Conformal field theory and the cohomology of the moduli space of stable bundles

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1 Introduction

Let Σ_g be a compact Riemann surface of genus $g \geq 2$, and let Λ be a line bundle over Σ_g of degree 1. Then the moduli space of rank 2 stable bundles V over Σ_g such that $\Lambda^2 V \cong \Lambda$ was shown by Seshadri [19] to be a nonsingular projective variety $\mathcal{N}_g$. Its rational cohomology ring $H^*(\mathcal{N}_g)$ is exceedingly rich and despite years of study has never quite been computed in full. The object of this paper is to give an essentially complete characterization of this ring, or at least to reduce the problem to a matter of linear algebra. In particular, we find a proof of Newstead's conjecture that $p_1^g(\mathcal{N}_g) = 0$. We also obtain a formula for the volume of $\mathcal{N}_g$, which can be regarded as a twisted version of the formula for the degree 0 moduli space recently announced by Witten.

The approach we shall take is not from algebraic geometry but from mathematical physics: it relies on the SU(2) Wess-Zumino-Witten model, which is a functor Z_k associating a finite-dimensional vector space to each Riemann surface with marked points. The relationship with the moduli space is that when the "level" k of the functor is even, the vector space associated to Σ_g with no marked points can be identified with $H^0(\mathcal{N}_g; L^{k/2})$, where L is a fixed line bundle over $\mathcal{N}_g$. Now the work of Verlinde provides us with a means of calculating the dimension of any vector space arising from our functor, and in particular $\dim H^0(\mathcal{N}_g; L^{k/2})$, which we shall denote $D(g,k)$. On the other hand Newstead [14] found explicit generators for $H^*(\mathcal{N}_g)$, and we can also express $D(g,k)$ in terms of them using a Riemann-Roch theorem. Equating the two formulas enables us to evaluate any monomial in the generators on the fundamental class of $\mathcal{N}_g$, and by Poincaré duality this is sufficient, at least in principle, to determine the ring structure of $H^*(\mathcal{N}_g)$.

In the discussion above one important point has been skated over. The SU(2) WZW model is of course associated to bundles of degree 0, not degree 1, so in order to make use of Verlinde's work it is necessary to formulate a "twisted" version of the field theory. This is carried out in §2, but the crucial properties of the twisted theory, analogous to those which make the ordinary theory a modular functor, are not proved in this paper. Rather, we will confine ourselves to exploring the consequences of these claims, and hope to return to justify them in a later paper.

An outline of the remaining sections goes as follows. In §3 we review those parts of Verlinde's work we shall need, show how they must be modified in the twisted case, and work out some explicit formulas for $D(g,k)$. In §4 we study the cohomology of $\mathcal{N}_g$, which we regard throughout as a space of representations via the theorem of Narasimhan and Seshadri. We define Newstead's generators α, β, and ψ_i of $H^*(\mathcal{N}_g)$

and, using Poincaré duality, show that the entire ring structure is determined by those pairings of the form $(\alpha^m\beta^n\gamma^p)[\mathcal{N}_g]$ where $\gamma = 2\sum\psi_i\psi_{i+g}$. A further application of Poincaré duality enables us to eliminate γ recursively from the pairings, so that we need only evaluate $(\alpha^m\beta^n)[\mathcal{N}_g]$. This is calculated in §5 using the results of §3 and the Hirzebruch-Riemann-Roch theorem. Finally we extract Newstead's conjecture and the volume formula from our results.

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2 The WZW model

We begin with a speedy review of the SU(2) Wess-Zumino-Witten model and its gluing rules. The reader is referred to [6], [7], or [17] for a more leisurely account. We must first fix a positive integer k, called the *level*. Let Σ_g be a compact connected Riemann surface of genus g, with marked points $x_1,\ldots,x_p \in \Sigma_g$. We choose a *labelling* of the marked points, that is, we associate to each one an irreducible representation V of $\mathrm{SL}(2,\mathbf{C}) = \mathrm{SU}(2)_{\mathbf{C}}$ with $\dim_{\mathbf{C}} V \leq k+1$. Such representations are determined up to isomorphism by their dimension: we will write V_n for the representation of dimension $n+1$, and denote by $(\Sigma_g; x_i, n_i)$ the Riemann surface with points x_i labelled by V_{n_i}. To each such labelled Riemann surface we wish to associate a complex vector space $Z_k(\Sigma_g; x_i, n_i)$.

Let P be a smooth principal $\mathrm{SL}(2,\mathbf{C})$ bundle over Σ_g; then P is unique up to isomorphism, and the set $\mathcal{A}$ of compatible holomorphic structures on the associated rank 2 vector bundle U is an affine space modelled on $\Omega^{0,1}_{\mathbf{C}}(\Sigma_g; \mathrm{ad}\ P)$. There exists a line bundle $\mathcal{L}\to\mathcal{A}$ whose fibre at any holomorphic structure is the determinant line of the associated $\overline{\partial}$-operator on U. It is easy to see that the usual action of the gauge group $\mathcal{G} = \Gamma(\mathrm{Ad}\ P)$ on $\mathcal{A}$ lifts to $\mathcal{L}$, so we have an action of $\mathcal{G}$ on the tensor power $\mathcal{L}^k$. Next, at each marked point x_i, choose an identification of the fibre $(\mathrm{Ad}\ P)_{x_i}$ with $\mathrm{SL}(2,\mathbf{C})$. (Actually, Ad P always has a global trivialization in the present case, so we may as well just use a restriction of that.) Then $\mathcal{G}$ also acts on each V_{n_i} via evaluation at the point x_i. We may now define our vector space at level k as a space of $\mathcal{G}$-equivariant sections:

$$Z_k(\Sigma_g; x_i, n_i) = \Gamma_{\mathcal{G}}(\mathcal{L}^k \otimes \bigotimes_i V_{n_i}).$$

If the genus g is at least 2, then the set $\mathcal{A}_s$ of semistable holomorphic structures has complement of codimension at least 2 (see p. 569 of [1]), so we may regard $Z_k(\Sigma_g; x_i, n_i)$ as the space of holomorphic sections of the twisted quotient

$$\mathcal{L}^k|_{\mathcal{A}_s} \otimes_{\mathcal{G}} \bigotimes_i V_{n_i},$$

which is a vector bundle over the moduli space $\mathcal{A}_s/\mathcal{G}$ of semistable $\mathrm{SL}(2,\mathbf{C})$ bundles over Σ_g. However, some care is needed as the moduli space has singularities.

