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Units of ring spectra, orientations, and Thom spectra via rigid infinite loop space theory

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Abstract

We extend the theory of Thom spectra and the associated obstruction theory for orientations in order to support the construction of the E_∞ string orientation of tmf, the spectrum of topological modular forms. Specifically, we show that, for an E_∞ ring spectrum A, the classical construction of $gl_1 A$, the spectrum of units, is the right adjoint of the functor

$$\Sigma^\infty_+\Omega^\infty \colon \operatorname{ho}(\text{connective spectra}) \longrightarrow \operatorname{ho}(E_\infty \text{ ring spectra}).$$

To a map of spectra

$$f \colon b \longrightarrow bgl_1 A,$$

we associate an E_∞ A-algebra Thom spectrum Mf, which admits an E_∞ A-algebra map to R if and only if the composition

$$b \longrightarrow bgl_1 A \longrightarrow bgl_1 R$$

is null; the classical case developed by May, Quinn, Ray, and Tornehave arises when A is the sphere spectrum. We develop the analogous theory for A_∞ ring spectra: if A is an A_∞ ring spectrum, then to a map of spaces

$$f \colon B \longrightarrow BGL_1 A,$$

we associate an A-module Thom spectrum Mf, which admits an R-orientation if and only if

$$B \longrightarrow BGL_1 A \longrightarrow BGL_1 R$$

is null. Our work is based on a new model of the Thom spectrum as a derived smash product.

1. *Introduction*

In a forthcoming paper [**3**], three of us (Ando, Hopkins, Rezk) construct an E_∞ string orientation of tmf, the spectrum of topological modular forms: more precisely, we construct a map of E_∞ ring spectra from the Thom spectrum $MO\langle 8\rangle$, also known as $MString$, to the

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spectrum tmf, whose value on homotopy rings refines the Witten genus from $\pi_* MString$ to the ring of integral modular forms for $\mathrm{SL}_2\mathbb{Z}$. As explained by Hopkins in his ICM address [**14**], the argument requires a new formulation of the obstruction theory for orientations of [**27**] in terms of the adjoint relationship between the units of a commutative ring spectrum and $\Sigma^\infty_+\Omega^\infty$. A central goal of this paper is to establish this formulation.

This new picture of the obstruction theory is motivated by a description of the Thom spectrum originally due to the fourth author. Another purpose of the paper is to study this construction of the Thom spectrum. For example, we use it to extend the classical theory by developing an obstruction theory for orientations of A_∞ ring spectra. We also use it to build Thom spectra in situations more general than stable spherical fibrations; these more general situations give rise to twisted generalized cohomology. To carry out these extensions, we use certain relatively recently developed 'rigid' point-set models for A_∞ (and E_∞) spaces.

1.1. *Recollection of the discrete case*

We begin by describing the algebraic model that motivates our approach. Let R be a discrete ring and let $G = GL_1 R$. A bundle of free rank-1 R-modules over X is classified by a map $f\colon X \to BG$; let $P \to X$ be the associated principal G-bundle. We would like to attach an R-module 'Thom spectrum' Mf to this situation, in such a way that trivializations of P over X can be understood in terms of R-module maps $Mf \to R$.

For simplicity, we will further assume that X is discrete. Then P is the G-set $P = \coprod_{x\in X} P_x$, and we can form the R-module 'algebraic Thom spectrum'

$$Mf = \mathbb{Z}[P] \otimes_{\mathbb{Z}[G]} R. \tag{1.1}$$

Formation of the tensor product uses the fact that the adjunction

$$\mathbb{Z}\colon (\text{sets}) \rightleftarrows (\text{abelian groups})$$

induces an adjunction

$$\mathbb{Z}\colon (G\text{-sets}) \rightleftarrows (\mathbb{Z}[G]\text{-modules}),$$

so $\mathbb{Z}[P]$ is a $\mathbb{Z}[G]$-module. Also, $\mathbb{Z}$ restricts to give an adjunction

$$\mathbb{Z}\colon (\text{groups}) \rightleftarrows (\text{rings})\colon GL_1, \tag{1.2}$$

whose counit is the natural ring homomorphism

$$\mathbb{Z}[G] \longrightarrow R. \tag{1.3}$$

Using these adjunctions, one checks easily that

$$(R\text{-modules})(Mf, R) \cong (G\text{-sets})(P, R),$$

and with respect to this isomorphism, the set of *orientations* of Mf is the subset

$$\begin{array}{ccc} (R\text{-modules})(Mf,R) & \xrightarrow{\cong} & (G\text{-sets})(P,R) \\ \uparrow & & \uparrow \\ (\text{orientations})(Mf,R) & \xrightarrow{\cong} & (G\text{-sets})(P,G) \end{array}$$

which in turn is isomorphic to the set of trivializations of the principal G-bundle $P \to X$.

1.2. *The space of units and orientations*

Our approach to the Thom spectrum functor develops the approach sketched above for a general space X and A_∞ ring spectrum R. Following [**27**], when R is an A_∞ ring spectrum in the sense of [18], we can define the space of units of R to be the pullback in the diagram of (unpointed) spaces

$$\begin{array}{ccc} GL_1R & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R. \end{array}$$

If X is any space, then

$$[X, GL_1 R] = \{f \in R^0(X_+) \,|\, \pi_0 f(X) \subset (\pi_0 R)^\times\} = R^0(X_+)^\times,$$

which provides a justification for the definition. More conceptually, we show in § 2 that this definition of units can be interpreted as the space of automorphisms of R (as an R-module).

Working with the models of [18], we have continuous (that is, topologically enriched) adjunctions (analogous to (1.2))

$$(\text{group-like } A_\infty \text{ spaces}) \underset{GL_1}{\rightleftarrows} (A_\infty \text{ spaces}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} (A_\infty \text{ ring spectra})\colon GL_1, \tag{1.4}$$

where the right-hand adjunction is a special case of [18, p. 366]. Thus, one can make sense of a map of A_∞ ring spectra $\Sigma^\infty_+ GL_1 R \to R$ analogous to (1.3).

However, classical technology does not make it straightforward to describe the adjunction

$$\Sigma^\infty_+\colon (\text{right } \Omega^\infty R\text{-modules}) \rightleftarrows (\text{right } R\text{-modules})\colon \Omega^\infty$$

and, moreover, since $GL_1 R$ is not a topological group or monoid but rather only a group-like A_∞ space, it is not immediately apparent how to form the (quasi)fibration

$$GL_1 R \longrightarrow EGL_1 R \longrightarrow BGL_1 R,$$

and then make sense of the construction (1.1).

Our strategy, which we carry out in § 3, is to use a 'rigid' model of A_∞ spaces. Specifically, we use a model of spaces equipped with a symmetric monoidal product such that strict monoids for this product are precisely A_∞ spaces [7].

In this setting, we can form a version of $GL_1 R$ which is a group-like monoid, and then model $EGL_1 R \to BGL_1 R$ as a quasifibration with an action of $GL_1 R$. Given a map

$$f\colon B \longrightarrow BGL_1 R,$$

$GL_1 R$ acts on the pullback P in the diagram

$$\begin{array}{ccc} P & \longrightarrow & EGL_1 R \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BGL_1 R \end{array}$$

and the spectrum $\Sigma^\infty_+ P$ becomes a right $\Sigma^\infty_+ GL_1 R$-module. We can then imitate (1.1) to form an R-module Thom spectrum as the derived smash product

$$Mf \overset{\text{def}}{=} \Sigma^\infty_+ P \wedge^{\mathrm{L}}_{\Sigma^\infty_+ GL_1 R} R.$$

With this definition, we find that

$$(\text{right } R\text{-modules})(Mf, R) \simeq (\text{right } GL_1 R\text{-spaces})(P, \Omega^\infty R), \tag{1.5}$$

where here (and in the remainder of this subsection) we are referring to derived mapping spaces.

The space of *orientations* of Mf is the subspace of R-module maps $Mf \to R$ which correspond to

$$(\text{right } GL_1 R\text{-modules})(P, GL_1 R) \subset (\text{right } GL_1 R\text{-modules})(P, \Omega^\infty R)$$

under the weak equivalence (1.5). That is, we have a homotopy pullback diagram

$$\begin{array}{ccc} (\text{orientations})(Mf, R) & \xrightarrow{\simeq} & (\text{right } GL_1R\text{-spaces})(P, GL_1R) \\ \downarrow & & \downarrow \\ (\text{right } R\text{-modules})(Mf, R) & \xrightarrow{\simeq} & (\text{right } GL_1R\text{-modules})(P, \Omega^\infty R). \end{array}$$

We obtain an obstruction-theoretic characterization of the space of orientations $Mf \to R$ as follows: it is weakly equivalent to the derived space of lifts in the diagram

$$\begin{array}{ccc} P & \longrightarrow & EGL_1R \\ \downarrow & \nearrow & \downarrow \\ B & \xrightarrow[f]{} & BGL_1R. \end{array}$$

We are able to use this to recover the classical picture of an orientation and also the Thom isomorphism.

Recall that a stable spherical fibration is classified by a map $B \to BF$, where $F = \operatorname{colim}_V h\mathrm{Aut}(S^V)$ (and the colimit is over finite-dimensional subspaces of $\mathbb{R}^\infty$ and inclusions). The space BF gives a particularly convenient model for BGL_1S. The generalized construction we study in this paper associates an R-module Thom spectrum Mf to a map $f\colon B \to BGL_1R$ for any ring spectrum R; f need not classify a stable spherical fibration.

To compare to the classical situation, we suppose that f does arise from a stable spherical fibration as the composite

$$f\colon B \xrightarrow{g} BGL_1S \xrightarrow{BGL_1\iota} BGL_1R.$$

It follows directly from the definition that $Mf \simeq Mg \wedge^{\mathrm{L}} R$.

We define an R-orientation of Mg to be a map of spectra $Mg \to R$ such that the induced map of R-modules $Mf \to R$ is an orientation as above. We then can show that the space of R-orientations of Mg is the space of indicated lifts in the diagram

$$\begin{array}{ccccc} P & \longrightarrow & B(S,R) & \longrightarrow & EGL_1R \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ B & \longrightarrow & BGL_1S & \longrightarrow & BGL_1R \end{array}$$

where $B(S,R)$ is the pullback in the solid diagram. This generalizes to the A_∞ case the work of May, Quinn, Ray, and Tornehave [**27**].

Remark 1.1. In the companion paper [**2**], we prove that when g classifies a stable spherical fibration, then the spectrum Mg constructed in this paper coincides with the Thom spectrum associated to g via the theory of [**18**].

1.3. *The spectrum of units and E_∞ orientations*

To see how our constructions work when R is an E_∞ ring spectrum, once again it is illuminating first to consider the discrete case. Suppose that R is a commutative ring. Then $G = GL_1R$ is an abelian group, and we can choose a model of BG that is an abelian group as well.

Now suppose that X is a discrete abelian group, and $f\colon X \to BG$ is a homomorphism. Then, in the pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

$P \cong G \times X$ is an abelian group, and so the discrete 'Thom spectrum'

$$Mf = \mathbb{Z}[P] \otimes_{\mathbb{Z}[G]} R \cong R[X]$$

is a commutative ring: indeed it is the pushout in the diagram of commutative rings

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & R \\ \downarrow & & \downarrow \\ \mathbb{Z}[P] & \longrightarrow & Mf \end{array}$$

where the homomorphism $\mathbb{Z}[G] \to R$ is the counit of the adjunction

$$\mathbb{Z}\colon (\text{abelian groups}) \rightleftarrows (\text{commutative rings})\colon GL_1,$$

which is the restriction to abelian groups of the adjunction (1.2).

