THE NASH BLOW-UP OF A TORIC VARIETY

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In a nutshell:

Is there a canonical resolution of the singularities of a projective variety?

Possibly — by iterating a construction called the Nash blow-up.

No compelling reason to believe except that it works in dimension 1 + 2.

We study special case of toric varieties — don't prove it even there, but amass enormous empirical evidence.

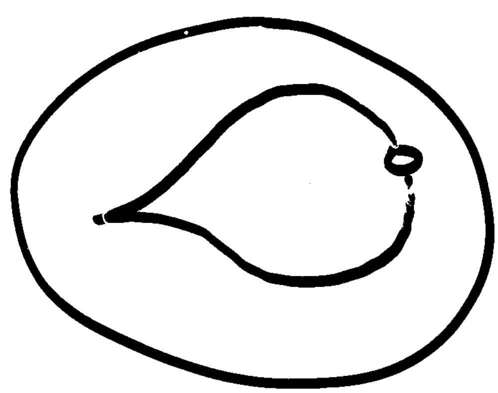
A projective variety

$$X = \{[x] \in \mathbb{CP}^n \mid f_1(x) = \dots = f_k(x) = 0\},$$

$f_i \in \mathbb{C}[x_0, .., x_n]$ homogeneous.

A quasi-projective variety is $Y \setminus Z$, Y, Z projective.

E.g. $x^2 z - y^3 = 0$, $z \neq 0$ in $\mathbb{CP}^2$:



Any projective X has a smooth locus X^{sm}, dense open set where it is a complex manifold.

Thm (Resolution of singularities)
$\forall X \; \exists$ smooth $\tilde{X}$ and $\pi : \tilde{X} \to X$
proper morphism which is an
isomorphism over X^{sm}.

E.g. X as before, $\tilde{X} = \mathbb{C}$,
$\pi(t) = [t^3, t^2, 1]$.

Is there a choice of $\tilde{X}$
that is in any way canonical?

Normalization $\pi : \hat{X} \longrightarrow X$ is
smooth for X a curve, but
not e.g. for $\{x^2+y^2+z^2=0\} \subset$
$\mathbb{CP}^3$.

Defn: the Gauss map $\underline{| d = \dim X^s}$

$$\mathcal{G} : X^{sm} \longrightarrow Gr(d+1, n+1)$$

is the morphism taking each point to its tangent plane.

Defn: the Nash blow-up of X is the (normalization of)

$$\overline{\Gamma(\mathcal{G})} \subset X \times Gr(d+1, n+1).$$

Clearly π : Nash $X \longrightarrow X$ proper.

Question: is any X desingularized by a finite number of Nash blow-ups?

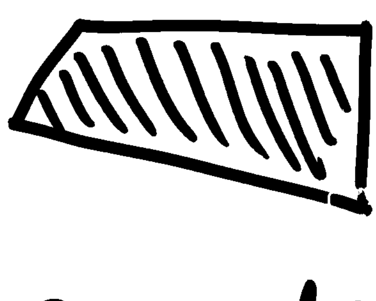
Trivially yes if $d=1$.
Non-trivially yes if $d=2$
(Gonzalez-Sprinberg, Spivakovsky).
Wide open if $d \geq 3$.

Idea: look at special case of toric varieties where it becomes a problem in polyhedral geometry.

Defn: A toric variety is a normal variety of dim d acted on by $(\mathbb{C}^\times)^d$ with a dense orbit.

E.g. $\mathbb{C}^n$, $\mathbb{CP}^n$, $\{xy - z^2 = 0\} \subset \mathbb{C}^3$

Quasi-projective toric varieties classified by polyhedra in $\mathbb{Q}^d$, i.e. loci (bdd or unbdd) defined by finitely many affine inequalities.



The correspondence:

P polyhedron $\longrightarrow$

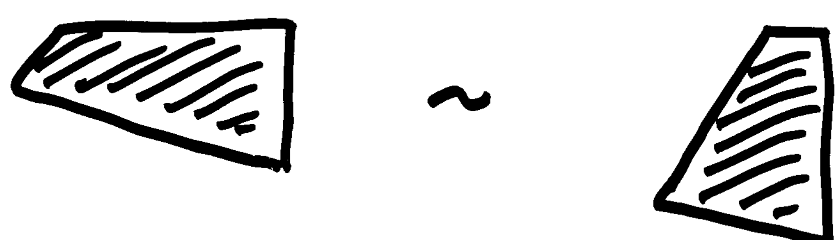
cone $C(P) := \overline{\{(\lambda, \lambda x) \mid \lambda \in \mathbb{Q}_+, x \in P\}} \subset \mathbb{Q} \times \mathbb{Q}^d$