In fact, all we will really want to know about $Z_k(\Sigma_g; x_i, n_i)$ is its dimension, which we shall denote $z_k(\Sigma_g; x_i, n_i)$. Now it is well known to physicists that Z_k is an example of a *modular functor*. In particular, it satisfies the two propositions below, which have important implications for the computation of $z_k(\Sigma_g; x_i, n_i)$.

(1) Proposition. *The value of* $z_k(\Sigma_g; x_i, n_i)$ *is independent of the complex structure on* Σ_g.

Consequently, it is independent of the positions of the x_i and of the trivializations of the fibres over them, because any two choices can be exchanged by a suitable smooth map. Thus we will be justified in henceforth writing simply $z_k(\Sigma_g; n_i)$ for $z_k(\Sigma_g; x_i, n_i)$.

The second proposition gives two *gluing axioms*, which describe the behaviour of z_k under *(i)* the addition of a handle and *(ii)* connect-sum. In light of (1), we don't need to worry about the choice of complex structure.

(2) Proposition.

(i) $\quad z_k(\Sigma_{g+1}; n_i) = \sum_{n=0}^{k} z_k(\Sigma_g; n, n, n_i)$.

(ii) $\quad z_k(\Sigma_{g+g'}; n_i, n_j) = \sum_{n=0}^{k} z_k(\Sigma_g; n, n_i)\, z_k(\Sigma_{g'}; n, n_j)$.

Proofs are essentially contained in [20]. □

The definition of Z_k given above is well known. However, as it stands the definition involves $\mathrm{SL}(2, \mathbf{C})$ bundles over Σ_g, which are topologically trivial. Since we are interested in non-trivial bundles over Σ_g, we shall need to modify and extend the definition slightly. Fortunately, there is a straightforward way to do this. We consider $(\Sigma_g; x_i, n_i)$ as before. Let $\hat{P}$ be a smooth principal $\mathrm{PSL}(2, \mathbf{C})$ bundle over Σ_g with $w_2(\hat{P}) \neq 0$; again, $\hat{P}$ is unique up to isomorphism. The associated $\mathbf{CP}^1$ bundle lifts to a rank 2 vector bundle $\hat{U}$ of degree 1. Let $\hat{\mathcal{A}}$ denote the set of compatible holomorphic structures on $\hat{U}$ which induce a fixed holomorphic structure ω on $\Lambda^2 \hat{U}$. (We didn't need to make this explicit in the untwisted case, because $\Lambda^2 U$ is canonically trivial.) As before we let $\hat{\mathcal{L}}$ be the determinant line bundle. However, we need to be careful in our definition of the gauge group; we let $\hat{\mathcal{G}} = \Gamma(\widetilde{\mathrm{Ad}}\ \hat{P})$, where $\widetilde{\mathrm{Ad}}\ \hat{P}$ is the adjoint bundle with fibre $\mathrm{SL}(2, \mathbf{C})$, *not* $\mathrm{PSL}(2, \mathbf{C})$. As in the untwisted case, we choose identifications of $(\widetilde{\mathrm{Ad}}\ \hat{P})_{x_i}$ with $\mathrm{SL}(2, \mathbf{C})$, and define

$$\hat{Z}_k(\Sigma_g; x_i, n_i) = \Gamma_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}^k \otimes \bigotimes_i V_{n_i}).$$

Again, we write $\hat{z}_k(\Sigma_g; x_i, n_i) = \dim \hat{Z}_k(\Sigma_g; x_i, n_i)$. In analogy with the properties of z_k discussed above, we propose the following.

(3) Claim. *The value of* $\hat{z}_k(\Sigma_g; x_i, n_i)$ *is independent of the complex structure on* Σ_g.

Consequently, it is again independent of the positions of the x_i and the trivializations of the fibres, as well as the choice of the holomorphic structure ω on $\Lambda^2 \hat{U}$. As in the untwisted case, we shall henceforth write simply $\hat{z}_k(\Sigma_g; n_i)$ for $\hat{z}_k(\Sigma_g; x_i, n_i)$.

(4) Claim.

(i) $\hat{z}_k(\Sigma_{g+1}; n_i) = \sum_{n=0}^{k} \hat{z}_k(\Sigma_g; n, n, n_i)$.

(ii) $\hat{z}_k(\Sigma_{g+g'}; n_i, n_j) = \sum_{n=0}^{k} \hat{z}_k(\Sigma_g; n, n_i)\; z_k(\Sigma_{g'}; n, n_j)$.

Despite their crucial importance for us, we will not attempt to prove these claims here. However, it appears that the proofs of the analogous statements given by Tsuchiya et al. [20] could be adapted to this case. To prove part *(ii)* of (2), for example, they define a sheaf $\mathcal{S}$ over a local universal family of curves of genus $g + g'$ near a curve $\Sigma_g \vee \Sigma_{g'}$ with an ordinary double crossing. The sheaf is so constructed that its stalk over the generic smooth curve is

$$\mathcal{S}|_{\Sigma_{g+g'}} = Z_k(\Sigma_{g+g'}; n_i, n_j),$$

but over the singular curve is

$$\mathcal{S}|_{\Sigma_g \vee \Sigma_{g'}} = \bigoplus_{n=0}^{k} Z_k(\Sigma_g; n, n_i) \otimes Z_k(\Sigma_{g'}; n, n_j).$$

The desired result is then obtained by showing that $\mathcal{S}$ is locally free. The key step in adapting such a proof to the twisted case would be to note that there exists a principal $\mathrm{PSL}(2, \mathbf{C})$ bundle $\mathcal{P}$ over the local universal family whose restriction to the generic curve is twisted, but whose restriction to the singular curve $\Sigma_g \vee \Sigma_{g'}$ is twisted over Σ_g, but untwisted over $\Sigma_{g'}$. Of course, similar reasoning would suggest that

(5) $$z_k(\Sigma_{g+g'}; n_i, n_j) = \sum_{n=0}^{k} \hat{z}_k(\Sigma_g; n, n_i)\; \hat{z}_k(\Sigma_{g'}; n, n_j),$$

and we shall see in the next section that this is indeed the case.

