# RESOLUTION OF SINGULARITIES

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# Strong resolution theorem

For every $X$ (char. 0) there is
$f : X' \to X$ such that

(1) $X'$ smooth,
(2) $f$: composite of smooth blow ups,
(3) isomorphism over $X^{ns}$,
(4) $f^{-1}(\operatorname{Sing} X)$ is normal crossings,
(5) functorial on smooth morphisms,
(6) functorial on field extensions.

Hironaka
Giraud
Villamayor, Bravo, Encinas
Bierstone and Milman
Encinas and Hauser
Włodarczyk

**Example**

Resolving $S := (x^2 + y^3 - z^6 = 0)$

(Secret: single elliptic curve $(E^2) = -1$)

Method: $H := (x = 0)$ and use $S \cap H$.

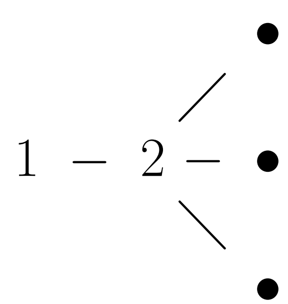
*Step 1.* $\operatorname{mult}(S \cap H) = (y^3 - z^6 = 0) = 3$

but came from multiplicity 2

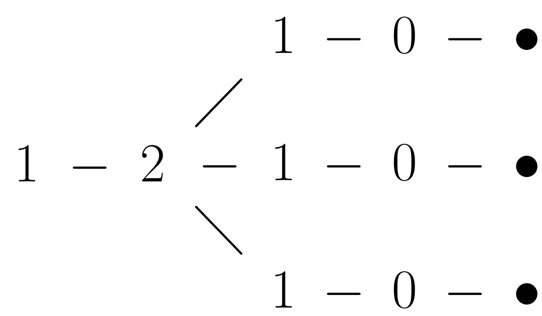
blow up until the mult. drops below 2.

2 blow ups to achieve this:

| $S$ | coordinates |
|---|---|
| $x^2 + y^3 - z^6$ | |
| $x_1^2 + (y_1^3 - z_1^3)z_1$ | $x_1 = \frac{x}{z}, y_1 = \frac{y}{z}, z_1 = z$ |
| $x_2^2 + (y_2^3 - 1)z_2^2$ | $x_2 = \frac{x_1}{z_1}, y_2 = \frac{y_1}{z_1}, z_2 = z_1.$ |

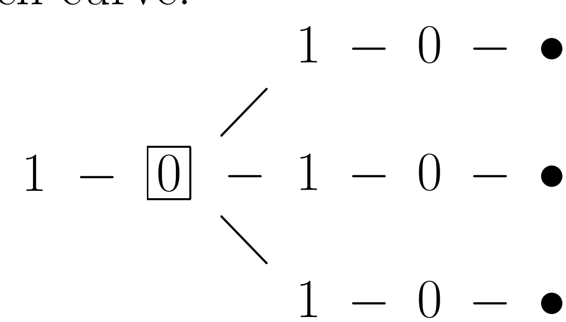


*Step 2.* Make $S \cap H$ disjoint from positive coeff. exceptional curves



*Step 3.* Blow up exceptional curves with multiplicity $\geq 2$.

one such curve:



where the boxed curve is elliptic.

**Problem 1.**

Get too many curves.
Higher dimensions: no minimal resolution, we do not know which resolution is simple

No solution.

**Problem 2.**

Reduction: from surfaces in $\mathbb{A}^3$ to curves in $\mathbb{A}^2$,
**but** exceptional curves and multiplicities treated differently.

Solution: marked ideals $(I, m)$.

# Problem 3.

$S$ has multiplicity $< 2$ along the birational transform of $H$, but what happens **outside** $H$?

Example: $H' := (x - z^2 = 0)$

$$\begin{array}{cc} S & H' \\ x^2 + y^3 - z^6 & x - z^2 \\ x_1^2 + (y_1^3 - z_1^3)z_1 & x_1 - z_1 \\ x_2^2 + (y_2^3 - 1)z_2^2 & x_2 - 1 \end{array}$$

singular point not on $H'$

Solution: careful choice of $H$

**maximal contact**

## Problem 4.

Too many singularities on $H$

Example: $H'' := (x - z^3 = 0)$.
$x^2 + y^3 - z^6 = (x - z^3)(x + z^3) + y^3$
so $S|_{H''}$: triple line.