Turning to spaces and spectra, the adjunction (1.4) restricts to an adjunction

$$(\text{group-like } E_\infty \text{ spaces}) \underset{GL_1}{\rightleftarrows} (E_\infty \text{ spaces}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} (E_\infty \text{ ring spectra})\colon GL_1.$$

In the E_∞ case there is the additional classical fact (for example, see [**25**]) that the category of group-like E_∞ spaces is a model for connective spectra: therefore, if R is an E_∞ ring spectrum, then there is a spectrum $gl_1 R$ such that $GL_1 R \simeq \Omega^\infty gl_1 R$. Putting all this together, we see that the functor gl_1 participates as the right adjoint in an adjunction

$$\Sigma^\infty_+ \Omega^\infty\colon \text{ho}((-1)\text{-connected spectra}) \rightleftarrows \text{ho}(E_\infty \text{ ring spectra})\colon gl_1, \tag{1.6}$$

which preserves the homotopy types of derived mapping spaces.

In contrast to the A_∞ setting, this adjunction can be constructed by assembling results in the literature, particularly work of May. However, as we worked through this, we found it very useful to reformulate the statements and proofs in a way that reflects advances in the state of the art since the original work was done. In § 5, we give a modern proof of this adjunction, carefully rederiving and explaining the many classical results involved.

Assuming this development, in § 4 we work out the theory of E_∞ Thom spectra generalizing our new model of A_∞ Thom spectra and establish results about orientations as used in the construction of the String orientation of tmf.

Let R be an E_∞ ring spectrum and suppose that b is a spectrum over $bgl_1 R = \Sigma gl_1 R$. Let p be the homotopy pullback

$$\begin{array}{ccc} gl_1 R & = & gl_1 R \\ \downarrow & & \downarrow \\ p & \longrightarrow & egl_1 R \simeq * \\ \downarrow & & \downarrow \\ b & \xrightarrow{f} & bgl_1 R. \end{array} \tag{1.7}$$

The E_∞ R-algebra Thom spectrum Mf of $f\colon b \to bgl_1 R$ is then defined to be the homotopy pushout in the diagram of E_∞ R-algebras

$$\begin{array}{ccc} R \wedge \Sigma^\infty_+ \Omega^\infty gl_1 R & \longrightarrow & R \\ \downarrow & & \downarrow \\ R \wedge \Sigma^\infty_+ \Omega^\infty p & \longrightarrow & Mf \end{array} \tag{1.8}$$

where the top map is induced from the counit of the adjunction (1.6). Since the homotopy pushout of E_∞ ring spectra coincides with the derived smash product, this generalizes the definition in the A_∞ setting.

For the obstruction theory, suppose that $\varphi\colon R \to A$ is a map of E_∞ ring spectra. Then we have the solid commutative diagram

$$\begin{array}{ccc} gl_1 R & \longrightarrow & gl_1 A \\ \downarrow & \nearrow & \downarrow \\ p & \longrightarrow & egl_1 A \simeq * \\ \downarrow & \nearrow & \downarrow \\ b & \xrightarrow{\tilde{\varphi}\circ f} & bgl_1 A \end{array} \tag{1.9}$$

where we write $\tilde{\varphi}\colon bgl_1 R \to bgl_1 A$ for the induced map.

Using the adjunction (1.6), we prove that there is a homotopy pullback diagram of derived mapping spaces (where $\mathscr{S}$ denotes the category of spectra)

$$\begin{array}{ccc} (E_\infty\ R\text{-algebras})(Mf, A) & \longrightarrow & \mathrm{Map}_{\mathscr{S}}(p, gl_1 A) \\ \downarrow & & \downarrow \\ \{\varphi\} & \longrightarrow & \mathrm{Map}_{\mathscr{S}}(gl_1 R, gl_1 A). \end{array}$$

That is, the space of R-algebra maps $M \to A$ is weakly equivalent to the space of lifts in diagram (1.9).

1.4. *Twisted generalized cohomology*

Our R-module Thom spectra locate 'twisted generalized cohomology' in stable homotopy theory; from this point of view $BGL_1 R$ classifies the twists. Let

$$f\colon X \longrightarrow BGL_1 R$$

be a map and let Mf be the associated R-module Thom spectrum. The *f-twisted R-homology* of X is

$$R^f_k X \stackrel{\text{def}}{=} \pi_0 \,\mathrm{Map}_{R\text{-mod}}(\Sigma^k R, Mf) \cong \pi_k Mf,$$

while the *f-twisted R-cohomology* of X is

$$R^k_f X \stackrel{\text{def}}{=} \pi_0 \,\mathrm{Map}_{R\text{-mod}}(Mf, \Sigma^k R).$$

If f factors as

$$f\colon X \xrightarrow{g} BGL_1 S \xrightarrow{i} BGL_1 R, \tag{1.10}$$

then we have $Mf \simeq (Mg) \wedge^{\mathrm{L}} R$ and so

$$\begin{aligned} R^f_k(X) &= \pi_k Mf \cong \pi_k(Mg \wedge^{\mathrm{L}} R) = R_k Mg, \\ R^k_f(X) &= \pi_0 \,\mathrm{Map}_{R\text{-mod}}(Mf, \Sigma^k R) \cong \pi_0 \,\mathrm{Map}_{S\text{-mod}}(Mg, \Sigma^k R) \cong R^k Mg. \end{aligned}$$

That is, the f-twisted homology and cohomology coincide with the untwisted R-homology and cohomology of the usual Thom spectrum of the spherical fibration classified by g. Thus, the constructions in this paper exhibit twisted generalized cohomology as the cohomology of a generalized Thom spectrum. In general, the twists correspond to maps $X \to BGL_1 R$; the ones that arise from Thom spectra of spherical fibrations are the ones that factor as in (1.10). We discuss the relationship to other approaches to twisted generalized cohomology in [**1**].

1.5. *Historical remarks and related work*

In his 1970 MIT notes [**36**] (in the version available at http://www.maths.ed.ac.uk/~aar/books/gtop.pdf, see the note on p. 236), Sullivan introduced the classical obstruction theory for orientations and suggested that Dold's theory of homotopy functors [**11**] could be used to construct the space $B(S, R)$ of R-oriented spherical fibrations. He also mentioned that the technology to construct the delooping $BGL_1 R$ was on its way. Soon thereafter, May, Quinn, Ray, and Tornehave [**27**] constructed the space $BGL_1 R$ in the case where R is an E_∞ ring spectrum, and described the associated obstruction theory for orientations of spherical fibrations.

Various aspects of the theory of units and Thom spectra have been revisited by a number of authors as the foundations of stable homotopy theory have advanced. For example, Schlichtkrull [**32**] studied the units of a symmetric ring spectrum, and May and Sigurdsson [**29**] have studied units and orientations from the perspective of their categories of parameterized spectra. Recently, May [**28**] has prepared an authoritative paper revisiting operad (ring) spaces and operad (ring) spectra from a modern perspective, which has substantial overlap with some of the review of the classical foundations in § 5.

2. *The space of units*

In this section, we recall the classical definition of $GL_1 R$ and explain how to use modern categories of spectra to interpret the units as a model for the derived space of homotopy automorphisms of the ring spectrum R. This preliminary work provides necessary foundations for our analysis of our new construction of the Thom spectrum functor in § 3. We do not make any particular claim to novelty in this section; in particular, May and Sigurdsson provide an excellent discussion of the situation in [**29**, § 22.2] (although note that our use of End and Aut is slightly different from theirs), and the conceptual description we give is of course implicit in the original definition in [**27**].

Given an A_∞ or E_∞ ring spectrum R in the classical sense (for example, see [**18**]), the classical construction of the group-like A_∞ or E_∞ space $GL_1 R$ is as follows.

Definition 2.1. The space $GL_1 R$ is the pullback in the diagram

$$\begin{array}{ccc} GL_1 R & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R. \end{array}$$

(Since the right-hand vertical map is a fibration, the pullback computes the homotopy pullback.)

If X is any space, then

$$[X, GL_1 R] = \{f \in R^0(X_+) \,|\, \pi_0 f(X) \subset (\pi_0 R)^\times\} = R^0(X_+)^\times,$$

which provides a justification for this definition.

We now explain how to interpret GL_1R as the space of homotopy automorphisms of R as an R-module. To begin, we need to work in a modern category of spectra, in order to have a sensible category of R-modules. Assume that $\mathcal{S}$ is a suitable symmetric monoidal topological model category of spectra, and let R be an S-algebra, that is, a monoid in $\mathcal{S}$. The category of R-modules inherits a model structure, and by the space of homotopy automorphisms of R, we mean the subspace of the derived mapping space $\mathrm{Map}_{R\text{-mod}}(R,R)$ consisting of weak equivalences.

In order to make this notion homotopically meaningful, we need to ensure that the mapping space has the right homotopy type.

Definition 2.2. If R' is a cofibrant–fibrant S-algebra, and M is a cofibrant–fibrant R'-module, then the *space of endomorphisms* of M is

$$\mathrm{End}(M) \stackrel{\mathrm{def}}{=} \mathrm{Map}_{R'\text{-mod}}(M,M).$$

This has a product induced by composition, and by definition the *space of homotopy automorphisms* of M is the subspace of group-like components: that is, $\mathrm{Aut}(M) = GL_1\,\mathrm{End}(M)$ is the pullback in the diagram

$$\begin{array}{ccc} \mathrm{Aut}(M) & \longrightarrow & \mathrm{End}(M) \\ \downarrow & & \downarrow \\ (\pi_0(\mathrm{End}(M))^{\times}) & \longrightarrow & \pi_0\,\mathrm{End}(M). \end{array}$$

Since M is cofibrant and fibrant, we can equivalently define $\mathrm{Aut}(M)$ to be the subspace of $\mathrm{End}(M)$ consisting of the homotopy equivalences.

If R is an arbitrary algebra, then the *derived space of endomorphisms* of R is the homotopy type

$$\mathrm{End}(R) = \mathrm{End}(R^{\circ}) \stackrel{\mathrm{def}}{=} \mathrm{Map}_{R'\text{-mod}}(R^{\circ}, R^{\circ}),$$

where R' is a cofibrant–fibrant replacement of R as an algebra, and R° is a cofibrant–fibrant replacement of R' as a module over itself. The *derived space of homotopy automorphisms* of R is the homotopy type of the subspace

$$\mathrm{Aut}(R) = \mathrm{Aut}(R^{\circ}) \subset \mathrm{End}(R^{\circ})$$

of homotopy equivalences of R°.

In analogy with the notation GL_1R, we have elected to use the notation $\mathrm{Aut}(R)$ for the space of homotopy automorphisms of R°, even though it is not a strict group. As defined, we have presented $\mathrm{Aut}(R)$ as a group-like topological or simplicial monoid. In practice, it is easier to access this homotopy type if we let R^c be a cofibrant replacement of R', and R^f a fibrant replacement. Then we have a weak homotopy equivalence of spaces

$$\mathrm{End}(R) \simeq \mathrm{Map}_{R'\text{-mod}}(R^c, R^f),$$

with $\mathrm{Aut}(R)$ equivalent to the subspace of weak equivalences.