$\longrightarrow$ graded semigroup algebra
$\mathbb{C}[C(P) \cap \mathbb{Z}^{d+1}]$

$\longrightarrow$ toric variety $X(P) :=$
$\operatorname{Proj} \mathbb{C}[C(P) \cap \mathbb{Z}^{d+1}]$

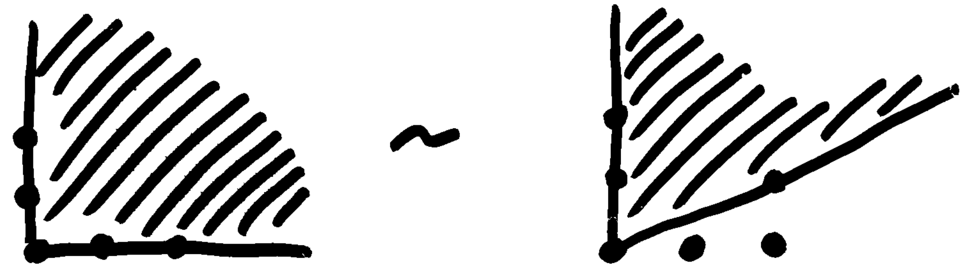
Modulo equivalences:

(1) Translating facets without changing combinatorics of P only changes polarization of $X(P)$, i.e. embedding in $\mathbb{CP}^n$:



(2) Acting by $GL(n, \mathbb{Z})$ only changes $(\mathbb{C}^\times)^d$-action by an element of $\operatorname{Aut}(\mathbb{C}^\times)^d$.

E.g. acting by $\begin{pmatrix}1&0\\2&1\end{pmatrix} \Longrightarrow$



$$X(P) = \mathbb{C}^2$$

In fact, any $X(P)$ has open cover $X(P_i)$, where P_i are cones obtained by localizing (at each vertex), viz.:

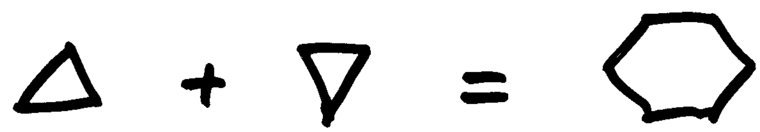
$$X(P) = X(P_1) \cup X(P_2) \cup X(P_3) \cup X(P_4)$$

On the other hand, if P is a cone, $X(P)$ is smooth $\Longleftrightarrow \exists\, A \in GL(n, \mathbb{Z}) \mid AP = \mathbb{Q}_+^d$.

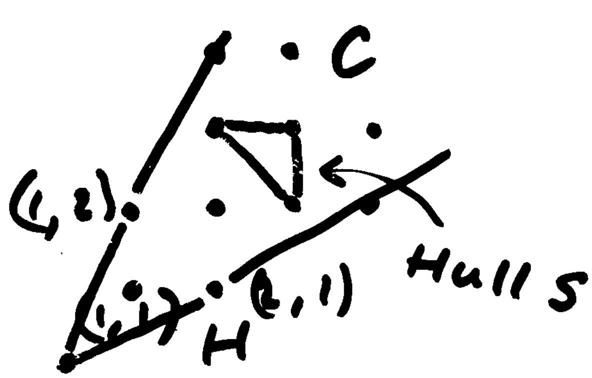
So easy to decide if $X(P)$ smooth.

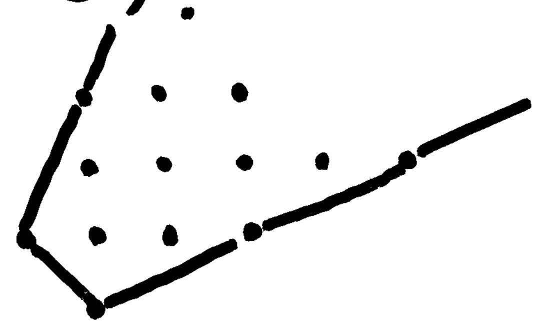
Defn: For polyhedra $P, Q \subset \mathbb{Q}^d$ the Minkowski sum is

$$P + Q := \{p + q \in \mathbb{Q}^n \mid p \in P,\ q \in Q\}.$$



Thm: Given cone $C \subset \mathbb{Q}^d$, let H be the Hilbert basis, i.e. unique minimal generating set, of the semigroup $C \cap \mathbb{Z}^d$.
Let $S = \{v_1 + \cdots + v_d \mid v_i \text{ lin. indep in } H\}$.
Then the Nash blow-up of $X(C)$ is $X(C + \operatorname{Hull} S)$.