Really a problem?
Yes: induction ruined

Solution: *coefficient ideal* $C(S)$

(i) resolving $S$ is equivalent to "resolving" $C(S)$, and

(ii) resolving the traces $C(S)|_H$ does not generate extra blow ups for $S$

## Problem 5.

$H$ not unique

e.g. automorphisms of $S$

$(x, y, z) \mapsto (x + y^3, y\sqrt[3]{1 - 2x - y^3}, z)$

Even with maximal contact choice of $H$,
$S \cap H$ depends on $H$

Solution: ideal $W(S)$ such that

(i) resolving $S$ is equivalent to
resolving $W(S)$, and

(ii) $W(S)|_H$ are analytically isomorphic
for all maximal contact $H$.

**Problem 6.**

(i) Many choices remain.
functorial but not "canonical"

(ii) Computationally hopeless.
Exponential increase in degrees and generators at each step.

No solutions

# Principalization

Data: $X$ *smooth* variety,
$I \subset \mathcal{O}_X$ ideal sheaf,
$E = \sum_i E_i$ normal crossing divisor with *ordered* index set

Blow ups: smooth centers,
normal crossing with $E$

## Strong principalization theorem

For every $(X, I, E)$ (char. 0) there is $f : X' \to X$ such that

(1) $f^* I \subset \mathcal{O}_{X'}$ locally principal,
(2) $f$: composite of smooth blow ups,
(3) isomorphism over $X \setminus \operatorname{cosupp} I$,
(4) $f^{-1}(E \cup \operatorname{cosupp} I)$ is normal crossing,
(5) functorial on smooth morphisms,
(6) functorial on field extensions,
(7) functorial on closed embeddings.

## Strong principalization $\Rightarrow$ Resolution

*Projective case*

take $X \hookrightarrow \mathbb{P}^N$, $N \geq \dim X + 2$.
$I \subset \mathcal{O}_{\mathbb{P}^N}$ ideal sheaf of $X$, $E = \emptyset$

Principalize $(\mathbb{P}^N, I, \emptyset)$.

$I$ is not principal along $X$,
so at some point, the
birational transform $X'$ of $X$ is blown up.

**But:** we blow up only smooth centers,
so $X'$ is smooth.

*Uniqueness?* Local question.

*Lemma.* Let $X \hookrightarrow \mathbb{A}^n$, $X \hookrightarrow \mathbb{A}^m$
be closed embeddings. Then

$X \hookrightarrow \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$, and
$X \hookrightarrow \mathbb{A}^m \hookrightarrow \mathbb{A}^{n+m}$

differ by an automorphism of $\mathbb{A}^{n+m}$.

$\operatorname{ord}_x I :=$ **order** of vanishing of $I$ at $x$
max-ord $I :=$ maximum $\{\operatorname{ord}_x I : x \in X\}$

blow up $Z$ to get $\pi : B_Z X \to X$
typical chart $Z = (x_1 = \cdots = x_r = 0)$
$g(x_1, \ldots, x_n)$ pulls back to
$\pi^* g := g(x_1' x_r', \ldots, x_{r-1}' x_r', x_r', x_{r+1}, \ldots, x_n)$.
if $\operatorname{ord}_Z I = s$ then
$g' := (x_r')^{-s} g(x_1' x_r', \ldots, x_{r-1}' x_r', x_r', x_{r+1}, \ldots, x_n)$.

*Lemma.* max-ord $g' \leq 2$ max-ord $g - s$.

| Our blow ups for the triple $(X, I, E)$: $Z$ smooth, normal crossing with $E$, $\operatorname{ord}_Z I =$ max-ord $I = m$. |
|---|

**New triple** $(X_1, I_1, E_1)$
$X_1 = B_Z X$ with $F \subset B_Z X$ except. div.
$I_1 = \pi_*^{-1} I := \mathcal{O}_{B_Z X}(mF) \cdot \pi^* I$
$E_1 = \pi_*^{-1} E + F$ (last divisor)

by lemma: max-ord $I_1 \leq$ max-ord $I$.