We now compare $\mathrm{Aut}(R)$ to GL_1R, in the setting of the S-modules of [**12**]. Let $\mathscr{S}$ be the Lewis–May–Steinberger category of spectra, let $\mathscr{S}[\mathbb{L}]$ denote the category of $\mathbb{L}$-spectra, let $\mathscr{M}_S$ denote the associated topological model category of S-modules, and write $U\colon \mathscr{M}_S \to \mathscr{S}$ for the forgetful functor.

PROPOSITION 2.3. *Let R be a cofibrant S-algebra or commutative S-algebra in $\mathscr{M}_S$. Then there are natural zig-zags of equivalences*

$$\mathrm{End}(R) \simeq \Omega^\infty UR \quad \textit{and} \quad \mathrm{Aut}(R) \simeq GL_1 R,$$

and a zig-zag of natural equivalences between the inclusion of derived mapping spaces

$$\mathrm{Aut}(R) \longrightarrow \mathrm{End}(R),$$

and the inclusion map

$$GL_1 R \longrightarrow \Omega^\infty R.$$

Proof. In the model structure on R-modules, all objects are fibrant. Thus, we can use R for R^f. In the notation of [**12**], $S \wedge_{\mathscr{L}} \mathbb{L}\Sigma^\infty S$ is a cofibrant replacement for S as an S-module and $R \wedge_S \mathbb{L}\Sigma^\infty S$ is a cofibrant replacement for R as an R-module. So the derived mapping space $\mathrm{Map}_{\mathscr{M}_R}(R^c, R^f)$ is given by

$$\begin{aligned}\mathrm{Map}_{\mathscr{M}_R}(R \wedge_S \mathbb{L}\Sigma^\infty S^0, R) &\cong \mathrm{Map}_{\mathscr{M}_S}(S \wedge_S \mathbb{L}\Sigma^\infty S^0, R)\\ &\cong \mathrm{Map}_{\mathscr{S}[\mathbb{L}]}(\mathbb{L}\Sigma^\infty S^0, F_{\mathscr{L}}(S, R))\\ &\cong \mathrm{Map}_{\mathscr{S}}(\Sigma^\infty S^0, F_{\mathscr{L}}(S, R))\\ &\cong \Omega^\infty F_{\mathscr{L}}(S, R),\end{aligned}$$

where $F_{\mathscr{L}}(-,-)$ denotes the mapping space in $\mathbb{L}$-spectra.

By [**12**, § I, Corollary 8.7], the natural map of $\mathbb{L}$-spectra

$$R \longrightarrow F_{\mathscr{L}}(S, R)$$

is a weak equivalence of $\mathbb{L}$-spectra, and so of spectra. The weak equivalence

$$\mathrm{Map}_{\mathscr{M}_R}(R \wedge_S \mathbb{L}\Sigma^\infty S^0, R) \simeq \Omega^\infty R$$

follows since Ω^∞ preserves weak equivalences. By comparing pullback diagrams, it is then straightforward to see that the subspace of R-module weak equivalences corresponds to $GL_1 R$. □

The proof of the preceding proposition illustrates how useful it is that in the Lewis–May–Steinberger and Elmendorf–Kriz–Mandell–May categories of spectra, an algebra or commutative algebra R is automatically fibrant as a module over itself, so that $\Omega^\infty R$ is homotopically meaningful. In particular, since $GL_1 R$ is identified as a subspace of $\Omega^\infty R$, it is evident how to identify the multiplicative structure on $GL_1 R$. As we shall see in § 3, this simplifies our analysis substantially.

REMARK 2.4. In the setting of a category of diagram spectra $\mathscr{C}$ (for example, orthogonal spectra), the situation is somewhat more complicated. For an associative S-algebra R, one can carry out a similar analysis after passing to a cofibrant–fibrant replacement of R as an S-algebra, and the pullback description of $GL_1 R$ in fact yields a genuine topological monoid [**29**, 22.2.3]. But the situation for commutative S-algebras in the diagrammatic setting is different. The model structure on commutative S-algebras is lifted from the positive model structure on (orthogonal) spectra, and in this model structure the underlying S-module of a cofibrant–fibrant commutative S-algebra will not be fibrant; indeed its zero space will be S^0, and so

$$\mathrm{Map}_{\mathscr{C}}(S, R) = S^0 \neq \mathrm{Map}_{\mathscr{C}}(S^0, R^f) \simeq h\,\mathrm{End}(R).$$

Of course, one can instead replace the given commutative S-algebra by an associative S-algebra instead, but in this case it is impossible to recover the E_∞ structure on GL_1R. To describe GL_1R in this setting requires a different construction; see [**32**] or [**19**] for a description.

The problem that arises above is a manifestation of Lewis's theorem [**17**] about the nature of symmetric monoidal categories of spectra. If $S = \Sigma^\infty S^0$ is cofibrant (as it is in diagram categories of spectra), then the zero space of a cofibrant–fibrant commutative S-algebra must not be homotopically meaningful, as otherwise we could make a cofibrant–fibrant replacement S' of S, and

$$\mathrm{Map}_{\mathscr{C}}(S, S') \simeq QS^0$$

would realize QS^0 as a commutative topological monoid. On the other hand, if the zero space of a cofibrant–fibrant commutative S-algebra is homotopically meaningful, then S cannot be cofibrant, and the $(\Sigma^\infty, \Omega^\infty)$ adjunction must take a modified form (as it does in the setting of Elmendorf–Kriz–Mandell–May spectra).

3. A_∞ *Thom spectra and orientations*

In this section, we describe a new model of the Thom spectrum functor and apply it to the study of orientations of A_∞ ring spectra. The technical foundation of our model is recent work on 'rigid' models of infinite loop spaces that constructs symmetric monoidal categories of 'spaces' such that monoids and commutative monoids model A_∞ and E_∞ spaces. There are now several well-developed categories of rigid spaces, notably $*$-modules, the space-level analog of Elmendorf–Mandell–Kriz–May S-modules, and $\mathcal{I}$-spaces, the space-level analog of symmetric spectra [**7**].

We work with $*$-modules, because the version of the $(\Sigma^\infty, \Omega^\infty)$ adjunction in this setting is technically felicitous for dealing with units, as explained in § 2. The essential strategy is to adapt the operadic smash product of [**12**, **16**] to the category of spaces. Specifically, we produce a symmetric monoidal product on a model of the category $\mathscr{T}$ of topological spaces such that monoids for this product are precisely A_∞ spaces; in particular, this allows us to work with models of GL_1R which are strict monoids for the new product. The observation that one could carry out the program of [**12**] in the setting of spaces is due to Mike Mandell, and was worked out in the thesis of the second author [**6**]. A detailed presentation of the theory (along with complete proofs) has appeared in [**7**] (and see also [**19**]).

In order to alleviate the burden on the reader, below we give a very streamlined exposition focused on the precise properties we need, with careful citations. The results we need that are not in the literature are proved below.

3.1. *The categories of* $\mathbb{L}$*-spaces and* $*$*-modules*

We begin by reviewing the linear isometries operad [**12**, § I.3]. Fix a countably infinite-dimensional real vector space $\mathscr{U}$ topologized as the colimit of its finite-dimensional subspaces, and let $\mathscr{L}(k)$ denote the space of linear isometries $\mathscr{U}^{\oplus k} \to \mathscr{U}$, given the usual function space topology. Observe that $\mathscr{L}(0)$ is a point and $\mathscr{L}(1)$ is a monoid with unit given by the identity map $\mathscr{U} \to \mathscr{U}$. Each space $\mathscr{L}(k)$ has a free (right) action of Σ_k by permutations and is contractible, and the structure maps induced from the direct sum of linear isometries make the collection $\{\mathscr{L}(k)\}$ into an E_∞ operad. If we ignore the permutations, then the linear isometries operad is an A_∞ operad.

Let $\mathscr{T}$ denote the category of compactly generated weak Hausdorff spaces. We define an $\mathbb{L}$-space to be a space with an action of $\mathscr{L}(1)$, and write $\mathscr{T}[\mathbb{L}]$ for the category of $\mathbb{L}$-spaces. Mimicking the definition of the smash product of $\mathbb{L}$-spectra (in the development of Elmendorf–Kriz–Mandell–May), we have an associative and commutative product $X \boxtimes Y$ on the category

$\mathscr{T}[\mathbb{L}]$ (see [7, 4.1,4.2]) given by the coequalizer in the diagram

$$\mathscr{L}(2) \times (\mathscr{L}(1) \times \mathscr{L}(1)) \times (X \times Y) \underset{1\times\xi}{\overset{\gamma\times 1}{\rightrightarrows}} \mathscr{L}(2) \times X \times Y \longrightarrow X \times_{\mathcal{L}} Y.$$

Here ξ denotes the map using the $\mathbb{L}$-algebra structure of X and Y, and γ denotes the operad structure map $\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1) \to \mathscr{L}(2)$. The left action of $\mathscr{L}(1)$ on $\mathscr{L}(2)$ induces an action of $\mathscr{L}(1)$ on $X \boxtimes Y$.

There is a corresponding internal mapping object $F_{\mathcal{L}}(-,-)$ (see [7, 4.3,4.4]). The product is weakly unital, in the sense that, for any $\mathbb{L}$-space X, there is a natural map $* \boxtimes X \to X$ which is a weak equivalence [7, 4.5 and 4.6].

Theorem 3.1 [7, 4.16]. *The category $\mathscr{T}[\mathbb{L}]$ of $\mathbb{L}$-spaces is a topological model category with weak equivalences the underlying equivalences of spaces. The forgetful functor $\mathscr{T}[\mathbb{L}] \to \mathscr{T}$ is the right adjoint of a Quillen equivalence.*

The product $\boxtimes$ is a version of the cartesian product of spaces; specifically, for X and Y cofibrant objects of $\mathbb{T}[\mathbb{L}]$ there is a canonical map induced by the universal property of the product

$$X \boxtimes Y \longrightarrow X \times Y$$

that is a weak equivalence [7, 4.24]. This suggests that to study A_∞ spaces one might consider the category of monoids in $\mathscr{T}[\mathbb{L}]$, that is, the category $(\mathscr{T}[\mathbb{L}])[\mathbb{T}]$ of algebras in $\mathscr{T}[\mathbb{L}]$ over the associated monad

$$\mathbb{T}X = \bigvee_{n\geqslant 0} \underbrace{X \boxtimes \ldots \boxtimes X}_{n\geqslant 0}.$$

These monoids model A_∞ spaces structured by the linear isometries operad:

Proposition 3.2 [7, 4.8]. *Let $\mathbb{A}$ denote the monad on the category $\mathscr{T}$ associated to the non-symmetric linear isometries operad. Then the category $\mathscr{T}[\mathbb{A}]$ of A_∞ algebras is equivalent to the category $(\mathscr{T}[\mathbb{L}])[\mathbb{T}]$.*

Remark 3.3. As one would hope, commutative monoids in $\mathscr{T}[\mathbb{L}]$ are E_∞ spaces. However, since we do not need the commutative theory herein, we have chosen to omit discussion of it.

In order to have a symmetric monoidal category, we restrict to the unital objects. We define the category $\mathscr{M}_*$ of $*$-modules to be the full subcategory of $\mathbb{L}$-spaces such that the unit map $* \boxtimes X \to X$ is a homeomorphism [7, 4.9]. When restricted to $\mathscr{M}_*$, we will continue to write $\boxtimes$ for the product and $F_\boxtimes(-,-)$ for the internal mapping object $* \boxtimes F_{\mathcal{L}}(-,-)$. We then have the following result.

