Discussion in the case $r = 3$

$$\frac{y_i}{x_i} \mapsto \frac{y_i}{x_i} + t_i, \quad (t_1, \ldots, t_n) \in \mathbf{C}^n$$

$\exists$ 3 independent linear combinations

$$X = \sum a_i \frac{y_i}{x_i}, \quad Y = \sum b_i \frac{y_i}{x_i}, \quad Z = \sum c_i \frac{y_i}{x_i}$$

which are G invariants.

$$\tilde{X} = (\prod x_i)X, \tilde{Y} = (\prod x_i)Y, \tilde{Z} = (\prod x_i)Z,$$

and $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ generate

$$\mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n]^G.$$

Hence

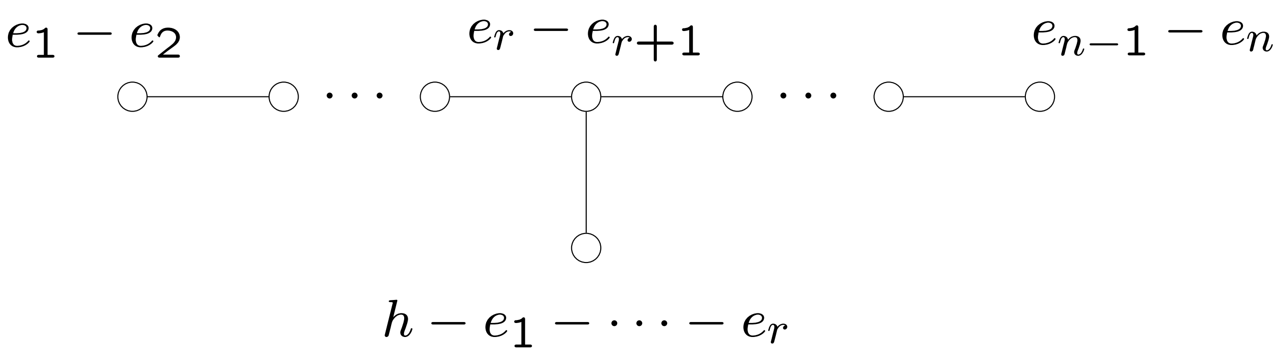
$$S^G = \mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \tilde{X}, \tilde{Y}, \tilde{Z}]^G \cap S.$$

Form $F(\tilde{X}, \tilde{Y}, \tilde{Z})$ on $\mathbf{P}^2$ vanishes at $p_1 \Leftrightarrow$ $F(\tilde{X}, \tilde{Y}, \tilde{Z})/x_i \in S$

§3 Proof of Nagata direction

$\frac{1}{r}+\frac{1}{s}+\frac{1}{2}\leq 1 \Rightarrow \ S_{2n}^G$ not finitely generated

Action of Weyl group $W(T_{r,s,2})$ on $H^2(X_G,\mathbb{Z})$ (monodromy)



A_{n-1} part $\leftrightarrow$ permutation of $p_1,\ldots,p_n$

Extra root $h - e_1 - \cdots - e_r$
$\leftrightarrow$ standard Cremona transformation

$$\mathbf{P}^{r-1} \leftarrow \cdots \rightarrow \mathbf{P}^{r-1}$$

$$(x_1:\cdots:x_r)\leftrightarrow(\frac{1}{x_1}:\cdots:\frac{1}{x_r})$$

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{2} \leq 1$$

$$\Downarrow$$

$W(T_{r,s,2})$ is infinite and X_G has infinitely many "(-1)-divisors"

(Simplest is the case $r = 3$ and $s = 3$: $Bl_9\mathbf{P}^2$ has infinitely many (-1)$\mathbf{P}^1$'s.)

$$\Downarrow$$

Effective semi-group $\mathrm{Eff}\, X_G \subset \mathrm{Pic}\, X_G \simeq \mathbb{Z}^{n+1}$ is not finitely generated

$$\Downarrow$$

$TC(X_G)$ is not finitely generated

§4 Proof of f.g. direction

$\frac{1}{r}+\frac{1}{s}+\frac{1}{2}>1 \Rightarrow$ S^G_{2n}, or $TC(X_G)$, finitely generated