# Solution of Problem 2

**marked ideals** $(I, m)$

Aim: for $Z \subset H \subset X$,
$(\pi_H)_*^{-1}(I|_H, m) :=$ trace of $\pi_*^{-1} I$ on $B_Z H$.

| Our blow ups for the triple $(X, I, m, E)$: $Z$ smooth, normal crossing with $E$, $\operatorname{ord}_Z I \geq m$. |
|---|

**New triple** $(X_1, I_1, m, E_1)$

$X_1 = B_Z X$ with $F \subset B_Z X$ except. div.
$(I_1, m) = \pi_*^{-1}(I, m) := \mathcal{O}_{B_Z X}(mF) \cdot \pi^* I$
$E_1 = \pi_*^{-1} E + F$ (last divisor)

Note: for $m = \text{max-ord}\, I$:

| blow up seqs. of order $m$ for $(X, I)$ $\|\|$ blow up seqs. of order $\geq m$ for $(X, I, m)$ |
|---|

## Order reduction for ideals

For $(X, I, E)$ and $m = \text{max-ord}\, I$,
there is $(X', I', E')$ and $\Pi : X' \to X$ s.t.

(1) $\Pi$ is composite of order $m$ blow ups

$$\Pi : (X', I', E') = (X_r, I_r, E_r) \xrightarrow{\pi_{r-1}} \cdots$$
$$(X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E),$$

(2) $\text{max-ord}\, I' < m$, and
(3) functoriality properties.

## Order reduction for marked ideals

For $(X, I, m, E)$, there is
$(X', I', m, E')$ and $\Pi : X' \to X$ s.t.

(1) $\Pi$ is composite of order $\geq m$ blow ups

$$\Pi : (X', I', m, E') = (X_r, I_r, m, E_r) \xrightarrow{\pi_{r-1}} \cdots$$
$$\cdots \xrightarrow{\pi_0} (X_0, I_0, m, E_0) = (X, I, m, E),$$

(2) $\text{max-ord}\, I' < m$, and
(3) functoriality properties.

# Spiraling induction

| |
|---|
| Order reduction, marked ideals, dim $= n-1$ |
| $\Downarrow$ |
| Order reduction, ideals, dim $= n$ |
| $\Downarrow$ |
| Order reduction, marked ideals, dim $= n$ |

Hard: first arrow
Easy: second arrow

| |
|---|
| Order reduction |
| $\Downarrow$ |
| Principalization |

Proof: In $m$ steps, reduce order to 0:
$\Pi_*^{-1} I = \mathcal{O}_{X'}$. Thus
$\Pi^* I = \mathcal{O}_{X'}(-\sum c_i E_i)$ for some $c_i$.

# Structure of the proof

Step 1. Solve Problem 2 using
*marked ideals*

Step 2. Solve Problem 3 using
*maximal contact*

Step 3. Solve Problem 4 for
*$D$-balanced* ideals

Step 4. Solve Problem 5 for
*MC-invariant* ideals

Step 5. Given $I$, find $W(I)$ such that
(i) order reduction for $(X, I, E)$
is equivalent to
order reduction for $(X, W(I), m!, E)$,
(ii) $W(I)$ is $D$-balanced and MC-invariant

Step 6. Complete the spiraling induction.

## Derivative ideals

$$D(I) := \left(\frac{\partial g}{\partial x} : g \in I, x : \text{loc. coord.}\right)$$
$$D^{r+1}(I) := D(D^r(I))$$

$D$ lowers order by 1, so

$$D^r(I, m) := (D^r(I), m - r)$$

## Key computation

Blow up $Z = (x_1 = \cdots = x_r = 0)$:

$$y_1 = \frac{x_1}{x_r}, \ldots, y_{r-1} = \frac{x_{r-1}}{x_r}, y_r = x_r, \ldots, y_n = x_n$$

$$\begin{aligned}
\pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m-1\big) &= \tfrac{\partial}{\partial y_j}\pi_*^{-1}(f,m) \quad \text{for } j < r,\\
\pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m-1\big) &= y_r\tfrac{\partial}{\partial y_j}\pi_*^{-1}(f,m) \quad \text{for } j > r,\\
\pi_*^{-1}\big(\tfrac{\partial}{\partial x_r} f, m-1\big) &= y_r\tfrac{\partial}{\partial y_r}\pi_*^{-1}(f,m)\\
&\quad -y_r \textstyle\sum_{i<r} \tfrac{\partial}{\partial y_i}\pi_*^{-1}(f,m)+\\
&\quad +m \cdot \pi_*^{-1}(f,m)(-1)
\end{aligned}$$

