Mathematics V1207x
Honors Mathematics A

Practice Final Exam
December 21, 2015

Turn off all electronic devices.

Brief visits to the men's or women's room (just out the door to your left) are OK, but one at a time only, and you must tell me before you go.

Write your name, "Honors Math A, Prof. Thaddeus," and the number of blue books on the cover of each blue book.

Write your name on the attendance sheet when it comes around.

Write all answers and work in your blue books. Do not hand in this sheet.

On each page you use, write the number of the problem *in a circle in the margin.*

You may do more than one problem on a single page. Just put all numbers by their problems.

You may do the problems out of order, but this is discouraged as it can lead to misgrading.

When there is any doubt, state briefly but clearly what statements from Apostol, lecture, or assignments you are using.

You may use, without comment, facts from logic and high-school algebra.

In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.

PART A: True/False. Decide whether the given statement is true or false, and give a brief reason for your answer (sketch of proof or counterexample). Keep it simple — one or two lines should suffice! 3 points each: 1 for answer, 2 for reason.

1. If f is continuous on $[a, b]$, then so is $|f|$.

2. If f is differentiable on $[a, b]$, then so is $|f|$.

3. If $\lim_{n\to 0} |a_n| = 0$, then $\sum_{n=0}^{\infty} a_n$ converges.

4. The series $\sum_{n=0}^{\infty} e^{-n^2}$ converges.

5. The series $\sum_{n=0}^{\infty} \sin(nx)/2^n$ converges to a continuous function of x.

6. If $A \in M_{m\times n}$ is a matrix such that $T_A = 0$ (where T_A is the corresponding linear map $\mathbb{R}^n \to \mathbb{R}^m$), then $A = 0$.

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PART B: Statements. State in a complete sentence including all hypotheses. 5 points each.

7. State the small-span theorem.

8. State the theorem on uniform convergence and integration.

9. Define (a) convergence of a series; (b) absolute convergence of a series.

10. Define the standard basis vectors in $\mathbb{R}^n$.

PART C: Proofs. Give complete proofs of as many as possible. 10 points each.

11. Let $f : S \to T$ be a bijection of sets, and let $g : T \to S$ be a function such that $g \circ f = \mathrm{id}_S$. Prove that $f \circ g = \mathrm{id}_T$.

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f(0)$ and $f(2)$ are positive, but $f(1)$ is negative. Show that $f'(x) = 0$ for some x.

13. Let $f(x) = \sin x + 2x$. Prove that $f : \mathbb{R} \to \mathbb{R}$ is a bijection.

14. Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$.

15. Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = n$ if $0 < x < 1/n$ and $f_n(x) = 0$ otherwise. Show that $f_n \to 0$ pointwise but not uniformly.

16. Prove that for every linear map $f : \mathbb{R}^n \to \mathbb{R}$, there exists $B \in \mathbb{R}^n$ such that for all $X \in \mathbb{R}^n$, $f(X) = B \cdot X$.

17. Let V be the set consisting of all step functions $[0,1] \to \mathbb{R}$.

(a) Prove that V is a vector space.

(b) Prove that the indefinite integral gives a linear map $V \to \mathcal{F}([a,b], \mathbb{R})$.

(c) Is this map is injective? Why or why not?