3 Some calculations

In one respect the twisted theory described above is simpler than the familiar untwisted one: if the genus g is at least 2, then not only does the set $\hat{\mathcal{A}}_s$ have complement of codimension at least 2, as before, but now the moduli space $\hat{\mathcal{A}}_s/\hat{\mathcal{G}}$ is known [19] to be a compact complex manifold $\mathcal{N}_g$. The section $-I \in \hat{\mathcal{G}}$ now acts nontrivially on $\hat{\mathcal{L}}$ as well as the V_{n_i}, so that $\hat{\mathcal{L}}^k \otimes \bigotimes V_{n_i}$ descends to $\mathcal{N}_g$ if and only if $k + \sum n_i$ is even. When this is the case, we need have no qualms about identifying $\hat{Z}_k(\Sigma_g; n_i)$ with the space of holomorphic sections of a vector bundle over $\mathcal{N}_g$. In particular, let L denote the line bundle over $\mathcal{N}_g$ such that $\hat{\mathcal{L}}^2$ descends to L. Then if there are no marked points at all, we have for even k

$$\hat{Z}_k(\Sigma_g) = H^0(\mathcal{N}_g; L^{k/2}).$$

This is the link which will enable us to use $\hat{Z}_k$ to study the cohomology of $\mathcal{N}_g$. Hence our object in this section will be to give an explicit formula for $\hat{z}_k(\Sigma_g)$.

From (4) we deduce

$$\begin{aligned}\hat{z}_k(\Sigma_g) &= \sum_n \hat{z}_k(\Sigma_{g-1}; n, n) \\ &= \sum_{m,n} z_k(\Sigma_{g-1}; m, n)\ \hat{z}_k(\Sigma_0; m, n),\end{aligned}$$

where Σ_0 is of course just the 2-sphere. Thus, provided we understand $\hat{z}_k(\Sigma_0; m, n)$, we can reduce the computation of $\hat{z}_k$ to the computation of z_k, which is well understood and which we shall now describe.

Following Verlinde [21], we define the coefficients

$$N_{m_1,m_2,m_3} = z_k(\Sigma_0; m_1, m_2, m_3),$$

which are symmetric in all three indices. Define the $(k+1) \times (k+1)$ symmetric matrices N_m by

$$(N_m)_{i,j} = N_{m,i,j}.$$

Both here and in future we index the rows and columns starting with 0 instead of 1, for convenience. We then obtain the following formula for the torus with two marked points.

(6) Proposition. $z_k(\Sigma_1; m_1, m_2) = \mathrm{tr}\ N_{m_1} N_{m_2}$.

Proof.

$$\begin{aligned}z_k(\Sigma_1; m_1, m_2) &= \sum_m z_k(\Sigma_0; m, m, m_1, m_2) \\ &= \sum_{m,n} z_k(\Sigma_0; m, n, m_1)\ z_k(\Sigma_0; m, n, m_2) \\ &= \sum_{m,n} (N_{m_1})_{m,n} (N_{m_2})_{n,m} \\ &= \mathrm{tr}\, N_{m_1} N_{m_2}.\end{aligned}$$

□

Now define another $(k+1) \times (k+1)$ symmetric matrix M by $(M)_{i,j} = \mathrm{tr}\,(N_i N_j)$. This enables us to generalize the previous formula to arbitrary genus.

(7) Proposition. *For $g \geq 1$, $z_k(\Sigma_g; m_1, m_2) = (M^g)_{m_1,m_2}$.*

Proof by induction. Proposition (6) is the case $g = 1$. Then, if $z_k(\Sigma_{g-1}; m_1, m_2) = (M^{g-1})_{m_1,m_2}$, by attaching a handle we obtain

$$\begin{aligned}z_k(\Sigma_g; m_1, m_2) &= \sum_n z_k(\Sigma_{g-1}; m_1, n)\ z_k(\Sigma_1; n, m_2) \\ &= \sum_n (M^{g-1})_{m_1,n} (M)_{n,m_2} \\ &= (M^g)_{m_1,m_2}.\end{aligned}$$

□

(8) Corollary. $z_k(\Sigma_g) = \mathrm{tr}\, M^{g-1}$.

Proof. $z_k(\Sigma_g) = \sum z_k(\Sigma_{g-1}; m, m) = \sum (M^{g-1})_{m,m} = \mathrm{tr}\, M^{g-1}$. □

It is worth mentioning that the coefficients N_{m_1,m_2,m_3} can actually be calculated explicitly:

(9) PROPOSITION.

$$N_{m_1,m_2,m_3} = \begin{cases} 1 & \textit{if } |m_1 - m_2| \le m_3 \le \min(m_1+m_2, k-m_1-m_2) \textit{ and } m_1+m_2+m_3 \textit{ is even} \\ 0 & \textit{otherwise.} \end{cases}$$

The proof closest to our point of view appears in [7]; a fuller discussion can be found in [8]. However, we shall not use the coefficients in this form, but rather make use of the Verlinde conjecture, which states that the matrices N_m can be simultaneously diagonalized. Let S be the $(k+1) \times (k+1)$ matrix such that

$$(S)_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right).$$

Note that S is orthogonal and symmetric, so $S = S^{-1}$.

(10) PROPOSITION.

$$SN_mS = \mathrm{diag}\left(\frac{(S)_{n,m}}{(S)_{0,m}}\right).$$

Proofs are given in [2], [8], and [22]. □

(11) COROLLARY.

$$SMS = \mathrm{diag}\left(\frac{1}{(S)_{0,m}^2}\right).$$

Proof.

$$\begin{aligned}(SMS)_{i,j} &= \sum_{m,n} (S)_{i,m} \,\mathrm{tr}\,(N_m N_n)(S)_{n,j} \\ &= \sum_{m,n,p,q} (S)_{i,m} N_{m,p,q} N_{n,p,q} (S)_{n,j} \\ &= \sum_p (SN_p^2 S)_{i,j}.\end{aligned}$$

□

(12) COROLLARY.

$$z_k(\Sigma_g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{m=1}^{k+1} \frac{1}{(\sin\frac{m\pi}{k+2})^{2g-2}}.$$

Proof. (8) and (11). □

In order to find a similar formula for $\hat{z}_k$, as we said above, we need to understand $\hat{z}_k(\Sigma_0; m_1, m_2)$. This can be computed directly.

(13) PROPOSITION. $\hat{z}_k(\Sigma_0; m_1, m_2) = \delta_k^{m_1+m_2}$.

