Rationality,

Unirationality and

Rational Connectivity

References

János Kollár, *Rational Curves on Algebraic Varieties*

Olivier Debarre, *Higher Dimensional Algebraic Geometry*

To avoid complications: throughout, X will be a smooth projective variety of dimension n over $\mathbb{C}$.

For statements for singular varieties and varieties over field of positive characteristic or non-algebraically closed fields, see Kollár.

Definitions

X is *rational* if there exists a birational map $\mathbb{P}^n \dashleftarrow\dashrightarrow X$; that is, if its function field

$$\mathbb{C}(X) \;\cong\; \mathbb{C}(x_1, \ldots, x_n).$$

X is *unirational* if there exists a dominant rational map $\mathbb{P}^m \dashrightarrow X$ for some m; that is, if its function field

$$\mathbb{C}(X) \;\subset\; \mathbb{C}(x_1, \ldots, x_m).$$

Equivalently, X is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$; that is, if $\mathbb{C}(X) \;\subset\; \mathbb{C}(x_1, \ldots, x_n)$.

Rational Connectivity (Campana; Kollár-Miyaoka-Mori)

X is *rationally connected* if two general points $p, q \in X$ can be joined by a chain of rational curves.

This is equivalent to the a priori stronger condition that *any* finite subset of X is contained in a single rational curve; if $n \geq 3$ we can also require this curve C to be smooth.

For curves and surfaces, the three notions—rationality, unirationality and rational connectivity—coincide.

For higher-dimensional varieties, however, they behave quite differently. The primary goal of this talk will be to give an overview of the properties of rational connectivity, emphasizing the ways in which they differ from (known) properties of rationality and unirationality.

1. *There exists a "local" criterion for rational connectivity:*

Lemma. X is rationally connected iff there exists a map $f : \mathbb{P}^1 \to X$ such that the pullback f^*T_X is ample.

If $n \geq 3$, we can also say that X is rationally connected iff there exists a smooth rational curve $C \subset X$ with ample normal bundle.

As an application, we have:

2. *Rational connectivity is both an open and a closed condition*: we have

Lemma. If $\mathcal{X} \to B$ is a smooth, projective morphism, then the locus

$$\{b \in B : X_b \text{ is rationally connected}\}$$

is both open and closed in B.

It's not known whether rationality is open or closed, though work of Hassett suggests that the answer is "neither" (to be discussed later).

Closed: For any $p_0, q_0 \in X_0$, we can join nearby points $p_b, q_b \in X_b$ by a rational curve $C_b \subset X_b$; the limit of C_b as $b \to 0$ will contain a chain of rational curves in X_0 through p_0 and q_0.

Open: Say X_0 contains a smooth rational curve C with ample normal bundle. From the sequence

$$0 \to N_{C/X_0} \to N_{C/\mathcal{X}} \to (N_{X_0/\mathcal{X}})|_C \to 0$$

we see that the normal bundle of C in $\mathcal{X}$ is generated by global sections; thus C deforms to a curve $C_b \subset X_b$, whose normal bundle remains ample.

3. *There is a divisor-theoretic criterion for rational connectivity:* we have the

Theorem (C, K-M-M). If X is Fano—that is, if $-K_X$ is ample—then X is rationally connected.

Corollary. If $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree d, then X is rationally connected iff $d \leq n + 1$.

(Exercise: prove this corollary using only the two lemmas above.)

Compare this with the situation for rationality: as Kollár points out, no one has ever found a smooth rational hypersurface of degree 4 or more; we don't know if one exists.

As for unirationality, we have the

Theorem (Paranjape, Srinivas): For any d, there exists $N = N(d)$ such that for $n \geq N$, a general hypersurface $X \subset \mathbb{P}^{n+1}$ is unirational.

This was extended to any smooth hypersurface (with substantially larger N) by Mazur, Pandharipande, -

4. There is (conjecturally, at least) a numerical criterion for rational connectivity: it's not hard to see that if X is rationally connected, then all contravariant holomorphic tensors on X vanish, and Mumford has conjectured that the converse should be true as well, i.e.,

Conjecture 1. X is rationally connected iff

$$h^0(X, (T_X^*)^{\otimes m}) = 0 \quad \forall m > 0$$

In this connection, we should also mentioned the related

Conjecture 2. X is uniruled iff

$$h^0(X, K_X^{\otimes m}) = 0 \quad \forall m > 0$$

5. *There exists a "quantitative version" of rational connectivity.*

Suppose we define an equivalence relation on the points of X by saying that $p \sim q$ if p and q are connected by a chain of rational curves. We could try to construct a quotient of X by this relation—that is, a map

$$\pi : X \ \rightarrow \ Z$$

with the properties that
- the fibers of π are rationally connected; and
- all rational curves in X are contained in fibers of π.

Such a quotient can't possibly exist in general (for example, a K3 surface has countably many rational curves).

But suppose we weaken the requirements on the map $\pi : X \to Z$, to

• the fibers of π are rationally connected; and

• *almost* all rational curves in X are contained in fibers of π, in the sense that for very general $z \in Z$, any rational curve in X meeting X_z lies in X_z.