(A) Case division

r	1		2		3			4	5
s		1		2	3	4	5	3	3
$T_{r,s,2}$	A_n	A_n	D_n	D_n	E_6	E_7	E_8	E_7	E_8
X_G				Bl_n $\mathbf{P}^{n-3}$					
pf				moduli of bundles on Y_G					
Y_G				pointed $\mathbf{P}^1$					

(B) Birational geometry

Basic technique:

L line bundle

$|L|$ base point free $\Rightarrow \bigoplus_{n\geq 0} H^0(L^n)$ finitely generated

L semiample ($\Leftrightarrow$ some multiple is base point free) $\Rightarrow$ the same

$L_1, \ldots, L_k$ semin-ample line bundles on X $\Rightarrow \bigoplus_{n_1,\ldots,n_k\geq 0} H^0(L_1^{n_1} \otimes \cdots \otimes (L_k^{n_k})$ finitely generated

Theorem $\frac{1}{r} + \frac{1}{s} + \frac{1}{2} > 1, \quad r \geq 3$

(1) Eff X_G is finitely generated.

(2) $\exists$ decomposition Mov $X_G = \bigcup_i C_i$ into finitely many chambers such that each C_i is generated by finitely many semi-ample linebundles on a variety X_i isomorphic to X_G in codimension one.

(C) Moduli of bundles

$G \subset \mathbf{C}^n$ $\quad s$-dimensional general linear subspace

$$Y_G = (\mathbf{P}^* G; q_1, \ldots, q_n)$$

Coble dual, or Gale transformation of

$$(\mathbf{P}_*(\mathbf{C}^n/G); p_1, \ldots, p_n)$$

Restriction of

$$\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(1)) \curvearrowright \bigoplus_i (\mathcal{O} \oplus \mathcal{O}(1))_{q_i}$$

to the unipotent part ($\simeq \mathbf{G}_a^2$) is Nagata action.

$V/\mathbf{G}_a^2 \cdot \mathbf{G}_m^{n+1}$ is isomrphic to X_G in codimsnsion one.

$s = 3, n = 7, 8$ $\quad V/\mathbf{G}_a^2 \cdot \mathbf{G}_m^{n+1}$ is isomorphic to moduli of 2-bundles on $Bl_{q_1,\ldots,q_n}\mathbf{P}^2$ in codimnsion one.

$s = 2$ $\quad V/\mathbf{G}_a^2 \cdot \mathbf{G}_m^{n+1}$ is isomorphic to moduli of parabolic 2-bundles on n-pointed projective line $(\mathbf{P}^1 : q_1, \ldots, q_n)$ in codimnsion one.

Theorem is proved by moduli change under variation of polarizations.

§5 Generalization by H. Naito and open problem

Fix a decomposition

$$\{1,2,\ldots,n\} = \coprod_{j=1}^{m} N_j, \quad |N_j| \geq 1.$$

$$\mathbf{C}^n \curvearrowright \mathbf{C}[x_1,\ldots,x_m,y_1,\ldots,y_n] =: S_{m+n}$$

$$(t_1,\ldots,t_n) \qquad \begin{cases} x_i \mapsto x_i & 1 \leq i \leq n \\ y_i \mapsto y_i + t_i x_j & i \in N_j \end{cases}$$

($|N_j| = 1 \, \forall j \Rightarrow$ Nagata action)

$G \subset \mathbf{C}^n$ s-dimensional general linear subspace, $r := n - s$ (codimension)

Geometrization Theorem

$r \geq |N_j| + 2 \quad \forall j \Rightarrow S^G_{m+n} \simeq TC(X_G)$

$$X_G = Bl_{L_1,\ldots,L_m}\mathbf{P}^{r-1}$$

$\mathbf{P}^{r-1}$ is projectivization of $\mathbf{C}^n/G$ and L_j is image of $\mathbf{C}^{N_j} \subset \mathbf{C}^n$.

Problem When is S^G_{m+n} finitely generated?