Proof. In this case the action of $\hat{\mathcal{G}}$ on $\hat{\mathcal{A}}$ is well understood [1]. The orbits are indexed by positive integers n; $\hat{\mathcal{A}}_n$ consists of those holomorphic structures which are isomorphic to $\mathcal{O}(-n) \oplus \mathcal{O}(n+1)$. The complex codimension of $\hat{\mathcal{A}}_n$ in $\hat{\mathcal{A}}$ is $2n$. In particular $\hat{\mathcal{A}}_0$ is a dense open orbit. Hence any equivariant section over $\hat{\mathcal{A}}_0$ will extend over $\hat{\mathcal{A}}$ by the Hartogs theorem. That is,

$$\hat{Z}_k(\Sigma_0; m_1, m_2) = \Gamma_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}^k|_{\hat{\mathcal{A}}_0} \otimes V_{m_1} \otimes V_{m_2}).$$

However, since $\hat{\mathcal{A}}_0$ is a single orbit of $\hat{\mathcal{G}}$, such an equivariant section is determined by its value at a single fixed $a \in \hat{\mathcal{A}}_0$. Conversely, any element of $\hat{\mathcal{L}}_a^k \otimes V_{m_1} \otimes V_{m_2}$ which is fixed by $\hat{\mathcal{G}}_a$, the stabilizer of a, can be moved around by $\hat{\mathcal{G}}$ to give an equivariant section. Hence

$$\hat{Z}_k(\Sigma_0, m_1, m_2) = \Gamma_{\hat{\mathcal{G}}_a}(\hat{\mathcal{L}}_a^k \otimes V_{m_1} \otimes V_{m_2}). \tag{14}$$

Now $\hat{\mathcal{G}}_a$ consists precisely of the ad P-compatible automorphisms of $\hat{U}$ which are holomorphic with respect to a. Since $a \in \hat{\mathcal{A}}_0$ this means that $\hat{\mathcal{G}}_a$ is isomorphic to the group of holomorphic automorphisms of $\mathcal{O} \oplus \mathcal{O}(1)$ having determinant 1 on each fibre. If x_1, x_2 are our two marked points, we may choose a basis $\{s_1, s_2\}$ of $H^0(\Sigma_0; \mathcal{O}(1))$ such that s_1 vanishes at x_1 and s_2 vanishes at x_2. Then we can write

$$\hat{\mathcal{G}}_a = \left\{ \begin{pmatrix} z & 0 \\ y_1 s_1 + y_2 s_2 & z^{-1} \end{pmatrix} : z \in \mathbf{C}^\times, y_i \in \mathbf{C} \right\}, \tag{15}$$

where the rows and columns of the matrix correspond to the decomposition $\mathcal{O} \oplus \mathcal{O}(1)$.

Now since $H^1(\Sigma_0; \mathcal{O} \oplus \mathcal{O}(1)) = 0$, the fibre of the determinant line bundle $\mathcal{L}$ at a is just

$$\mathcal{L}_a^2 = \Lambda^3 H^0((\Sigma_0; \mathcal{O} \oplus \mathcal{O}(1))^*.$$

A straightforward computation shows that, with respect to the presentation in (15), an element of $\hat{\mathcal{G}}_a$ acts on $\mathcal{L}_a$ by multiplication by z. On the other hand, for $i = 1, 2$, the vanishing property of s_i at x_i implies that, under an appropriate identification of $(\widetilde{\mathrm{Ad}}\ \hat{P})_{x_i}$ with $\mathrm{SL}(2, \mathbf{C})$, $\hat{\mathcal{G}}_a$ acts on V_{n_i} via the homomorphism $\hat{\mathcal{G}}_a \longrightarrow \mathrm{SL}(2, \mathbf{C})$ given by

$$\begin{pmatrix} z & 0 \\ y_1 s_1 + y_2 s_2 & z^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} z & 0 \\ y_i & z^{-1} \end{pmatrix}.$$

Let us try to determine the right-hand side of (14) explicitly. Fix a nonzero $t \in \hat{\mathcal{L}}_a^k$; then any element of $\hat{\mathcal{L}}_a^k \otimes V_{m_1} \otimes V_{m_2}$ is of course of the form $t \otimes v$ for $v \in V_{m_1} \otimes V_{m_2}$. The 1-parameter subgroups

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbf{C}^\times \right\} \text{ and } \left\{ \begin{pmatrix} 1 & 0 \\ y_i s_i & 1 \end{pmatrix} : y_i \in \mathbf{C} \right\} \text{ for } i = 1, 2$$

generate $\hat{\mathcal{G}}_a$, so for a section ϕ to be $\hat{\mathcal{G}}_a$-equivariant it is necessary and sufficient to be equivariant with respect to each of these. Now we saw that $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot t = z^k t$, so for equivariance we must have $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot v = z^{-k} v$ for that v such that $\phi(a) = t \otimes v$. In infinitesimal terms, this means that $v \in (V_{m_1} \otimes V_{m_2})(-k)$, the $-k$-weight space of $V_{m_1} \otimes V_{m_2}$ as a representation of the Lie algebra $\mathrm{sl}(2, \mathbf{C})$. A typical element $\begin{pmatrix} 1 & 0 \\ y_1 s_1 & 1 \end{pmatrix}$ of the second subgroup acts trivially on $\hat{\mathcal{L}}_a^k$ and on V_{m_2}; on V_{m_1} it acts as $\begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C})$. Hence for equivariance v must belong to $L_{m_1} \otimes V_{m_2}$, where $L_{m_1} \subset V_{m_1}$ is the subspace on which $\begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix}$ acts trivially. Infinitesimalizing again we recognize L_{m_1} as the lowest weight space $V_{m_1}(-m_1)$. In a similar way we deduce $v \in V_{m_1} \otimes V_{m_2}(-m_2)$. Now since v determines ϕ, we can identify

$$\begin{aligned}
\Gamma_{\hat{\mathcal{G}}_a}(\hat{\mathcal{L}}_a^k \otimes V_{m_1} \otimes V_{m_2}) &= (V_{m_1} \otimes V_{m_2})(-k) \cap V_{m_1}(-m_1) \otimes V_{m_2} \cap V_{m_1} \otimes V_{m_2}(-m_2) \\
&= (V_{m_1} \otimes V_{m_2})(-k) \cap V_{m_1}(-m_1) \otimes V_{m_2}(-m_2) \\
&= (V_{m_1} \otimes V_{m_2})(-k) \cap (V_{m_1} \otimes V_{m_2})(-m_1 - m_2),
\end{aligned}$$

which clearly has dimension $\delta_k^{m_1+m_2}$, q.e.d. □

(16) Corollary. *$\hat{z}_k(\Sigma_g; m_1, m_2) = (M^g J)_{m_1,m_2}$, where J is the $(k+1) \times (k+1)$ matrix with $(J)_{i,j} = \delta_k^{i+j}$.*

Proof. $\hat{z}_k(\Sigma_g; m_1, m_2) = \sum_{m=0}^{k} z_k(\Sigma_g; m_1, m)\ \hat{z}_k(\Sigma_0; m, m_2)$; then use (7) and (13). (In particular, this corollary proves (5), because $J^2 = I$.) □

(17) Corollary. *For $g \geq 1$, $\hat{z}_k(\Sigma_g) = \widetilde{\mathrm{tr}}\ M^{g-1}$, where we define $\widetilde{\mathrm{tr}}\ A = \sum_{i=0}^{k} (A)_{i,k-i}$.*

Of course, this "twisted trace" is not preserved by the diagonalization, so to actually evaluate $\hat{z}_k$ we notice instead that J is also diagonalized by S; in fact, $SJS = \mathrm{diag}\ (-1)^n$. This gives us an analogue of (12).