Theorem 3.4 [7, 4.17]. *The category $\mathscr{M}_*$ of $*$-modules is a closed symmetric monoidal topological model category, with product $\boxtimes$, unit $*$, and internal hom $F_\boxtimes(-,-)$. The weak equivalences are the maps that are underlying weak equivalences of spaces. The forgetful functor $\mathscr{M}_* \to \mathscr{T}$ is the right adjoint of a Quillen equivalence.*

All objects in the model structure on $\mathscr{M}_*$ are fibrant [7, 4.18]. The inclusion $\mathscr{M}_* \to \mathscr{T}[\mathbb{L}]$ has a right adjoint given by the functor $* \boxtimes X$. It is formal that right adjoints on $\mathscr{T}[\mathbb{L}]$ can therefore be lifted to $\mathscr{M}_*$ by applying this functor.

The monad $\mathbb{T}$ restricts to $\mathscr{M}_*$, and the model structure on $\mathscr{M}_*$ lifts to a topological model structure on $\mathscr{M}_*[\mathbb{T}]$ in which the weak equivalences and fibrations are determined by the forgetful functor $\mathscr{M}_*[\mathbb{T}] \to \mathscr{M}_*$ (see [7, 4.19]).

LEMMA 3.5 [7, 4.12]. *Let M be an A_∞ algebra in $\mathscr{T}$ over the linear isometries operad. Then $* \boxtimes_{\mathcal{L}} M$ is a monoid in $\mathscr{M}_*$.*

Associated to a monoid M in $\mathscr{M}_*$, we can consider the category of modules. If G is a monoid in $\mathscr{M}_*$, then a G-module is an object of $\mathscr{M}_*$ equipped with a map

$$G \boxtimes P \longrightarrow P$$

satisfying the usual associativity and unit conditions. We write $\mathscr{M}_G$ for the category of G-modules.

THEOREM 3.6. *The category $\mathscr{M}_G$ is a topological model category in which the fibrations and weak equivalences are determined by the forgetful functor $\mathscr{M}_G \to \mathscr{M}_*$.*

Proof. The proof of this result is analogous to the proof of [7, 4.16]. Specifically, we regard the category $\mathscr{M}_G$ as the category of algebras in $\mathscr{M}_*$ over the monad $G \boxtimes (-)$. Since $\mathscr{M}_*$ is a cofibrantly generated topological model category with all objects fibrant (and satisfying suitable smallness hypotheses on the generating cofibrations), we can apply the standard techniques for lifting model structures to monadic algebras (for example, see [7, 4.15]). □

Let $\mathcal{A}$ denote either the category $\mathscr{T}[\mathbb{L}]$ or $\mathscr{M}_*$. As usual, we say that a monoid M in $\mathcal{A}$ is group-like if $\pi_0(M)$ is a group. Let $(\mathcal{A}[\mathbb{T}])^\times$ denote the full subcategory of $\mathcal{A}[\mathbb{T}]$ consisting of group-like objects. Because an A_∞ space is precisely a monoid in $\mathscr{T}[\mathbb{L}]$, Definition 2.1 can be interpreted as a functor

$$GL_1 \colon \mathscr{M}_*[\mathbb{T}] \longrightarrow (\mathscr{T}[\mathbb{L}])[\mathbb{T}]^\times.$$

Composing with $* \boxtimes (-)$ produces a functor

$$GL_1 \colon \mathscr{M}_*[\mathbb{T}] \longrightarrow (\mathscr{M}_*[\mathbb{T}])^\times,$$

which is the right adjoint to the inclusion $(\mathscr{M}_*[\mathbb{T}])^\times \to \mathscr{M}_*[\mathbb{T}]$.

Given a monoid M, we will be interested in the bar construction. Thus, we will need to employ geometric realization in $\mathscr{M}_*$. Given a simplicial object $X_\bullet$ in $\mathscr{M}_*$, there is a natural homeomorphism $U|X_\bullet| \cong |UX_\bullet|$ (see [7, 4.26]), where here U denotes the forgetful functor to spaces. As in [7, § 3.1], we say that a simplicial object $X_\bullet$ in $\mathscr{M}_*$ is good if the degeneracies are h-cofibrations in the following sense: a morphism $X \to Y$ in $\mathcal{A}$ is an h-cofibration if the map

$$(X \boxtimes I) \coprod_X Y \longrightarrow Y \boxtimes I$$

has a retract. In this case, the underlying simplicial space $UX_\bullet$ is good in the classical sense [34, § A].

Given a monoid M in $\mathscr{M}_*[\mathbb{T}]$ and right and left M-modules X and Y, we can define the bar construction as the geometric realization in $\mathscr{M}_*$ of the simplicial object with k-simplices

$$B_k(X, M, Y) = X \boxtimes \underbrace{M \boxtimes M \boxtimes \cdots \boxtimes M}_{k} \boxtimes Y,$$

with the usual simplicial structure maps induced by the multiplication on M and the action maps on X and Y. In particular, we define

$$B_\boxtimes G = |B_\bullet(*, G, *)| \quad \text{and} \quad E_\boxtimes G = |B_\bullet(*, G, G)|.$$

Note that $E_\boxtimes G$ becomes a G-module via the action on the right and the map $\pi\colon E_\boxtimes G \to B_\boxtimes G$ becomes a map of G-modules when we give $B_\boxtimes G$ the trivial action

$$B_\boxtimes G \boxtimes G \longrightarrow B_\boxtimes G \boxtimes * \longrightarrow B_\boxtimes G.$$

By inspection of the definition of $\boxtimes$ (see [7, 4.1]), we see that the fiber at the basepoint of this map is precisely the realization of the simplicial object with k-simplices $G \boxtimes * \boxtimes \cdots \boxtimes *$, which is homeomorphic to G. Again let $\mathcal{A}$ denote one of the categories $\mathscr{T}[\mathbb{L}]$ or $\mathscr{M}_*$. We say that an object of $\mathcal{A}[\mathbb{T}]$ is a well-based monoid in $\mathcal{A}$ if the unit map $1_\mathcal{A} \to M$ is an h-cofibration. When M is a well-based monoid, these simplicial objects are good [7, 3.2].

In order to understand the homotopy type of $B_\boxtimes G$, we recall that we have a continuous strong symmetric monoidal functor $Q\colon \mathscr{T}[\mathbb{L}] \to \mathscr{T}$ that is the left adjoint to the functor that gives a space the trivial $\mathbb{L}$-action [7, 4.13 and 4.14]. The functor Q comes equipped with a natural transformation $U \to Q$, which is a weak equivalence when applied to cofibrant objects in $\mathscr{T}[\mathbb{L}]$, $\mathscr{M}_*$, or $\mathscr{M}_*[\mathbb{T}]$ (see [7, 4.27]). In fact, we have the following comparison results.

THEOREM 3.7. *The functor Q induces a Quillen equivalence between $\mathscr{M}_*$ and $\mathscr{T}$, and a Quillen equivalence between $\mathscr{M}_G$ and $QG\mathscr{T}$ (where the latter is equipped with the standard model structure determined by the underlying equivalences).*

Proof. The proof of [7, 4.27] shows that the left adjoint functor $Q\colon \mathscr{M}_* \to \mathscr{T}$ preserves cofibrations and weak equivalences between cofibrant objects. Therefore, Q is a left Quillen adjoint. Since Q is strong symmetric monoidal, it lifts to a functor $Q\colon \mathscr{M}_G \to QG\mathscr{T}$. Since the model structure on $\mathscr{M}_G$ is lifted from $\mathscr{M}_*$, an analogous elaboration of the argument for [7, 4.27] shows that Q is a Quillen left adjoint in this setting as well. Since the right adjoint preserves all weak equivalences and Q preserves weak equivalences between cofibrant objects, in each case Q induces a Quillen equivalence. □

Since Q is a continuous left adjoint, it commutes with geometric realization. As a particular consequence, we see that $QB_\boxtimes G \cong BQG$, where B denotes the usual bar construction for the topological monoid QG. Since $UB_\boxtimes G \to QB_\boxtimes G$ is a weak equivalence, this identifies the bar construction as the usual one applied to the rectification QG. This comparison allows us to show that the map $E_\boxtimes G \to B_\boxtimes G$ is a quasifibration in the following sense.

THEOREM 3.8. *Let G be a group-like cofibrant object in $\mathscr{M}[\mathbb{T}]$ which is well-based. Then the map $UE_\boxtimes G \to UB_\boxtimes G$ is a quasifibration of spaces.*

Proof. By the remarks above, $QE_{\boxtimes}G \cong E(QG)$ and $QB_{\boxtimes}G \cong B(QG)$. By naturality, there is a commutative diagram

$$\begin{array}{ccc} UE_{\boxtimes}G & \longrightarrow & E(QG) \\ \big\downarrow{\scriptstyle U\pi} & & \big\downarrow{\scriptstyle Q\pi} \\ UB_{\boxtimes}G & \xrightarrow{f} & B(QG) \end{array}$$

For any $p \in UB_{\boxtimes}G$, $(U\pi)^{-1}(p) = UG$; $\pi^{-1}(p) = G$ by inspection of the definition of $\boxtimes$ (see [7, 4.1]). Furthermore, $(Q\pi)^{-1}(fp) = QG$, and the map between them is induced from the natural transformation $U \to Q$. Writing $F(U\pi)_p$ for the homotopy fiber of $U\pi$ at p and $F(Q\pi)_{fp}$ for the homotopy fiber of $Q\pi$ at fp, we have a commutative diagram

$$\begin{array}{ccc} UG \cong (U\pi)^{-1}(p) & \longrightarrow & F(U\pi)_p \\ \big\downarrow & & \big\downarrow \\ QG \cong (Q\pi)^{-1}(fp) & \longrightarrow & F(Q\pi)_{fp} \end{array}$$

where the horizontal maps are the natural inclusions of the actual fiber in the homotopy fiber. The hypotheses on G ensure that the vertical maps are weak equivalences: on the left, this follows directly because G is cofibrant, and on the right, we use the fact that $UE_{\boxtimes}G \to QE_{\boxtimes}G$ and $UB_{\boxtimes}G \to QB_{\boxtimes}G$ are weak equivalences since U and Q commute with geometric realization and all the simplicial spaces involved are proper. Furthermore, since QG is a group-like topological monoid with a non-degenerate basepoint, $Q\pi$ is a quasifibration [26, 7.6], and so the inclusion of the actual fiber of $U\pi$ in the homotopy fiber of $U\pi$ is an equivalence. That is, the bottom horizontal map is an equivalence. Thus, we deduce that the top horizontal map is an equivalence and so that $U\pi$ is a quasifibration. $\square$

As one would expect from the definition of the category $\mathscr{M}_*$, the category of Elmendorf–Kriz–Mandell–May S-modules is the natural model for the stabilization. Specifically, the $(\Sigma^\infty_+, \Omega^\infty)$ adjunction on the category $\mathscr{S}$ of Lewis–May–Steinberger spectra and the natural equivalence $\mathcal{L}(1) \ltimes \Sigma^\infty_+ X \cong \Sigma^\infty_+(\mathcal{L}(1) \times X)$ gives rise to an adjunction $(\Sigma^\infty_{\mathbb{L}+}, \Omega^\infty_{\mathbb{L}})$ connecting $\mathscr{T}[\mathbb{L}]$ and the Elmendorf–Kriz–Mandell–May category of $\mathbb{L}$-spectra [19, 7.2]. To model this in the setting of $*$-modules, for an S-module M we define

$$\Omega^\infty_S M = * \boxtimes_{\mathcal{L}} \Omega^\infty_{\mathbb{L}} F_{\mathcal{L}}(S, M),$$

where $F_{\mathcal{L}}(S, M)$ is the mapping $\mathbb{L}$-spectrum [19, 7.4]. (See also [4, § 6] for discussion of this adjunction.) We then obtain the following result.