Such a map is called a *maximal rationally connected fibration*, or *mrc fibration* for short; and we have the

Theorem (C, K-M-M). The mrc fibration of X exists and is unique up to birational equivalence.

The target Z of the mrc fibration of X is called the *mrc quotient* of X.

We can also define the *rational dimension* of X as

$$\mathrm{rdim}(X) \; = \; \dim X - \dim Z,$$

or simply as the maximal dimension of a rationally connected subvariety of X through a very general point of X. We see that

$$0 \; \leq \; \mathrm{rdim}(X) \; \leq \; \dim(X)$$

with

- $\mathrm{rdim}(X) > 0$ iff X is uniruled, and

- $\mathrm{rdim}(X) = \dim(X)$ iff X is rationally connected.

6. *Rational connectivity behaves well in fibrations.* Simply, we have the

Theorem (Graber, Starr, -). Let $X \rightsquigarrow Y$ be a dominant map of varieties, with general fiber F. If Y and F are rationally connected, then X is.

This is certainly not the case for rationality: for example, consider the projection of a cubic threefold $X \subset \mathbb{P}^4$ onto a line.

This theorem follows from the (very slightly stronger)

Theorem. Let $f : X \to B$ be a proper morphism of varieties, with B a smooth curve. If the general fiber F of f is rationally connected, then f has a section.

(More about this later)

Two corollaries of this theorem, pointed out by Kollár:

Corollary. If X is any variety, and $f : X \dashrightarrow Z$ its mrc fibration, then Z is not uniruled.

In other words, every variety is uniquely (up to birational equivalence) expressible as a fibration, with rationally connected fibers, over a non-uniruled variety.

Corollary. Conjecture 2 implies Conjecture 1.

An arithmetic question

If X is defined over a number field K and is rational or unirational over K, then its K-rational points are dense. More generally, if X is unirational over $\overline{K}$, it is *potentially dense*: for some finite extension L of K, $X(L)$ is dense.

Question. Is the same true for rationally connected varieties?

More generally, what arithmetic consequences can we deduce from the geometric condition of rational connectivity?

An embarrassing confession

Now that we've spent all this time differentiating the conditions of rationality, unirationality and rational connectivity, it's time to confess:

We don't actually know that the classes of unirational varieties and rationally connected varieties are distinct.

We don't logically need to know the answer to this, but it does raise some interesting questions.

How can we distinguish between unirational varieties and rationally connected varieties?

One possible way:

If X is unirational, then X contains lots of rational subvarieties of every dimension between 1 and $n = \dim(X)$.

This raises the

Question. Does a rationally connected variety necessarily contain any rational subvarieties of dimension > 1?

Consider the following three varieties:

- A general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d = n + 1$, with n large;

- A double cover $X \rightarrow \mathbb{P}^n$ branched over a general hypersurface $B \subset \mathbb{P}^n$ of degree $2n$, again with n large; and

- A general hypersurface $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, d)$, with d large.

The first two are rationally connected by virtue of having $-K_X$ ample; the third by virtue of being a conic bundle over $\mathbb{P}^2$ via projection on the second factor.

Question. Are any of these three varieties unirational?

More specifically, we can ask:

Do the first two contain any rational surfaces?

Does the third contain any rational surfaces other than preimages of rational curves in the second factor?

Totally bogus (but possibly sugges-
tive) **dimension count**:

By way of a preliminary: suppose we
didn't know the theorem of Campana
and Kollár-Miyaoka-Mori, and wanted
to guess whether a hypersurface $X \subset
\mathbb{P}^{n+1}$ of degree d was rationally con-
nected. We could ask:

What is the expected dimension of the
space of rational curves $C \subset X$ of de-
gree e?

The space of rational curves $C \subset \mathbb{P}^{n+1}$ of degree e has dimension

$$(e+1)(n+2) - 4.$$

The number of conditions for such a curve to lie on X is

$$e \cdot d + 1.$$

So the expected dimension of the family of rational curves of degree e on X is

$$e(n + 2 - d) + n - 3.$$

(We could also obtain this number as the Euler characteristic $\chi(N_{C/X})$ of a rational curve C of degree e on a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d.)

Open problem: Is this true for a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d?

A related question, proposed by de Jong:

Question: In case $e = 1$, is this true for an arbitrary smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \leq n + 1$?

Outcome: If the expected dimension of the family of rational curves of degree e on X is

$$e(n + 2 - d) + n - 3,$$

then

- If $d < n+2$, we expect lots of rational curves, more as e increases; and

- If $d \geq n + 2$, we expect that the rational curves on X will never fill up X.

In other words, the dimension count suggests the correct conclusion.

Now let's try it for rational surfaces, for example images of maps $f : \mathbb{P}^2 \to \mathbb{P}^{n+1}$ given by polynomials of degree e:

The space of such surfaces has dimension
$$\binom{e+2}{2}(n+2) - 9.$$
The number of conditions for such a surface to lie on X is
$$\binom{ed+2}{2}.$$
So the expected dimension of the family of such surfaces on X is
$$\frac{n+2-d^2}{2} \cdot e^2 + O(e).$$

Of course, this proves nothing. So we'll just leave it as an open problem:

For which d does a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d contain rational surfaces?