(18) Corollary.

$$\hat{z}_k(\Sigma_g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{m=1}^{k+1} \frac{(-1)^{m+1}}{(\sin \frac{m\pi}{k+2})^{2g-2}}.$$ □

This formula is the only result obtained so far which will be needed in the sequel. Hence to simplify notation we will in future denote $\hat{z}_k(\Sigma_g)$ simply by $D(g,k)$.

Although it is far from apparent from (18), for fixed g $D(g,k)$ is a polynomial in k with rational coefficients. We shall conclude this section by proving that. We shall closely imitate similar calculations for the untwisted case which appear in an unpublished letter by Don Zagier. Similar results have also been obtained by Dowker [5].

(19) Proposition. *For fixed $g \geq 2$ and even k, $D(g,k)$ equals the coefficient of x^{3g-3} in the power series expansion of*

$$\left(-\frac{k+2}{2}x\right)^{g-1} \left(\frac{x}{\sinh x}\right)^{2g-2} \frac{(k+2)x}{\sinh(k+2)x}.$$

Proof. We first rewrite (18) in terms of roots of unity:

$$D(g,k) = \tfrac{1}{2}(-2k-4)^{g-1} \sum_{\substack{\zeta^{2k+4}=1 \\ \zeta \neq \pm 1}} \frac{-\zeta^{k+2}}{(\zeta - \zeta^{-1})^{2g-2}}.$$

Substituting $\zeta^2 = \lambda$, we get

$$D(g,k) = (-2k-4)^{g-1} \sum_{\substack{\lambda^{k+2}=1 \\ \lambda\neq 1}} \frac{-\lambda^{g-1}\lambda^{(k+2)/2}}{(\lambda-1)^{2g-2}}.$$

Hence

$$\begin{aligned} \frac{D(g,k)}{(-2k-4)^{g-1}} &= \sum_{\substack{\lambda^{k+2}=1 \\ \lambda\neq 1}} \mathrm{Res}_{z=\lambda}\left[\frac{-z^{g-1}}{(z-1)^{2g-2}} \frac{(k+2)z^{(k+2)/2}}{z^{k+2}-1} \frac{dz}{z}\right] \\ &= \mathrm{Res}_{z=1}\left[\frac{z^{g-1}}{(z-1)^{2g-2}} \frac{(k+2)z^{(k+2)/2}}{z^{k+2}-1} \frac{dz}{z}\right] \end{aligned}$$

for $g \geq 2$ by the residue theorem, since then the only poles of the expression in square brackets are at the $(k+2)$nd roots of unity. Substituting $z = e^{2x}$ in the final expression gives

$$\begin{aligned} \frac{D(g,k)}{(-2k-4)^{g-1}} &= \mathrm{Res}_{x=0}\left[\left(\frac{1}{2\sinh x}\right)^{2g-2} \frac{k+2}{\sinh(k+2)x} dx\right] \\ &= \begin{array}{c}\text{coefficient} \\ \text{of } x^{2g-2} \text{ in}\end{array} \frac{1}{2^{2g-2}}\left(\frac{x}{\sinh x}\right)^{2g-2} \frac{(k+2)x}{\sinh(k+2)x}. \end{aligned}$$

□

4 The cohomology of $\mathcal{N}_g$

Let Σ_g be a Riemann surface of genus $g \geq 2$. Fix a holomorphic line bundle Λ over Σ_g of degree 1. Then the moduli space $\mathcal{N}_g$ of stable rank 2 holomorphic vector bundles V over Σ_g with $\Lambda^2 V = \Lambda$ is a compact complex $3g-3$-manifold. We wish to study the structure of the rational cohomology ring $H^*(\mathcal{N}_g)$. In §5, this will essentially be determined completely using proposition (19). First, though, we need to review what is already known about $\mathcal{N}_g$, and to prove some necessary lemmas. A broader discussion of some of the topics mentioned can be found in [1].

Roughly speaking, we wish to identify $\mathcal{N}_g$ as a space of representations of $\pi_1(\Sigma_g)$. So let us choose loops $e_1, e_2, \ldots, e_{2g}$ on Σ_g which generate $\pi_1(\Sigma_g)$ in the usual way, so that $\prod e_i e_{i+g} e_i^{-1} e_{i+g}^{-1} \sim 1$. If we cut out a small disc $D \subset \Sigma_g$, then $\prod e_i e_{i+g} e_i^{-1} e_{i+g}^{-1}$ is homotopic to the boundary circle of D, but is no longer contractible. Hence we lose the relation, and $\pi_1(\Sigma_g - D)$ is the free group on the generators e_i. Now consider the map $\mu : \mathrm{SU}(2)^{2g} \to \mathrm{SU}(2)$ defined by

$$(A_1, A_2, \ldots, A_{2g}) \longmapsto \prod_{i=1}^{g} A_i A_{i+g} A_i^{-1} A_{i+g}^{-1}$$

and in particular the subspace $S_g = \mu^{-1}(-I) \subset \mathrm{SU}(2)^{2g}$. It can be shown [11] that $-I$ is a regular value of μ, so S_g is a smooth $6g-3$-submanifold of $\mathrm{SU}(2)^{2g}$. We may regard an element $\omega \in S_g$ as a representation of $\pi_1(\Sigma_g - D)$ sending the boundary circle to $-I$. It must be irreducible, since if it were reducible to an abelian subgroup, it would send the boundary circle to I. Such a representation gives us a flat connection on an SU(2) bundle over $\Sigma_g - D$ having holonomy $-I$ around D. By passing to the associated rank 2 vector bundle and gluing in a fixed twisted unitary connection over D, we can extend our flat connection ω to a connection on a unitary vector bundle $V \to \Sigma_g$ of degree 1. Then the $(0,1)$ part of the associated covariant derivative is a Cauchy-Riemann operator which induces a holomorphic structure on V. Now the diagonal conjugation action of $\mathrm{SU}(2)/\pm 1 = \mathrm{SO}(3)$ on $\mathrm{SU}(2)^{2g}$ clearly preserves S_g, and by Schur's lemma the restriction of the action is free. Hence the quotient $S_g/\mathrm{SO}(3)$ is a smooth $6g-6$-manifold. But if two representations in S_g are conjugate, then the induced connections on V are isomorphic, and hence so are the holomorphic structures. Hence to each point in $S_g/\mathrm{SO}(3)$ we can associate an isomorphism class of holomorphic bundles over Σ_g. In this context, the celebrated theorem of Narasimhan and Seshadri [18] can be stated as follows.