**Corollary:** $\Pi_*^{-1}\big(D^j(I,m)\big) \subset D^j\big(\Pi_*^{-1}(I,m)\big)$

# Solution of Problem 3

**Corollary:** Any order $\geq m$ blow up seq.

$$\Pi : (X', I', m, E') = (X_r, I_r, m, E_r) \xrightarrow{\pi_{r-1}} \cdots$$
$$\cdots \xrightarrow{\pi_0} (X_0, I_0, m, E_0) = (X, I, m, E),$$

gives order $\geq j$ blow up seq.

$$\Pi : (X', J', j, E') = (X_r, J_r, j, E_r) \xrightarrow{\pi_{r-1}} \cdots$$
$$\xrightarrow{\pi_0} (X_0, J_0, j, E_0) = (X, D^{m-j}(I), j, E).$$

**Maximal contact:** $j = 1$ case:

$MC(I) = D^{m-1}(I)$ maximal contact ideal

max-ord $MC(I) = 1$, so for general $h \in I_x$
$H := (h = 0)$ is smooth at $x$ and
if $H$ is smooth (ok on open subset) then

# Going down theorem

| |
|:---:|
| Blow up seqs. of order $m$ for $(X, I)$ |
| $\bigcap$ |
| Blow up seqs. of order $\geq m$ for $(H, I\|_H, m)$ |

# Tuning ideals

**Corollary:** Any order $\geq m$ blow up seq. starting with $(X, I, m, E)$
gives order $\geq \sum_i j_i$ blow up seq. starting with

$$(X, \prod_i D^{m-j_i}(I), \sum_i j_i, E).$$

Definition:

$$W(I) := \left\langle \prod_j \big(D^{m-j}(I)\big)^{c_j} : \sum j \cdot c_j \geq m! \right\rangle$$

Since $W(I) \supset I^{(m-1)!}$, we get

**Theorem**

Order reduction for $(X, I, m, E)$.

$$\Updownarrow$$

Order reduction for $(X, W(I), m!, E)$.

# Derivatives and restriction

*Problem.* Multiplicity jumps in restriction
e.g. $(xy - z^n)|_{(y=0)}$

Defn. $\operatorname{cosupp}(I, m) = \{x : \operatorname{ord}_x I \geq m\}$.

*Problem again:*

$S \cap \operatorname{cosupp}(I, m) \subset \operatorname{cosupp}(I|_S, m)$
and $=$ holds only for $m = 1$.

**Theorem.** $S \subset X$ smooth, then

$$S_r \cap \operatorname{cosupp}\big(\Pi_*^{-1}(I, m)\big) =$$
$$= \bigcap_{j=0}^{m} \operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big((D^j I)|_S, m - j\big)$$

Solution attempt:

$\operatorname{cosupp}(I, m) = \operatorname{cosupp}(D^{m-1}(I), 1)$

*Other problem:* Set $S := (x_1 = 0)$, then
$D(I|_S) \subsetneq D(I)|_S$ since $\partial/\partial x_1$ is lost.

Solution:

(i) Set $D_{\log S} := \langle x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \rangle$
then: $D(I|_S) = D_{logS}(I)|_S$.

(ii) $D^s(I) =$ (well defined as filtration)

$$= D^s_{\log S}(I) + D^{s-1}_{\log S}\left(\frac{\partial I}{\partial x_1}\right) + \cdots + \left(\frac{\partial^s I}{\partial x_1^s}\right)$$

Restrict to $S$:

$$\big(D^s I\big)|_S = D^s\big(I|_S\big) + D^{s-1}\left(\frac{\partial I}{\partial x_1}|_S\right) + \cdots + \left(\frac{\partial^s I}{\partial x_1^s}|_S\right)$$