THEOREM 3.9 [19, 7.5]. *There is a strong symmetric monoidal left Quillen functor*

$$\Sigma^\infty_{\mathbb{L}+} \colon \mathscr{M}_* \longrightarrow \mathscr{M}_S.$$

The corresponding lax symmetric monoidal right Quillen adjoint is

$$\Omega^\infty_S \colon \mathscr{M}_S \longrightarrow \mathscr{M}_*.$$

A consequence of Theorem 3.9 is the following generalization.

Corollary 3.10. *The adjunction* $(\Sigma^\infty_{\mathbb{L}+}, \Omega^\infty_S)$ *specializes to a Quillen adjunction*

$$\Sigma^\infty_{\mathbb{L}+} \colon \mathscr{M}_G \leftrightarrows \mathscr{M}_{\Sigma^\infty_+ G} \colon \Omega^\infty_S.$$

3.2. *Thom spectra*

Assembling adjunctions from the previous section, we have the following structured version of the adjunction (1.4):

$$(\mathscr{M}_*[\mathbb{T}])^\times \underset{GL_1}{\rightleftarrows} \mathscr{M}_*[\mathbb{T}] \underset{\Omega^\infty_S}{\overset{\Sigma^\infty_{\mathbb{L}+}}{\rightleftarrows}} \mathscr{M}_S[\mathbb{T}] \colon GL_1. \tag{3.1}$$

Taking a cofibrant replacement $(GL_1R)^c$ in the category $\mathscr{M}_*[\mathbb{T}]$, we have a composite map of S-modules

$$\gamma \colon \Sigma^\infty_{\mathbb{L}+}((GL_1R)^c) \longrightarrow \Sigma^\infty_{\mathbb{L}+} GL_1R \longrightarrow R,$$

where the second map is the counit of the adjunction in equation (3.1).

Using Corollary 3.10, we conclude the following lemma.

Lemma 3.11. *The map* γ *gives* R *the structure of a left* $\Sigma^\infty_{\mathbb{L}+}((GL_1R)^c)$*-module.*

We can now give the definition of the Thom spectrum functor. For convenience, assume that R is a cofibrant S-algebra. We will regard the input as a map

$$f \colon X \longrightarrow B_\boxtimes((GL_1R)^c)$$

of $*$-modules; this entails no loss of generality, as we now explain. Suppose that we are given more classical input data in the form of a map of spaces $f \colon X \to B_\boxtimes GL_1R$. By adjunction, this is equivalent to a map $f' \colon \mathbb{L}X \to B_\boxtimes GL_1R$ in $\mathscr{T}[\mathbb{L}]$. Applying $* \boxtimes_{\mathcal{L}} (-)$ yields a map of $*$-modules

$$f'' \colon * \boxtimes_{\mathcal{L}} \mathbb{L}X \longrightarrow * \boxtimes_{\mathcal{L}} B_\boxtimes GL_1R \cong B_\boxtimes GL_1R.$$

Finally, we take the (homotopy) pullback in the diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & B_\boxtimes((GL_1R)^c) \\
\downarrow \simeq & & \downarrow \simeq \\
* \boxtimes_{\mathcal{L}} \mathbb{L}X & \xrightarrow{f''} & B_\boxtimes(GL_1R)
\end{array}$$

where the right-hand vertical map is induced by the cofibrant replacement $(GL_1R)^c \to GL_1R$ in $\mathscr{M}_*[\mathbb{T}]$.

Definition 3.12. Let $f \colon X \to B_\boxtimes((GL_1R)^c)$ be a map in $\mathscr{M}_*$. The *Thom spectrum* of f is the functor

$$M \colon \mathscr{M}_*/B_\boxtimes((GL_1R)^c) \longrightarrow \mathscr{M}_R,$$

given by

$$Mf \overset{\text{def}}{=} \Sigma^\infty_{\mathbb{L}+} P' \wedge_{\Sigma^\infty_{\mathbb{L}+}((GL_1R)^c)} R,$$

where here P' denotes a cofibrant replacement as a (right) $(GL_1R)^c$-module of the pullback P in the diagram

$$\begin{array}{ccc} P & \longrightarrow & E_{\boxtimes}((GL_1R)^c) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tilde{f}} & B_{\boxtimes}((GL_1R)^c) \end{array}$$

(here $\tilde{f}$ is a fibrant replacement of f).

By construction, $\Sigma^\infty_{\mathbb{L}+}P'$ is then a cofibrant $\Sigma^\infty_{\mathbb{L}+}((GL_1R)^c)$-module, so we are computing the derived smash product. Moreover, since R is cofibrant S-algebra, the resulting Thom spectrum Mf is a cofibrant R-module.

REMARK 3.13. Definition 3.12 constructs the Thom spectrum directly as a homotopical functor and a homotopical left adjoint. One might hope to construct a point-set Thom functor, which we then derive in the usual fashion, but because this definition involves the composite of a right adjoint equivalence (the pullback functor from $\mathscr{M}_*/B_{\boxtimes}((GL_1R)^c)$ to $\mathscr{M}_{(GL_1R)^c}$) and a left adjoint (the functor $\Sigma^\infty_{\mathbb{L}+}(-)\wedge_{\Sigma^\infty_{\mathbb{L}+}((GL_1R)^c)} R$ from $\mathscr{M}_{(GL_1R)^c}$ to $\mathscr{M}_R$), it is intricate (although possible) to give a model that can be derived without the intermediate cofibrant replacement step.

We now want to interpret the notion of orientation in this setting. We first observe that, for any right R-module T, there is a natural equivalence of mapping spaces

$$\mathrm{Map}_{\mathscr{M}_R}(Mf,T) \simeq \mathrm{Map}_{\mathscr{M}_{\Sigma^\infty_{\mathbb{L}+}((GL_1R)^c)}}(\Sigma^\infty_{\mathbb{L}+}P',T) \simeq \mathrm{Map}_{\mathscr{M}_{(GL_1R)^c}}(P',\Omega^\infty_S T).$$

Note that here we are computing derived mapping spaces because all objects are fibrant in all of the model categories involved. In particular, taking $T=R$, we have

$$\mathrm{Map}_{\mathscr{M}_R}(Mf,R) \simeq \mathrm{Map}_{\mathscr{M}_{(GL_1R)^c}}(P',\Omega^\infty_S R). \tag{3.2}$$

This gives rise to the following definition of the space of orientations of a Thom spectrum.

DEFINITION 3.14. The space of *orientations* of Mf is the subspace of components of the (derived) mapping space $\mathrm{Map}_{\mathscr{M}_R}(Mf,R)$ that correspond to

$$\mathrm{Map}_{\mathscr{M}_{(GL_1R)^c}}(P',GL_1R) \subseteq \mathrm{Map}_{\mathscr{M}_{(GL_1R)^c}}(P',\Omega^\infty_S R)$$

under the adjunction (3.2). That is, we form the homotopy pullback diagram

$$\begin{array}{ccc} (\mathrm{orientations})(Mf,R) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathscr{M}_{(GL_1R)^c}}(P',GL_1R) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathscr{M}_R}(Mf,R) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathscr{M}_{(GL_1R)^c}}(P',\Omega^\infty_S R). \end{array} \tag{3.3}$$

We can provide an obstruction-theoretic description of the space of orientations in terms of lifts in the diagram

$$\begin{array}{ccc} P & \longrightarrow & E_{\boxtimes}G \\ \downarrow & \nearrow & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & B_{\boxtimes}G. \end{array} \tag{3.4}$$

THEOREM 3.15. *Suppose that G is a cofibrant group-like monoid in $\mathscr{M}_*$, and f is a fibration. Then there is a natural zig-zag of weak equivalences between the derived mapping space $\mathrm{Map}_{\mathscr{M}_*/B_{\boxtimes}G}(f,\pi)$ of lifts in diagram (3.4) and the derived mapping space $\mathrm{Map}_{\mathscr{M}_G}(P,G)$.*

Proof. We will deduce this result from the corresponding result for group-like monoids (for example, see [**35**, 8.5]) using the functorial rectification process provided by the functor Q.

If G is group-like, then QG is a group-like topological monoid that has the homotopy type of a CW-complex and a non-degenerate basepoint. Therefore, applying Q and taking the homotopy pullback, we obtain a square of QG-spaces in $\mathscr{T}$

$$\begin{array}{ccc} \hat{P} & \longrightarrow & B(QG) \\ \downarrow & & \downarrow \\ QX & \longrightarrow & E(QG) \end{array}$$

such that there is a weak equivalence of derived mapping spaces

$$\mathrm{Map}_{\mathscr{T}/B(QG)}(QX, E(QG)) \simeq \mathrm{Map}_{(QG)\mathscr{T}}(\hat{P}, QG).$$

We now use Theorem 3.7. On the one hand, a straightforward extension of Theorem 3.7 implies that Q induces a Quillen equivalence between $\mathscr{M}_*/B_{\boxtimes}G$ and $\mathscr{T}/B(QG)$, and so there is an equivalence of derived mapping spaces

$$\mathrm{Map}_{\mathscr{M}_*/B_{\boxtimes}G}(X, E_{\boxtimes}G) \simeq \mathrm{Map}_{\mathscr{T}/B(QG)}(QX, E(QG)).$$

On the other hand, since Q also induces a Quillen equivalence between $\mathscr{M}_G$ and $QG\mathscr{T}$, there is an equivalence of derived mapping spaces

$$\mathrm{Map}_{\mathscr{M}_G}(P, G) \longrightarrow \mathrm{Map}_{QG\mathscr{T}}(QP, QG).$$

The proof of the theorem will be complete once we have shown that a cofibrant replacement of QP is naturally weakly equivalent to a cofibrant replacement of $\hat{P}$ as QG-spaces. Finally, this follows because either derived functor associated to a Quillen equivalence preserves homotopy limits up to a zig-zag of natural weak equivalences. Although this result is standard, the authors are not aware of a convenient reference and so we briefly remind the reader of the proof. The homotopy limit of shape D in the homotopical category $\mathcal{C}$ is the right derived functor $\mathrm{ho}(\mathcal{C}^D) \to \mathrm{ho}(\mathcal{C})$ of the right adjoint (which exists on the level of homotopical categories) of the constant diagram functor. Since equivalences of homotopical categories (or Quillen equivalences of cofibrantly generated model categories) induce equivalences on diagram categories (or Quillen equivalences of the projective model structure on the diagram categories), the result follows by lifting the isomorphism in the homotopy category to a weak equivalence between cofibrant–fibrant objects. □

Theorem 3.15 now has the following immediate consequence.

THEOREM 3.16. *The space of orientations of Mf is weakly equivalent to the space of lifts in diagram (3.4). In particular, the spectrum Mf is orientable if and only if $f\colon X \to B_{\boxtimes}((GL_1R)^c)$ is null homotopic.*

Since our construction of the Thom spectrum takes homotopic classifying maps to weakly equivalent spectra, Theorem 3.16 implies that an orientation gives rise to an equivalence $Mf \simeq \Sigma^{\infty}_{\mathbb{L}+}X \wedge R$. This is a version of the Thom isomorphism theorem, and we will give a description of the map inducing this equivalence below.