(20) Theorem. *All the holomorphic bundles constructed in this manner are stable, and the resulting map $\phi : S_g/\mathrm{SO}(3) \longrightarrow \mathcal{N}_g$ is a diffeomorphism.* □

Remark. We will in future identify $\mathcal{N}_g$ with $S_g/\mathrm{SO}(3)$. Under this identification, any diffeomorphism $f : \Sigma_g \to \Sigma_g$ induces a diffeomorphism $\hat{f} : \mathcal{N}_g \to \mathcal{N}_g$. (To be precise, we should require that f preserves a small neighbourhood of the disc D; but this is not a serious restriction, as any diffeomorphism of Σ_g is isotopic to one of this form.)

To obtain distinguished cohomology classes in $\mathcal{N}_g$, we construct a vector bundle $\mathcal{U}$ over $\mathcal{N}_g \times \Sigma_g$, as follows. Let $\tilde{\Sigma}_g$ be the universal cover of Σ_g, and consider the twisted quotient

$$\tilde{\Sigma}_g \times_{\pi_1(\Sigma_g)} (S_g \times \mathrm{sl}(2, \mathbf{C})).$$

The action on the right-hand factor is given by $h(\rho, v) = (\rho, \mathrm{ad}\ \rho(h) \cdot v)$, where we regard $\rho \in S_g$ as determining an SO(3)-representation of $\pi_1(\Sigma_g)$. This gives us a vector bundle over $S_g \times \Sigma_g$; to see that it descends to $\mathcal{N}_g \times \Sigma_g$, note that the conjugation action of SO(3) on S_g lifts to an action on this twisted quotient, given by

$$T \cdot (s \times (\rho, v)) = (s \times (T\rho T^{-1}, \mathrm{ad}\ T \cdot v)).$$

The resulting vector bundle $\mathcal{U}$ is clearly *natural* in the sense that, for any diffeomorphism $f : \Sigma_g \to \Sigma_g$, we have $(\hat{f} \times f)^*(\mathcal{U}) \cong \mathcal{U}$. (In fact, it is also *universal*, that is, it has a connection in the fibre direction whose restriction to $\{\omega\} \times \Sigma_g$ is isomorphic to ad ω. It is even possible to perform a similar construction in the holomorphic setting—for details see [1].) Because $\mathcal{N}_g$ is simply connected [13], we may write

$$c_2(\mathcal{U}) = -2\alpha\,\sigma + \beta - 4\psi,$$

where σ denotes the fundamental class in $H^2(\Sigma_g)$ and

$$\alpha \in H^2(\mathcal{N}_g);\ \ \beta \in H^4(\mathcal{N}_g);\ \ \psi \in H^3(\mathcal{N}_g) \otimes H^1(\Sigma_g).$$

(The scalar factors are inserted to agree with the conventions of [14].) The Poincaré duals of our loops e_i form a basis $e^1, e^2, \ldots, e^{2g}$ of $H^1(\Sigma_g, \mathbf{Z})$; using this, we can decompose

$$\psi = \sum_{i=1}^{2g} \psi_i e^i,$$

where $\psi_i \in H^3(\mathcal{N}_g)$. We can now state the following crucial result.

(21) THEOREM (Newstead). *The ring $H^*(\mathcal{N}_g)$ is generated by α, β, and the ψ_i.* □

Remark. Atiyah and Bott [1] later showed that the integral cohomology $H^*(\mathcal{N}_g, \mathbf{Z})$ is torsion-free, so that rational multiples of the same classes generate the integral cohomology. Moreover, the ψ_i are actually integral generators of $H^3(\mathcal{N}_g, \mathbf{Z})$, a fact which we will be needing later.

Since we have a set of generators, to determine the ring structure completely it suffices to find the relations. The usual commutation relations hold, so this amounts to deciding for which nonnegative integers $m_j, n_j, p_{i,j}$ we have

$$\sum_j \alpha^{m_j} \beta^{n_j} (\prod_i \psi_i^{p_{i,j}}) = 0. \tag{22}$$

But according to Poincaré duality, $\mu \in H^m(\mathcal{N}_g) = 0$ if and only if $\mu\nu = 0$ for all $\nu \in H^{6g-6-m}(\mathcal{N}_g)$. Hence in principle to decide when equality holds in (22) we need only evaluate the integers

$$\alpha^m \beta^n (\prod_i \psi_i^{p_i})[\mathcal{N}_g] \tag{23}$$

for those m, n, p_i such that

$$2m + 4n + 3\sum_i p_i = 6g - 6.$$

A more general discussion of these top-dimensional pairings, and of their analogues in four dimensions, can be found in [4]. Our goal will be to find an explicit formula for (23).

The first thing to notice is the following.

(24) PROPOSITION. *Let $1 \le i_0 \le g$. If $p_{i_0} > 1$, or if $p_{i_0} \ne p_{i_0+g}$, then the pairing in (23) is zero.*

Proof. The first part is easy, since the commutation relations imply $\psi_{i_0}^2 = 0$. This only leaves the case $p_{i_0} + p_{i_0+g} = 1$ for the second part. Now if $f : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism, then by naturality

$$\hat{f}^*(\alpha)^m \hat{f}^*(\beta)^n (\prod_i \hat{f}^*(\psi_i)^{p_i})[\mathcal{N}_g] = \alpha^m \beta^n (\prod_i \psi_i^{p_i})[\mathcal{N}_g].$$

However, since $(\hat{f} \times f)^* \mathcal{U} \cong \mathcal{U}$, we have

$$\hat{f}^*(\alpha) = \alpha, \;\; \hat{f}^*(\beta) = \beta, \;\; \sum_i \hat{f}^*(\psi_i) f^*(e^i) = \sum_i \psi_i e^i.$$