C + Hull S

E.g. in 2 dimensions:

⊛ cones $/GL(2,\mathbb{Z})$ of the form $\mathbb{Q}_+\langle(0,1),(b,a)\rangle$, $\frac{a}{b}\in\mathbb{Q}\cap[0,1)$

⊛ Hilbert basis is

$(0,1),(1,1),(b_1,a_1),\dots,(b_k,a_k)$, where $\frac{a_i}{b_i}=[c_1,\dots,c_i]$

= partial expansion of Hirzebruch-Jung continued fraction

$$\frac{a}{b}=1-\cfrac{1}{c_1-\cfrac{1}{c_2-\cfrac{1}{c_3-\cdots c_k}}}=:[c_1,\dots,c_k]$$

⊛ Except in a few cases, denominator b strictly decreases under Nash blow-up, since

$$\frac{a}{b} = [c_1, \dots, c_k]$$

$$\frac{a'}{b'} = [c_i, \dots, c_j], \quad 1 < i < j < k$$

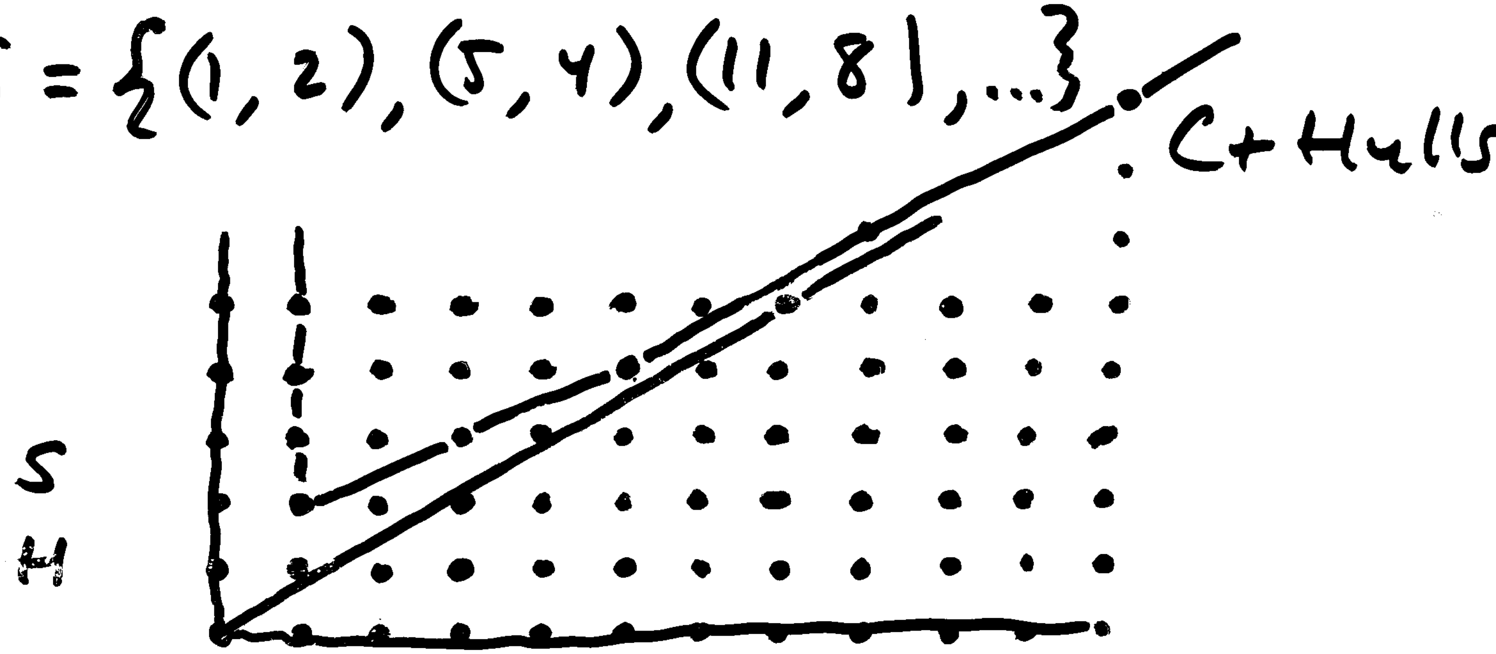
$$\Rightarrow b' < b.$$

E.g. $\frac{a}{b} = \frac{5}{7} = 1 - \cfrac{1}{4 - \cfrac{1}{2}}$

$\frac{a_1}{b_1} = 1 - \frac{1}{4} = \frac{3}{4}$

$H = \{(0,1), (1,1), (4,3), (7,5)\}$

$S = \{(1,2), (5,4), (11,8), \ldots\}$