3.3. *Orientations and the Thom isomorphism*

To make contact with familiar notions of orientation, we shall be more explicit about the adjunctions in Definition 3.14. For this, it is helpful to recapitulate some classical computations of Thom spectra in our setting.

LEMMA 3.17. *The Thom spectrum of the inclusion of a point*

$$* \longrightarrow B_{\boxtimes}((GL_1R)^c)$$

is a cofibrant R-module that is weakly equivalent to R. More generally, the Thom spectrum of a trivial map

$$X \longrightarrow * \longrightarrow B_{\boxtimes}((GL_1R)^c)$$

is weakly equivalent to $R \wedge \Sigma^{\infty}_{\mathbb{L}+}X$.

Proof. Let $* \to B_{\boxtimes}((GL_1R)^c)$ be the inclusion of a point. The Thom spectrum is $\Sigma^{\infty}_{+}P' \wedge_{\Sigma^{\infty}_{\mathbb{L}+}(GL_1R)^c} R$, where P' is a cofibrant replacement of the homotopy pullback

$$\begin{array}{ccc} P & \longrightarrow & E_{\boxtimes}((GL_1R)^c) \\ \downarrow & & \downarrow \\ * & \longrightarrow & B_{\boxtimes}((GL_1R)^c) \end{array}$$

as a $(GL_1R)^c$-module in $\mathscr{M}_*$. Since Theorem 3.8 implies that $UE_{\boxtimes}((GL_1R)^c) \to UB_{\boxtimes}((GL_1R)^c)$ is a quasifibration (with fiber $(GL_1R)^c$), it follows that $(GL_1R)^c \simeq P'$ as $(GL_1R)^c$-modules.

Consideration of the iterated pullback square

$$\begin{array}{ccccc} \tilde{P} & \longrightarrow & P & \longrightarrow & E_{\boxtimes}((GL_1R)^c) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & * & \longrightarrow & B_{\boxtimes}((GL_1R)^c) \end{array}$$

implies that $\tilde{P}$ is equivalent to $(GL_1R)^c \boxtimes X$ as a $(GL_1R)^c$-module, where X has the trivial action. □

In particular, we have the following corollary.

COROLLARY 3.18. *Since $E_{\boxtimes}((GL_1R)^c) \simeq *$, we have*

$$M(\pi\colon E_{\boxtimes}((GL_1R)^c)) \longrightarrow B_{\boxtimes}((GL_1R)^c) \simeq R,$$

as R-modules.

Now suppose that $f\colon X \to B_{\boxtimes}((GL_1R)^c)$ is a fibration of $*$-modules, let P be the pullback in the diagram

$$\begin{array}{ccc} P & \longrightarrow & E_{\boxtimes}((GL_1R)^c) \\ \downarrow & \overset{\tilde{a}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & B_{\boxtimes}((GL_1R)^c) \end{array} \tag{3.5}$$

and let $M = Mf$. If $\tilde{a}$ is a lift as indicated, then by functoriality passing to Thom spectra along $\tilde{a}$ induces a map of R-modules

$$a\colon Mf \longrightarrow R.$$

This is the orientation associated to the lift $\tilde{a}$.

Conversely, suppose that $a\colon Mf \to R$ is a map of R-modules. Each point $p \in P'$ (the cofibrant replacement of P as a $(GL_1R)^c$-module) determines a $(GL_1R)^c$-map

$$(GL_1R)^c \longrightarrow P',$$

and therefore a map of $\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c)$-modules

$$\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c) \longrightarrow \Sigma^{\infty}_{\mathbb{L}+}P'.$$

Passing to Thom spectra, this in turn yields a map of R-modules

$$j_p\colon \Sigma^{\infty}_{+}((GL_1R)^c) \wedge_{\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c)} R \cong R \to Mf \to R.$$

As p varies the j_p assemble; we take the adjoint of the composite

$$\begin{aligned} \Sigma^{\infty}_{\mathbb{L}+}P' &\longrightarrow F_{\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c)}(\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c), \Sigma^{\infty}_{\mathbb{L}+}P') \\ &\longrightarrow F_R(\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c) \wedge_{\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c)} R, \Sigma^{\infty}_{+}P' \wedge_{\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c)} R) \\ &\cong F_R(R, Mf) \xrightarrow{a} F_R(R,R), \end{aligned}$$

where the first map is the adjoint of the action map and the second map is induced by functoriality.

The argument for Proposition 2.3 shows that

$$\Omega^{\infty}_{S} F_R(R,R) \simeq \Omega^{\infty}_{S} R,$$

and the resulting map

$$j\colon P' \longrightarrow \Omega^{\infty}_{S} R$$

corresponds to a under the equivalence of derived mapping spaces

$$\mathrm{Map}_{\mathscr{M}_R}(Mf, R) \simeq \mathrm{Map}_{\mathscr{M}_{GL_1R}}(P', \Omega^{\infty}_{S} R).$$

Put another way, for each $q \in X$, Lemma 3.17 implies that the Thom spectrum M_q of $q \to X \to B_{\boxtimes}GL_1R$ is non-canonically weakly equivalent to R. Passing to Thom spectra gives a map

$$i_q\colon M_q \longrightarrow Mf \xrightarrow{a} R.$$

A choice of point $p \in P$ lying over q fixes an equivalence $R \simeq M_q$ making the diagram

$$\begin{array}{ccc} R & \xrightarrow{\simeq} & M_q \\ & {\scriptstyle j_p}\searrow \quad \swarrow{\scriptstyle i_q} & \\ & R & \end{array}$$

commute. Thus, we have the following analog of the standard description of Thom classes as in, for example, [**37**, Definition 14.5].

Proposition 3.19. *Suppose that* $a\colon Mf \to R$ *is a map of* R*-modules. Then the following are equivalent*:

(1) a *is an orientation*;
(2) *for each* $q \in X$, *the map of* R*-modules*

$$i_q\colon M_q \longrightarrow Mf \xrightarrow{a} R$$

is a weak equivalence;
(3) *for each* $p \in P$, *the map of* R*-modules*

$$j_p\colon R \longrightarrow Mf \xrightarrow{a} R$$

is a weak equivalence.

We conclude by discussing the Thom isomorphism in this setting. Let $f\colon X \to B_{\boxtimes}(GL_1R)^c$ be a fibration of $*$-modules and suppose that X is cofibrant in $\mathscr{M}_*$. Suppose that we are given an orientation in the form of a $(GL_1R)^c$-map

$$s\colon P' \longrightarrow (GL_1R)^c,$$

corresponding to an R-module map

$$a\colon Mf \longrightarrow R.$$

Consider the map

$$X \boxtimes X \xrightarrow{f\pi_2} B_{\boxtimes}((GL_1R)^c),$$

where here π_2 is the projection onto the second factor (induced from the composite $X \boxtimes X \to * \boxtimes X$). Passing to pullbacks, we obtain the commutative diagram

$$\begin{array}{ccccc} \tilde{P} & \longrightarrow & P & \longrightarrow & E_{\boxtimes}((GL_1R)^c) \\ \downarrow & & \downarrow & & \downarrow \\ X \boxtimes X & \longrightarrow & X & \longrightarrow & B_{\boxtimes}((GL_1R)^c). \end{array}$$

Since the map $P \boxtimes X \to X \boxtimes X$ induced from the map $P \to X$ and the projection map $P \boxtimes X \to P$ are compatible with the maps to X, the universal property of the pullback induces a map $P \boxtimes X \to \tilde{P}$. Passing to cofibrant replacements as $(GL_1R)^c$-modules gives us a map between cofibrant–fibrant $(GL_1R)^c$-modules; using Q and the argument for Theorem 3.15, we see that this map represents the identity map on $QX \times QP$ in the homotopy category of $Q((GL_1R)^c)$-spaces, and hence is a weak equivalence.

Let P' denote a cofibrant replacement of P as a $(GL_1R)^c$-module. Since P' and $X \boxtimes P'$ are cofibrant–fibrant objects, we can choose a map $P' \to X \boxtimes P'$ which lifts the homotopy class of the diagonal map $QP' \to QX \times QP'$. Passing to Thom spectra, we obtain the R-module Thom diagonal map

$$M \xrightarrow{\Delta} (\Sigma^{\infty}_{\mathbb{L}+}X) \wedge M.$$

Next, we form the composite

$$M \xrightarrow{\Delta} (\Sigma^{\infty}_{\mathbb{L}+}X) \wedge M \xrightarrow{1\wedge a} (\Sigma^{\infty}_{\mathbb{L}+}X) \wedge R, \tag{3.6}$$

as in [20].

To analyze this, we compose the orientation s with the map $P' \to P' \boxtimes X$ to obtain the composite map of $(GL_1R)^c$-modules

$$P' \longrightarrow X \boxtimes P' \longrightarrow X \boxtimes (GL_1R)^c.$$

Now, applying the functor $(-) \wedge_{\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c)} R$ produces the Thom diagonal equation (3.6). On the other hand, since s corresponds to a section of the map $P \to X$ induced by the universal property of the pullback, this composite is a weak equivalence of $(GL_1R)^c$-modules. Since $(-) \wedge_{\Sigma^{\infty}_{\mathbb{L}+}((GL_1R)^c)} R$ preserves weak equivalences of cofibrant $(GL_1R)^c$-modules, we obtain the following proposition.

Proposition 3.20. *If $a\colon Mf \to R$ is an orientation, then the map of right R-modules*

$$Mf \xrightarrow{\Delta} \Sigma^{\infty}_{\mathbb{L}+} X \wedge Mf \xrightarrow{1\wedge a} \Sigma^{\infty}_{\mathbb{L}+} X \wedge R$$

is a weak equivalence.

4. E_∞ *Thom spectra and orientations*

In this section, we describe the construction and orientation of E_∞ Thom spectra, generalizing the perspective of § 3. For an E_∞ ring spectrum R, the space of units GL_1R can be delooped to form a spectrum of units gl_1R. This is encoded in the basic adjunction

$$\Sigma^{\infty}_{+}\Omega^{\infty}\colon \text{ho}((-1)\text{-connected spectra}) \rightleftarrows \text{ho}\,\mathscr{S}[E_\infty]\colon gl_1, \tag{4.1}$$

which is proved in § 5; see Theorem 5.1. Here $\mathscr{S}[E_\infty]$ denotes the model category of Lewis–May–Steinberger E_∞ ring spectra. In order to support the generalization to R-algebras, we model $\mathscr{S}[E_\infty]$ via the Quillen equivalent model category $\mathscr{M}_S[\mathbb{P}]$ of Elmendorf–Mandell–Kriz–May commutative S-algebras, the connective spectra as a subcategory of $\mathscr{M}_S$, and $\Sigma^{\infty}_{+}\Omega^{\infty}$ as the composite $\Sigma^{\infty}_{\mathbb{L}+}\Omega^{\infty}_{S}$ (see Theorem 3.9).