Apply this to $\pi_*^{-1}(I, m)$ with chart
$y_1 = \frac{x_1}{x_r}, \ldots, y_{r-1} = \frac{x_{r-1}}{x_r}, y_r = x_r, \ldots, y_n = x_n$ :

$$D^s \pi_*^{-1}(I, m) = \sum_{j=0}^{s} D^{s-j}_{\log S_1} \left( \frac{\partial^j \pi_*^{-1}(I, m)}{\partial y_1^j} \right)$$

Usually diff. does **not** commute with birational transforms, **but** it does so for $\partial/\partial x_1$ and $\partial/\partial y_1$, so

$$D^s\pi_*^{-1}(I,m) = \sum_{j=0}^{s} D^{s-j}_{\log S_1}\pi_*^{-1}\left(\frac{\partial^j(I,m)}{\partial x_1^j}\right)$$

For a sequence of blow ups $\Pi$:

$$D^s\Pi_*^{-1}(I,m) = \sum_{j=0}^{s} D^{s-j}_{\log S_r}\Pi_*^{-1}\left(\frac{\partial^j(I,m)}{\partial x_1^j}\right)$$

increasing the summands on the right:

$$D^s\Pi_*^{-1}(I,m) = \sum_{j=0}^{s} D^{s-j}_{\log S_r}\Pi_*^{-1}(D^jI, m-j)$$

restricting to $S_r$:

$$\big(D^s\Pi_*^{-1}(I,m)\big)|_{S_r} =$$
$$= \textstyle\sum_{j=0}^{s} D^{s-j}(\Pi|_{S_r})_*^{-1}\big((D^jI)|_S, m-j\big)$$

For $s = m-1$, take cosupport to get the theorem.

# Solution of Problem 4.

*D-balanced:* $\big(D^j(I)\big)^m \subset I^{m-j} \quad \forall\, j < m$

## Going up theorem

$I$: $D$-balanced, $S \subset X$ smooth such that

(i) $S \not\subset \operatorname{cosupp}(I, m)$, $m = \text{max-ord}\, I$,

(ii) $E|_S$ is normal crossing,

**then:**

| |
|---|
| blow up seqs. of order $m$ for $(X, I, E)$. |
| $\bigcup$ |
| blow up seqs. of order $\geq m$ for $(S, I|_S, m, E|_S)$. |

Proof:

$$\begin{aligned}
&\operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big((D^jI)|_S, m-j\big)\\
&\quad = \operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big((D^jI)^m|_S, m(m-j)\big)\\
&\qquad\qquad (\text{since } \big(D^j(I)|_S\big)^m \subset (I|_S)^{m-j})\\
&\quad \supset \operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big(I^{m-j}|_S, m(m-j)\big)\\
&\quad = \operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big(I|_S, m\big)
\end{aligned}$$

Thus

$$S_r \cap \operatorname{cosupp}\big(\Pi_*^{-1}(I, m)\big) = \operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big(I|_S, m\big)$$

# Going up and down theorem

$I$: $D$-balanced, $H \subset X$ smooth such that

(i) $H$ is maximal contact,

(ii) $H \not\subset \operatorname{cosupp}(I, m)$, $m = \text{max-ord}\, I$

(iii) $E|_H$ is normal crossing,

**then:**

$$\begin{array}{c} \text{blow up seqs. of order } m \text{ for } (X, I, E) \\ \| \\ \text{blow up seqs. of order} \geq m \text{ for } (H, I|_H, m, E|_H) \end{array}$$

**Are we done?**

*Problem:* No global $H$, so

we have open cover $X = \cup X^i$,
on each: $H^i \subset X^i$, smooth max. contact

**How to patch?**

*Solution:*

Make sure blow ups do not depend on $H$.

$R = K[[x_1, \ldots, x_n]]$, $B \subset R$ ideal.
For any $b_i \in B$ and general $\lambda_i \in K$
$(x_1, \ldots, x_n) \mapsto (x_1 + \lambda_1 b_1, \ldots, x_n + \lambda_n b_n)$
is an automorphism.

*Lemma.* For $I \subset R$, equivalent:
(i) $I$ invariant under above automs.
(ii) $B \cdot D(I) \subset I$,
(iii) $B^j \cdot D^j(I) \subset I \ \forall \ j$.