Now it is easy to find a diffeomorphism of Σ_g such that $f^*(e^{i_0}) = -e^{i_0}$, $f^*(e^{i_0+g}) = -e^{i_0+g}$, but $f^*(e^i) = e^i$ for all other i. For example, a half twist of the surface in Figure 1 below the loop labelled has the desired properties. If $p_{i_0} + p_{i_0+g} = 1$, we conclude

$$\alpha^m \beta^n (\prod_i \psi_i^{p_i})[\mathcal{N}_g] = -\alpha^m \beta^n (\prod_i \psi_i^{p_i})[\mathcal{N}_g] = 0.$$

□

For $1 \leq i \leq g$, define $\gamma_i = \psi_i \psi_{i+g}$. Then the proposition above shows that all the pairings (23) are zero except those of the form

$$(\alpha^m \beta^n \gamma_{i_1} \gamma_{i_2} \ldots \gamma_{i_p})[\mathcal{N}_g] \tag{25}$$

where $m + 2n + 3p = 3g - 3$ and $1 \leq i_1 < i_2 < \cdots < i_p \leq g$. Actually, the value of (25) is independent of the choice of i_j. This follows from a diffeomorphism argument similar to that in the proof of (24), because any permutation of the handles in figure 1 can be realized by a diffeomorphism. Consequently, any pairing of the form (25) is equal to $(g-p)!/(2^p g!)$ times the even simpler expression

$$(\alpha^m \beta^n \gamma^p)[\mathcal{N}_g],$$

where $\gamma = 2\sum_{i=1}^{g} \gamma_i$. The class γ has the advantage of being independent of our choice of basis $\{e^i\}$, since $\psi^2 = \gamma\,\sigma$.

Still using the identification $\mathcal{N}_g = S_g/\mathrm{SO}(3)$, we can perform another construction that would not have made sense in the holomorphic setting. The map from Σ_g to Σ_{g-1} which collapses the ith handle to a point (see figure 2) is not holomorphic, but it induces an embedding $\eta_i : \mathcal{N}_{g-1} \to \mathcal{N}_g$. The image of this embedding consists precisely of those conjugacy classes of representations of $\pi_1(\hat{\Sigma}_g)$ which send the generators e_i and e_{i+g} to the identity. That is, $\eta_i(\mathcal{N}_{g-1}) = p(\pi_i^{-1}(I) \cap \pi_{i+g}^{-1}(I))$, where π_i is the restriction to S_g of the projection of $\mathrm{SU}(2)^{2g}$ on the ith factor, and p is the quotient map $S_g \to \mathcal{N}_g$. Since $\dim_{\mathbf{R}} \mathcal{N}_g = 6g - 6$, $\mathcal{N}_{g-1}$ has real codimension 6.

(26) Proposition. *$\eta_i(\mathcal{N}_{g-1})$ is Poincaré dual to γ_i. Consequently,*

$$\alpha^m \beta^n \gamma^p [\mathcal{N}_g] = 2g\ \alpha^m \beta^n \gamma^{p-1}[\mathcal{N}_{g-1}].$$

Proof. If we fix an element $(A_1, A_2, \ldots, A_{2g-2}) \in S_{g-1}$, then the map $\mathrm{SU}(2) \to S_g$ given by

$$T \longmapsto (A_1, \ldots, A_{i-1}, T, A_i, \ldots, A_{i+g-1}, T^{-1}, A_{i+g}, \ldots, A_{2g-2})$$

is a right inverse for π_i. There is a similar right inverse for π_{i+g}. Hence the pullbacks by π_i and π_{i+g} of the fundamental cohomology class of $\mathrm{SU}(2)$ are indivisible classes $\chi_i, \chi_{i+g} \in H^3(S_g, \mathbf{Z})$, Poincaré dual to $\pi_i^{-1}(I)$ and $\pi_{i+g}^{-1}(I)$ respectively. We claim that $\chi_i = \pm p^*(\psi_i)$ and $\chi_{i+g} = \pm p^*(\psi_{i+g})$.

To prove this, note that $\pi_i^{-1}(I)$ is a union of fibres of the $\mathrm{SU}(2)$-action on S_g, so $\chi_i = p^*(\hat{\psi}_i)$ for some $\hat{\psi}_i \in H^3(\mathcal{N}_g, \mathbf{Z})$. This $\hat{\psi}_i$ is unique, because we can see from the Leray-Serre spectral sequence (or even the Gysin sequence) that the natural map $H^3(\mathcal{N}_g, \mathbf{Z}) \to \mathbf{H^3(S_g, Z)}$ is injective. Now for any e^j in our basis for $H^1(\Sigma_g, \mathbf{Z})$ with

$j \neq i$, there exist diffeomorphisms $f : \Sigma_g \to \Sigma_g$ such that $f^*(e^i) = e^i$ but $f^*(e^j) \neq e^j$: the Dehn twists and half twists in figure 3 do the trick. Thus the only classes in $H^3(\mathcal{N}_g, \mathbf{Z})$ which are invariant under f^* for all such f are the integer multiples of ψ_i. But χ_i is invariant under such f^* by construction, and hence so is $\hat{\psi}_i$. Hence by the second part of (21), $\hat{\psi}_i$ is an integer multiple of ψ_i. Since both ψ_i and $\hat{\psi}_i$ are indivisible, we must have $\hat{\psi}_i = \pm\psi_i$. Likewise, $\hat{\psi}_{i+g} = \pm\psi_{i+g}$.

We shall not bother to pin down the sign, because for our purposes it is sufficient to note that the same sign holds for $\hat{\psi}_i$ and $\hat{\psi}_{i+g}$. This can be deduced from a diffeomorphism argument like the one above, using a Dehn twist that induces $e^i \mapsto e^{i+g}$ and $e^{i+g} \mapsto -e^i$ (see figure 4). Then $\chi_i\chi_{i+g} = p^*(\gamma_i)$, and so descending to $\mathcal{N}_g$ we obtain the first result.

As for the second, it is sufficient to show that $\eta^*(\alpha) = \alpha$, $\eta^*(\beta) = \beta$, and $\eta^*(\gamma) = \gamma$. This is straightforward from the construction of $\mathcal{U}$. □

We can use this proposition recursively to eliminate γ from the pairings. (Actually, to bootstrap we do need to know that $\gamma[\mathcal{N}_2] = 4$, but this follows from Poincaré duality.) Hence it now suffices to evaluate $\alpha^m\beta^n[\mathcal{N}_g]$ when $m + 2n = 3g - 3$. This will be carried out in the next section.