We begin by discussing the classical case of stable spherical fibrations. The counit of the adjunction above yields a map in $\text{ho}\,\mathscr{M}_S[\mathbb{P}] \cong \text{ho}\,\mathscr{S}[E_\infty]$

$$\epsilon\colon \Sigma^{\infty}_{\mathbb{L}+}\Omega^{\infty}_{S} gl_1S \longrightarrow S.$$

Assume that we are given a map

$$\zeta\colon b \longrightarrow bgl_1S,$$

where we write bgl_1S for Σgl_1S. Let $j = \Sigma^{-1}\zeta$ and form the diagram

$$\begin{array}{ccccc}
g & \xrightarrow{j} & gl_1S & = & gl_1S \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & Cj & \longrightarrow & egl_1S \simeq * \\
 & & \downarrow & & \downarrow \\
 & & b & \longrightarrow & bgl_1S
\end{array}$$

by requiring that the upper left and bottom right squares are homotopy cartesian. Note that we may also view b as an infinite loop map

$$f\colon B \longrightarrow BGL_1S.$$

Definition 4.1. The *Thom spectrum* of f, or of ζ, or of j, is the homotopy pushout $M\zeta$ of the diagram in $\mathscr{M}_S[\mathbb{P}]$

$$\begin{array}{ccccc} \Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S g & \xrightarrow{\Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S j} & \Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S gl_1 S & \xrightarrow{\epsilon} & S \\ \downarrow{\scriptstyle \Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S *} & & \downarrow & & \downarrow \\ S \simeq \Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S * & \longrightarrow & \Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S Cj & \longrightarrow & M\zeta \end{array} \tag{4.2}$$

which is to say that

$$M\zeta \cong \Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S Cj \wedge^{\mathrm{L}}_{\Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S gl_1 S} S \cong S \wedge^{\mathrm{L}}_{\Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S g} S.$$

Note that the left-hand square in diagram (4.2) is a homotopy pushout by definition of Cj and the fact that $\Sigma^\infty_{\mathbb{L}+}\Omega^\infty_S$ preserves homotopy pushouts. Also note that when writing this homotopy pushout, we are suppressing the choice of a point-set representative of the homotopy class ϵ. Since all objects are fibrant in the model structure on $\mathscr{M}_S[\mathbb{P}]$, it suffices to choose a cofibrant model for $\Omega^\infty_S gl_1 S$ (and subsequently of $\Omega^\infty_S g$) in the model structure on $\mathscr{M}_*[\mathbb{P}]$ (see [**7**, 4.19]).

Now suppose that R is a commutative S-algebra with unit $\iota\colon S \to R$; let $i = gl_1\iota$, and let $k = ij\colon g \to gl_1 R$, so that we have the solid arrows of the diagram

$$\begin{array}{ccccc} g & \xrightarrow{j} & gl_1 S & \longrightarrow & Cj \\ & {\scriptstyle k}\searrow & \downarrow{\scriptstyle i} & \swarrow{\scriptstyle u} & \\ & & gl_1 R & & \end{array} \tag{4.3}$$

in which the row is a cofiber sequence. The homotopy pushout diagram (4.2) and the adjunction (4.1) give the following.

Theorem 4.2. *The derived mapping space* $\mathrm{Map}_{\mathscr{M}_S[\mathbb{P}]}(M\zeta, R)$ *is equivalent to the fiber of the map of derived mapping spaces*

$$\mathrm{Map}_{\mathscr{M}_S}(Cj, gl_1 R) \longrightarrow \mathrm{Map}_{\mathscr{M}_S}(gl_1 S, gl_1 R)$$

over the basepoint associated to the map $i : gl_1 S \to gl_1 R$*. That is, the map* k *is the obstruction to the existence of an* E_∞ *map* $M\zeta \to R$*, and* $\mathrm{Map}_{\mathscr{M}_S[\mathbb{P}]}(M\zeta, R)$ *is weakly equivalent to the space of lifts in diagram* (4.3).

We have the following E_∞ analog of the usual Thom isomorphism.

Theorem 4.3. *If* $\mathrm{Map}_{\mathscr{M}_S[\mathbb{P}]}(M\zeta, R)$ *is non-empty* (*that is, if* k *is homotopic to the trivial map* $g \to gl_1 R$)*, then we have equivalences of derived mapping spaces*

$$\mathrm{Map}_{\mathscr{M}_S[\mathbb{P}]}(M\zeta, R) \simeq \Omega\,\mathrm{Map}_{\mathscr{M}_S}(g, gl_1 R) \simeq \mathrm{Map}_{\mathscr{M}_S}(b, gl_1 R) \simeq \mathrm{Map}_{\mathscr{M}_S[\mathbb{P}]}(\Sigma^\infty_{\mathbb{L}+}B, R).$$

More generally, suppose that R is a commutative S-algebra. There is a category $\mathscr{M}_R[\mathbb{P}]$ of commutative R-algebras; the functor $R \wedge_S (-)$ is the left adjoint of a Quillen pair connecting $\mathscr{M}_R[\mathbb{P}]$ and $\mathscr{M}_S[\mathbb{P}]$ (with right adjoint the forgetful functor). Therefore, we can consider the homotopical adjunction $(R \wedge \Sigma^\infty_{\mathbb{L}+}\Omega_S U, gl_1 U)$ connecting $\mathscr{M}_R$ and $\mathscr{M}_R[\mathbb{P}]$, where here U denotes both the forgetful functor $\mathscr{M}_R \to \mathscr{M}_S$ and $\mathscr{M}_R[\mathbb{P}] \to \mathscr{M}_S[\mathbb{P}]$, respectively. In further abuse of notation, we will suppress U and write $gl_1 R$ for $gl_1 UR$ and $\Sigma^\infty_{\mathbb{L}+}\Omega_S$ for $\Sigma^\infty_{\mathbb{L}+}\Omega_S U$.

Now, given a map

$$\zeta\colon b \longrightarrow bgl_1 R,$$

we obtain a map of cofiber sequences

$$\begin{array}{ccc} g & \longrightarrow & gl_1 R \\ \downarrow & & \downarrow \\ gl_1 R & = & gl_1 R \\ \downarrow & \nearrow & \downarrow \\ p & \longrightarrow & egl_1 R \simeq * \\ \downarrow & \nearrow & \downarrow \\ b & \xrightarrow{\zeta} & bgl_1 R \end{array}$$

in which $g = \Sigma^{-1} b$ and p is the fiber of $b \to bgl_1 R$.

Definition 4.4. The *R-algebra Thom spectrum* of ζ is the commutative R-algebra $M\zeta$ defined as the homotopy pushout in $\mathscr{M}_R[\mathbb{P}]$ of the diagram

$$\begin{array}{ccccc} R \wedge \Sigma^{\infty}_{\mathbb{L}+}\Omega^{\infty}_{S} g & \longrightarrow & R \wedge \Sigma^{\infty}_{\mathbb{L}+}\Omega^{\infty}_{S} gl_1 R & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ R \wedge \Sigma^{\infty}_{\mathbb{L}+}\Omega^{\infty}_{S} * & \longrightarrow & R \wedge \Sigma^{\infty}_{\mathbb{L}+}\Omega^{\infty}_{S} p & \longrightarrow & M\zeta. \end{array}$$

Again, note that the left-hand square is automatically a homotopy pushout in $\mathscr{M}_R[\mathbb{P}]$, which means that $M\zeta$ can be taken to be the homotopy pushout of the right-hand square or of the composite square.

The Thom R-algebra is a generalization of the Thom R-module of Definition 3.12.

Lemma 4.5. *The underlying R-module of the R-algebra Thom spectrum of ζ is weakly equivalent to the A_∞ Thom spectrum of $\Omega^\infty \zeta$.*

Proof. This follows from a check of the definitions given the fact that the homotopy pushout

$$B \longleftarrow A \longrightarrow C$$

in the category $\mathscr{M}_R[\mathbb{P}]$ is naturally weakly equivalent to the derived smash product $B \wedge^{\mathbb{L}}_A C$ (see [**12**, § VII.1.6]). ∎

Theorem 4.6. *Let A be a commutative R-algebra and write*

$$i\colon gl_1 R \longrightarrow gl_1 A,$$

for the induced map on unit spectra. The derived mapping space $\mathrm{Map}_{\mathscr{M}_R[\mathbb{P}]}(M\zeta, A)$ *is weakly equivalent to the fiber in the map of derived mapping spaces*

$$\mathrm{Map}_{\mathscr{M}_S}(p, gl_1 A) \longrightarrow \mathrm{Map}_{\mathscr{M}_S}(gl_1 R, gl_1 A), \tag{4.4}$$

at the basepoint associated to the map i.

Taking $A = R$, we see that the space of R-algebra orientations of $M\zeta$ is the space of lifts

$$\begin{array}{ccc} & & egl_1 R \\ & \nearrow & \downarrow \\ b & \xrightarrow[\zeta]{} & bgl_1 R. \end{array}$$

In this form, the obstruction theory is a generalization of the obstruction theory for orientations of A_∞ ring spectra in Theorem 3.16.

To make contact with the classical situation, let S be the sphere spectrum, and suppose that we are given a map

$$g\colon b \longrightarrow bgl_1 S,$$

so that $\Omega^\infty g$ classifies a stable spherical fibration.

Now suppose that R is a commutative S-algebra with unit $\iota\colon S \to R$, and let

$$f = bgl_1\iota \circ g\colon b \longrightarrow bgl_1 S \to bgl_1 R.$$

Then

$$Mf \simeq Mg \wedge^{\mathrm{L}} R,$$

and so extension of scalars induces an equivalence of derived mapping spaces

$$\mathrm{Map}_{\mathscr{M}_S[\mathbb{P}]}(Mg, R) \simeq \mathrm{Map}_{\mathscr{M}_R[\mathbb{P}]}(Mf, R).$$

If we let $b(S, R)$ be the homotopy pullback in the solid diagram

$$\begin{array}{ccccc} p & \longrightarrow & b(S,R) & \longrightarrow & * \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ b & \longrightarrow & bgl_1 S & \longrightarrow & bgl_1 R, \end{array} \tag{4.5}$$

then Theorem 4.6 specializes to a result of May, Quinn, Ray, and Tornehave [**27**].

Corollary 4.7. *The derived space of E_∞ maps $Mg \to R$ is weakly equivalent to the derived space of lifts in diagram* (4.5).

5. *Units after May–Quinn–Ray*

Our construction of the Thom spectrum in § 3 uses a model for the adjunction

$$(\text{group-like } A_\infty \text{ spaces}) \underset{GL_1}{\rightleftarrows} (A_\infty \text{ spaces}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} (A_\infty \text{ ring spectra})\colon GL_1,$$

which is a homotopical refinement of the standard adjunction

$$\mathbb{Z}\colon (\text{groups}) \rightleftarrows (\text{rings}).$$

For the E_∞ case, we use the E_∞ analog,

$$(\text{group-like } E_\infty \text{ spaces}) \underset{GL_1}{\rightleftarrows} (E_\infty \text{ spaces}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} (E_\infty \text{ ring spectra})\colon GL_1,$$

which is modeled on the analogous adjunction

$$\mathbb{Z}\colon (\text{abelian groups}) \rightleftarrows (\text{commutative rings}).$$

When A is an E_∞ ring spectrum, $GL_1 A$ is a group-like E_∞ space. Since group-like E_∞ spaces model connective spectra, it follows that there is a spectrum $gl_1 A$ such that

$$\Omega^\infty gl_1 A \simeq GL_1 A. \tag{5.1}$$

In this section, we give a precise model of the adjunction and combine it with a modernized version of the delooping result to prove the following theorem.

THEOREM 5.1. *The functors $\Sigma^\infty_+\Omega^\infty$ and gl_1 induce adjunctions*

$$\Sigma^\infty_+\Omega^\infty\colon \text{ho}((-1)\text{-connected spectra}) \rightleftarrows \text{ho}\,\mathscr{S}[E_\infty]\colon gl_1 \tag{5.2}$$

of categories enriched over the homotopy category of spaces.