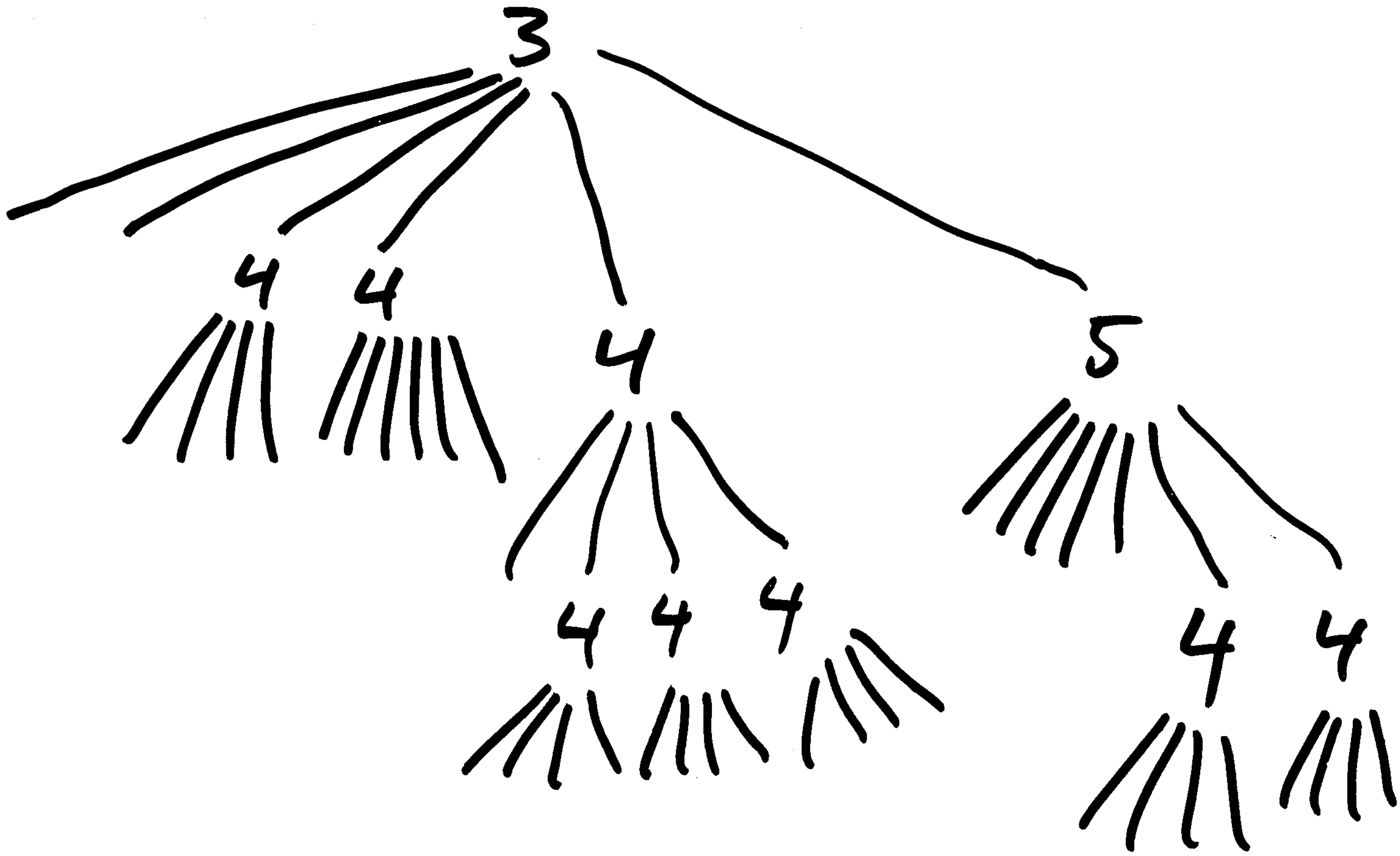


Localization P_2 at $(5,4)$

$(1,2) - (5,4) = (-4,-2) = 2(-2,-1)$

$(11,8) - (5,4) = (6,4) = 2(3,2)$

$\begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} = 1 \Rightarrow X(P_2)$ smooth


3
4
4
4
5
4
4
4
4
4

```
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  [3,3]((0,1,0),(2,3,3),(1,1,1)); lvl=1; h,r-ind=1,1
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    [3,3]((4,7,9),(5,8,10),(3,5,6)); lvl=2; h,r-ind=1,1
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      [3,3]((3,4,5),(2,3,4),(8,12,15)); lvl=3; h,r-ind=1,1
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      [3,3]((4,9,12),(3,7,9),(6,13,17)); lvl=3; h,r-ind=1,1
      [3,3]((2,4,5),(3,6,8),(6,13,17)); lvl=3; h,r-ind=1,1
```

indices of lattices

\# of faces

dimension

depth in tree of Nash blow-ups

linear functionals defining faces of cone