Proof of (iii) $\Rightarrow$ (i): Taylor expansion

$$f(x_1 + b_1, \ldots, x_n + b_n) =$$
$$= f(x_1, \ldots, x_n) + \sum_i b_i \frac{\partial f}{\partial x_i} +$$
$$+ \frac{1}{2} \sum_{i,j} b_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \ldots$$

**Definition.** $I$ is *MC-invariant* if
$$MC(I) \cdot D(I) \subset I$$

# Solution of Problem 5.

**Theorem** Assume:
$I$ is MC-invariant,
$H, H' \subset X$ max. contact, smooth at $x$,
$H + E$ and $H' + E$ both normal crossing

**Then** there is $\phi \in \mathrm{Aut}(\hat{X})$
(where $\hat{X}$ denotes completion) such that
(1) $\phi(\hat{H}) = \hat{H}'$ and $\phi(\hat{E}) = \hat{E}$,
(2) $\phi^*\hat{I} = \hat{I}$ and $\phi^*\big(\hat{I}|_{\hat{H}'}\big) = \hat{I}|_{\hat{H}}$,
(3) for any blow up sequence of order $m$
$(X_r, I_r, E_r) \to \cdots \to (X_0, I_0, E)$
$\phi$ lifts to $\phi_i \in \mathrm{Aut}(X_i \times_X \hat{X})$
which is identity on the center of the
next blow up $Z_i \times_X \hat{X}$.

Proof: Pick $x_2, \dots, x_n$ and $x_1, x_1' \in MC(I)$
such that $H = (x_1 = 0)$, $H' = (x_1' = 0)$,
and $E \subset (x_2 \cdots x_n = 0)$
Apply lemma to:

$$(x_1', x_2, \dots, x_n) \mapsto (x_1' + (x_1 - x_1'), x_2, \dots, x_n) = (x_1, x_2, \dots, x_n)$$

**Theorem.** $W(I)$ is
$D$-balanced and MC-invariant.

Proof. Remember that $W(I) =$
$= \left(\prod_j (D^{m-j}(I))^{c_j} : \sum j \cdot c_j \geq m!\right)$.
By product rule $D^s(W(I)) \subset$
$\subset \left(\prod_j (D^{m-j}(I))^{c_j} : \sum j \cdot c_j \geq m! - s\right)$.
Since $MC(W(I)) = MC(I) = D^{m-1}(I)$,
$MC(W(I))^s \cdot D^s(W(I)) \subset W(I)$.

$D$-balanced: $\left(D^s(W(I))\right)^{m!} \subset W(I)^{m!-s}$
Fix $\sum j \cdot c_j \geq m! - s$, then

$$\begin{aligned}\left(\prod_j (D^{m-j}(I))^{c_j}\right)^{m!} &= \prod_j (D^{m-j}(I)^{m!})^{c_j} \\ &\subset \prod_j (D^{m-j}(I)^{m!/j})^{jc_j} \\ &\subset \prod_j (W(I))^{jc_j} \\ &= W(I)^{\sum j \cdot c_j} \subset W(I)^{m!-s}.\end{aligned}$$

**but** this is *weakly $D$-balanced:*
$\left(D^s(W(I))\right)^{m!}$ is integral over $W(I)^{m!-s}$

2 solutions

(i) weakly $D$-balanced enough (Slide 33)
(ii) more work: $W(I)$ is $D$-balanced

Order reduction, marked ideals, dim $= n - 1$

$\Downarrow$

Order reduction, ideals, dim $= n$

Start with $(X, I, E)$

Step 1. Replace $I$ by $W(I)$, so assume:
$I$ is $D$-balanced and MC-invariant

Step 2. (Local case): there is a smooth maximal contact $H$.

Substep 2.1 (achieve $H+E$ normal crossing)
work with $(E_i, I|_{E_i}, m, (E \setminus E_i)|_{E_i})$
use Going up to get:
Supp $E_i$ disjoint from cosupp$(I, m)$.
Note: get new divisors $E_j$ but they are automatically normal crossing with any $H$

Substep 2.2 ($H + E$ nc along cosupp$(I, m)$)
restrict to $H$: $(H, I|_H, m, E|_H)$
use induction and Going up and down.