5 A formula for the pairings

So far, we have practically ignored the complex structure on Σ_g, and the complex structure it induces on $\mathcal{N}_g$. In this section we shall use that complex structure, together with what we already know, to calculate $(\alpha^m\beta^n\gamma^p)[\mathcal{N}_g]$. Recall that in §3, we gave a formula for $D(g,k) = \dim H^0(\mathcal{N}_g; L^{k/2})$, where L was a positive line bundle over $\mathcal{N}_g$, and k was an even number. However, instead of the approach we took through mathematical physics, we could have gone through algebraic geometry, using the Hirzebruch-Riemann-Roch theorem. This tells us that

$$\sum_i (-1)^i \dim H^i(\mathcal{N}_g; L^{k/2}) = (\operatorname{ch} L^{k/2} \operatorname{td} \mathcal{N}_g)[\mathcal{N}_g]. \tag{27}$$

Since the canonical bundle of $\mathcal{N}_g$ is negative [14], the left-hand side equals $D(g,k)$ by Kodaira vanishing. On the other hand, we can calculate the right-hand side in terms of α and β. Newstead [14] showed that $c_1(\mathcal{N}_g) = 2\alpha$ and $p(\mathcal{N}_g) = (1+\beta)^{2g-2}$. By restricting L to a projective subspace of $\mathcal{N}_g$ as in [14], it is not hard to show that $c_1(L) = \alpha$. As for $\operatorname{td} \mathcal{N}_g$, the identity

$$\frac{x}{1-e^{-x}} = \exp(\tfrac{1}{2}x)\frac{\frac{1}{2}x}{\sinh \frac{1}{2}x}$$

implies (see p. 117 of [16]) that, if y_i are the Pontrjagin roots,

$$\operatorname{td} = \exp(\tfrac{1}{2}c_1)\prod_i \frac{\frac{1}{2}\sqrt{y_i}}{\sinh \frac{1}{2}\sqrt{y_i}}.$$

Noting that $2g-2$ of the Pontrjagin roots of $\mathcal{N}_g$ are β and the rest are 0, we obtain

$$\operatorname{td} \mathcal{N}_g = \exp(\alpha)\left(\frac{\frac{1}{2}\sqrt{\beta}}{\sinh \frac{1}{2}\sqrt{\beta}}\right)^{2g-2}.$$

Then by (27)

$$D(g,k) = \left(\exp\left(\frac{k+2}{2}\alpha\right) \left(\frac{\frac{1}{2}\sqrt{\beta}}{\sinh \frac{1}{2}\sqrt{\beta}}\right)^{2g-2} \right) [\mathcal{N}_g] \tag{28}$$

for even k. For fixed g, this is a polynomial in k whose coefficients involve the pairings. On the other hand, our original formula (19) is likewise a polynomial in k. Substituting $\ell = k+2$ in (19) and (28), and equating coefficients of ℓ^m, we get

$$-(-\tfrac{1}{2})^{g-1} \, \frac{(2^{m-g+1}-2)}{(m-g+1)!} \, B_{m-g+1} \, P_{3g-3-m} = \frac{1}{m!\, 2^{3g-3}} \, P_{3g-3-m}(\alpha^m \beta^{\frac{1}{2}(3g-3-m)})[\mathcal{N}_g],$$

where B_i is the ith Bernoulli number and P_i is the coefficient of x^i in the power series expansion of $(x/\sinh x)^{2g-2}$. (We use the conventions under which $B_2 = 1/6$, $B_4 = -1/30$, etc., and we interpret $B_i = 0$ if $i < 0$, because $(k+2)x/\sinh(k+2)x$ has no pole at 0.) Cancelling P_{3g-3-m} and rearranging, we obtain

$$(\alpha^m \beta^n)[\mathcal{N}_g] = (-1)^{g-1} \frac{m!}{(m-g+1)!} \, 2^{2g-2} \, (2^{m-g+1} - 2) \, B_{m-g+1}. \tag{29}$$

In particular, we get the volume formula:

$$\operatorname{vol} \mathcal{N}_g = \frac{1}{(3g-3)!} \, \alpha^{3g-3}[\mathcal{N}_g] = \frac{1}{(2g-2)!} \, 2^{2g-2} \, (2^{2g-2} - 2) \, |B_{2g-2}|.$$

Finally, we combine (29) with (26) to obtain, for $m + 2n + 3p = 3g - 3$,

$$(\alpha^m \beta^n \gamma^p)[\mathcal{N}_g] = (-1)^{p-g} \frac{m!}{(m+p-g+1)!} \, \frac{g!}{(g-p)!} \, 2^{2g-2-p} \, (2^{m+p-g+1} - 2) \, B_{m+p-g+1}.$$

This agrees with the results published for genus 2 [13] and 3 [15]. For example, the number 224 obtained by Ramanan is just $\alpha^6[\mathcal{N}_3]$. It also enables us to prove Newstead's conjecture that $\beta^g = 0$. This conjecture first appeared in [14] and is discussed at greater length in [1]; there is no proof in the literature, but Frances Kirwan [12] has recently found a proof using equivariant cohomology. In any case, though, it is manifest from our formula, since $n > g-1$ implies $m+p-g+1 < 0$. Indeed, we get the more general and unsuspected result that $\beta^{g-q}\gamma^q = 0$ whenever $0 \leq q \leq g$. However, Newstead's other conjecture, proved by Gieseker [9], that $c_r(\mathcal{N}_g) = 0$ for $r > 2g-2$, is not at all obvious from this equation, since we don't know of a systematic way to compute the Chern classes in terms of α, β, γ.

It was promised in the introduction that the determination of the ring structure would be reduced to a problem in linear algebra, and this ought to be explained. To compute the relations in any degree q, we construct a matrix whose columns are indexed by the monomials in α, β, ψ_i of total degree q and whose rows are indexed by similar monomials of complementary total degree $6g-6-q$. The matrix entry at row μ, column ν is given by $\mu\nu[\mathcal{N}_g]$, as determined from the formula above and (24). The rank of this matrix is just the qth Betti number of $\mathcal{N}_g$. (To prove that this rank agrees with the formula for the Betti number given in [1] could be regarded as a hard exercise in number theory.) Finding the relations at degree q is equivalent to finding a basis for the null space of this matrix. Such a basis could be obtained by row-reduction, but without any short cuts this would be prohibitively cumbersome. Luckily, though, for many applications, such as those employing the Riemann-Roch theorem, the top-dimensional pairings are exactly what we need.

Note added in proof. Since this paper was written, Don Zagier has solved the problem posed in the last paragraph, to give a complete set of explicit relations between Newstead's generators.

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