*Patching problem:* If $X = X^1 \cup X^2$,
we do the same over $X^1 \cap X^2$,
**but** for blow ups whith centers over
$X^1 \setminus X^2$ or $X^2 \setminus X^1$
we dont know in which order

Step 3. (Quasi projective case)
$C_j \subset X : j \in J$ all possible images of
blow up centers for local order reductions.

*Claim.* $L$ sufficiently ample,
$h \in L \otimes MC(I)$ general, then
$(h = 0)$ has smooth point on every $C_j$.
$\Rightarrow X = \cup_s X^s$ such that
(i) smooth max. contact $H^s \subset X^s \; \forall \; s$,
(ii) each $X^s$ intersects each $C_j$.

**Thus:** order reduction for each $(X^s, I|_{X^s}, E|_{X^s})$
(i) involves every blow up,
(ii) with same total ordering.

**Hence:** automatically globalizes.

Step 4. (Algebraic space)

Write $u : U \to X$ étale, $U$ quasi projective order reduction for $(U, u^*I, u^*E)$ plus étale invariance:

descends to $X$.

(Note: we see that Step 3 was not needed)

Order reduction, ideals, $\dim = n$

$\Downarrow$

Order reduction, marked ideals, $\dim = n$

Difference between $\Pi_*^{-1} I$ and $\Pi_*^{-1}(I, m)$: ideal of exceptional divisor.

*monomial part:* $M(I) :=$ largest $\mathcal{O}_X(-\sum e_i E_i) \subset I$
*nonmonomial part:* $N(I) := M(I)^{-1} \cdot I$.

*Step 1.* (Achieve $\operatorname{max-ord} N(I) < m$)

This is just order reduction for $N(I)$.

Why not go down to $\operatorname{max-ord} N(I) = 0$?

*Answer:* Only mult $\geq m$ blow ups allowed.

So if $\operatorname{max-ord} N(I) = s < m$, we can blow up only points where $\operatorname{ord} I \geq m$.

Reduction trick:

$$\begin{array}{l} \operatorname{ord}_x J_1 \geq s \\ \operatorname{ord}_x J_2 \geq m \end{array} \Leftrightarrow \operatorname{ord}_x(J_1^m + J_2^s) \geq ms$$

*Step 2.* (Achieve max-ord $N(I) = 0$)

Apply order reduction to $N(I)^m + I^s$.

*Step 3.* (Take care of $I = M(I)$)

*Substep 3.1* Blow up $E_i$ with $\operatorname{ord}_{E_i} M(I) \geq m$.

Use index set order to make it functorial.

*Substep 3.2* Blow up $E_i \cap E_j$ with $\operatorname{ord}_{E_i \cap E_j} M(I) \geq m$.

Check: new exceptional divisors have order $< m$.

*Substep 3.3* Blow up $E_i \cap E_j \cap E_k$ with $\operatorname{ord}_{E_i \cap E_j \cap E_k} M(I) \geq m$.

Check: new pairwise intersections have order $< m$.

etc.

# Appendix: Integral dependence

$R$ : ring, $I \subset R$ ideal. $r \in R$ is *integral* over $I$ if

$$r^d + a_1 r^{d-1} + \cdots + a_d = 0 \quad \text{where} \quad a_j \in I^j.$$

All elements integral over $I$: *integral closure:* $\bar{I}$.

*Lemma.* If $J$ is integral over $I$ then $\operatorname{cosupp}(J, m) \supset \operatorname{cosupp}(I, m)$.

Proof. Assume, $r \in \bar{I}$ but $\operatorname{ord}_x r < \operatorname{ord}_x I$.

$$\operatorname{ord}_x(a_1 r^{d-1} + \cdots + a_d) \geq \min_j\{\operatorname{ord}_x(a_i r^i)\}$$
$$\geq (d-1)\operatorname{ord}_x r + \operatorname{ord}_x I > d \cdot \operatorname{ord}_x r,$$

which contradicts the equation.

*Lemma.* If $J$ integral over $I$ then $f_*^{-1}(J, m)$ is integral over $f_*^{-1}(I, m)$.
Proof: Lift the equation.

*Cor.* If $D^j(I)^m$ integral over $I^{m-j}$, then

$$\operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big(D^j(I)^m|_S, m(m-j)\big)$$
$$\supset \operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\big(I^{m-j}|_S, m(m-j)\big).$$

This is what we needed on Slide 22.