Note that the construction of this adjunction realizes the left adjoint as a composite of left Quillen adjoints and Quillen equivalences and the right adjoint as a composite of right Quillen adjoints and Quillen equivalences. As a consequence, the left adjoint preserves homotopy colimits and the right adjoint preserves homotopy limits.

REMARK 5.2. In fact, Theorem 5.1 can be formulated as an adjunction of ∞-categories

$$\Sigma^\infty_+\Omega^\infty\colon ((-1)\text{-connected spectra}) \rightleftarrows \mathscr{S}[E_\infty]\colon gl_1 .$$

See the companion paper [**2**] and the subsequent paper [**1**] for a description of such an approach to the Thom spectrum functor.

Throughout this section, we work in the classical categories $\mathscr{S}$ of Lewis–May–Steinberger spectra [**18**] and $\mathscr{S}[E_\infty]$ of E_∞ ring spectra. As we noted in § 4, it is often useful to restate this adjunction using modern models for these homotopy categories. Since composition with an equivalence of categories preserves the property of being a left or right adjoint, such a shift is harmless.

The reader will note that a proof of Theorem 5.1 can mostly be assembled from results scattered in the literature, particularly [**12**, **18**, **24**, **25**, **27**]. We wrote this section in order to consolidate this material and in order to present modernized treatments using the language of model categories.

REMARK 5.3. We note that May has prepared a review of the relevant multiplicative infinite loop space theory [**28**], which also includes the results we need.

5.1. E_∞ *spectra*

In this section, we review the notion of a C-spectrum, where C is an operad (in spaces) over the linear isometries operad. We also recall the fact that the homotopy category of E_∞ spectra is well defined, in the sense that if C and D are two E_∞ operads over the linear isometries operad, then the categories of C-spectra and D-spectra are connected by a zig-zag of continuous Quillen equivalences.

If C is an operad, then, for $k \geqslant 0$, we write $C(k)$ for the kth space of the operad. We also write C for the associated monad. Let $\mathscr{S} = \mathscr{S}_{\mathscr{U}}$ denote the category of spectra based on a universe $\mathscr{U}$, in the sense of [**18**]. Let $\mathscr{L}$ denote the linear isometries operad of $\mathscr{U}$, and let

$C \to \mathscr{L}$ be an operad over $\mathscr{L}$. Then

$$CV = \bigvee_{k \geqslant 0} C(k) \ltimes_{\Sigma_k} V^{\wedge k}$$

is the free C-algebra on V. We write $\mathscr{S}[C]$ for the category of C-algebras in $\mathscr{S}$, and we call its objects C-spectra.

In general, $C(*) \cong \Sigma^{\infty}_{+} C(0)$ is the initial object of the category of C-spectra. We shall say that C is *unital* if $C(0) = *$, so that $C(0) \cong S$ is the sphere spectrum.

Lewis–May–Steinberger work with unital operads and the free C-spectrum with prescribed unit. If $S \to V$ is a spectrum under the sphere, then we write $C_* V$ for the free C spectrum on V with unit $\iota\colon S \to V \to C_* V$. This is the pushout in the category of C-spectra in the diagram

$$\begin{array}{ccc} CS & \longrightarrow & C(*) = S \\ \Big\downarrow{\scriptstyle C\iota} & & \Big\downarrow \\ CV & \longrightarrow & C_* V. \end{array} \tag{5.3}$$

By construction, C_* participates in a monad on the category $\mathscr{S}_{S/}$ of spectra under the sphere spectrum.

As explained in [**12**, II, Remark 4.9],

$$\mathbb{S}(V) = S \vee V$$

defines a monad on $\mathscr{S}$, using the fold map $S \vee S \to S$, and we have an equivalence of categories

$$\mathscr{S}_{S/} \cong \mathscr{S}[\mathbb{S}].$$

It follows that there is a natural isomorphism

$$C(V) \cong C_* \mathbb{S}(V), \tag{5.4}$$

and [**12**, II, Lemma 6.1] an equivalence of categories

$$\mathscr{S}[C] \cong \mathscr{S}_{S/}[C_*].$$

We recall the following, which can be proved easily using the argument of [**12**, **23**], in particular an adaptation of the 'Cofibration Hypothesis' [**12**, § VII].

PROPOSITION 5.4. *The category of C-spectra has the structure of a cofibrantly generated topological model category, in which the forgetful functor to $\mathscr{S}$ creates fibrations and weak equivalences. If $\{A \to B\}$ is a set of generating (trivial) cofibrations of $\mathscr{S}$, then $\{CA \to CB\}$ is a set of generating (trivial) cofibrations of $\mathscr{S}[C]$.*

In particular, the category of C-spectra is cocomplete (this is explained on [**12**, pp. 46—49]), a fact we use in the following construction. Let $f\colon C \to D$ be a map of operads over $\mathscr{L}$, so there is a forgetful functor

$$f^*\colon \mathscr{S}[D] \longrightarrow \mathscr{S}[C].$$

We construct the left adjoint $f_!$ of f^* as a certain coequalizer in C-algebras; see [**12**, § II.6] for further discussion of this construction.

Denote by $m\colon DD \to D$ the multiplication for D, and let A be a C-algebra with structure map $\mu\colon CA \to A$. Define $f_!A$ to be the coequalizer in the diagram of D-algebras

$$\begin{array}{ccc} DCA & \overset{D\mu}{\rightrightarrows} & DA \longrightarrow f_!A \\ & {\scriptstyle Df}\searrow \qquad \nearrow{\scriptstyle m} & \\ & DDA & \end{array} \tag{5.5}$$

In fact, it is enough to construct $f_!A$ as the coequalizer in spectra. Then D, applied to the unit $A \to CA$, makes the diagram a reflexive coequalizer of spectra, and so $f_!A$ has the structure of a D-algebra, and as such is the D-algebra coequalizer [**12**, § II.6.6]. By construction, we have the following proposition.

Proposition 5.5. *The functor $f_!$ is a continuous left adjoint to f^*; moreover, for any spectrum V, the natural map*

$$f_!CV \longrightarrow DV \tag{5.6}$$

is an isomorphism.

Remark 5.6. Some treatments write $C \otimes V$ for the free C-algebra CV, and then $D \otimes_C A$ for $f_!A$.

About this adjoint pair there is the following well-known result, which follows from the fact that f^* preserves fibrations and weak equivalences.

Proposition 5.7. *Let $f\colon C \to D$ be a map of operads over $\mathscr{L}$. The pair $(f_!, f^*)$ is a continuous Quillen pair.*

It is folklore that all E_∞ operads over $\mathscr{L}$ give rise to the same homotopy theory. Over the years, various arguments have been given to show this, starting with May's use of the bar construction to model $f_!$ (see [**12**, § II.4.3] for the most recent entry in this line). We present a model-theoretic formulation of this result (under mild hypotheses on the operads) in the remainder of the subsection.

Proposition 5.8. *If $f\colon C \to D$ is a map of E_∞ or A_∞ operads, then $(f_!, f^*)$ is a Quillen equivalence. More generally, if each map*

$$f\colon C(n) \longrightarrow D(n)$$

is a weak equivalence of spaces, then $(f_!, f^*)$ is a Quillen equivalence.

Before giving the proof, we make a few remarks. Assume that f is a weak equivalence of operads. Since the pullback $f^*\colon \mathscr{S}[D] \to \mathscr{S}[C]$ preserves fibrations and weak equivalences, to show that $(f_!, f^*)$ is a Quillen equivalence, it suffices to show, that for a cofibrant C-algebra X, the unit of the adjunction $X \to f^*f_!X$ is a weak equivalence.

If $X = CZ$ is a *free* C-algebra, then $f_!X = f_!CZ \cong DZ$ by (5.6), and so the map in question is the natural map

$$CZ \longrightarrow DZ.$$

It follows from [**12**, Propositions X.4.7, X.4.9, and A.7.4] that if the operad spaces $C(n)$ and $D(n)$ are CW-complexes, and if Z is a wedge of spheres or disks, then $CZ \to DZ$ is a homotopy equivalence. In fact, this argument applies to the wider class of *tame* spectra, whose definition we now recall.

DEFINITION 5.9 [**12**, Definition I.2.4]. A prespectrum D is Σ-*cofibrant* if each of the structure maps $\Sigma^W D(V) \to D(V \oplus W)$ is a (Hurewicz) cofibration. A spectrum Z is Σ-*cofibrant* if it is isomorphic to one of the form LD, where D is a Σ-cofibrant prespectrum and L denotes the spectrification functor [**18**, I.2.2]. A spectrum Z is *tame* if it is homotopy equivalent to a Σ-cofibrant spectrum. In particular, a spectrum Z of the homotopy type of a CW-spectrum is tame.

For a general cofibrant X, the argument proceeds by reducing to the free case $X = CZ$. In this paper, we present an inductive argument due to Mandell [**21**]. A different induction of this sort appeared in [**22**] in the algebraic setting; that argument can be adapted to the topological context with minimal modifications.

Our induction will involve the geometric realization of simplicial spectra. As usual, we would like to ensure that a map of simplicial spectra

$$f_\bullet \colon K_\bullet \longrightarrow K'_\bullet$$

in which each $f_n \colon K_n \to K'_n$ is a weak equivalence yields a weak equivalence upon geometric realization. The required condition is that the spectra K_n and K'_n are tame: [**12**, Theorem X.2.4] says that the realization of weak equivalences of tame spectra is a weak equivalence if $K_\bullet$ and $K'_\bullet$ are 'proper' [**12**, § X.2.1]. Recall that a simplicial spectrum $K_\bullet$ is proper if the natural map of coends

$$\int^{D_{q-1}} K_p \wedge D(q,p)_+ \longrightarrow \int^{D_q} K_p \wedge D(q,p)_+ \cong K_q$$

is a Hurewicz cofibration, where D is the subcategory of Δ consisting of the monotonic surjections (that is, the degeneracies), and D_q is the full subcategory of D on the objects $0 \leqslant i \leqslant q$. This is a precise formulation of the intuitive notion that the inclusion of the union of the degenerate spectra $s_j K_{q-1}$ in K_q should be a Hurewicz cofibration.

Thus, to ensure that the spectra that arise in our argument are tame and the simplicial objects are proper, we make the following simplifying assumptions on our operads.

(1) We assume that the spaces $C(n)$ and $D(n)$ have the homotopy type of Σ_n-CW-complexes.

(2) We assume that $C(1)$ and $D(1)$ are equipped with non-degenerate basepoints.

We believe these assumptions are reasonable, insofar as they are satisfied by many natural examples; for instance, the linear isometries operad and the little n-cubes operad both satisfy the hypotheses above (see [**12**, XI.1.4, XI.1.7; **24**, 4.8], respectively). More generally, if $\mathcal{O}$ is an arbitrary operad over the linear isometries operad, then taking the geometric realization of the singular complex of the spaces $\mathcal{O}$ produces an operad $|S(\mathcal{O})|$ with the properties we require.

Goerss and Hopkins have proved two versions of Proposition 5.8 using resolution model structures to resolve an arbitrary cofibrant C-space by a simplicial C-space with free k-simplices for every k. A first version (unpublished) proves the proposition for Lewis–May–Steinberger spectra, avoiding our simplifying assumptions on the operads via a detailed study of 'flatness' for spectra (as an alternative to the theory of 'tameness'). A more modern treatment [**13**] works with operads of simplicial sets and symmetric spectra in topological spaces. In that case, as they explain, a key point is that if X is a cofibrant spectrum, then $X^{(n)}$ is a *free* Σ_n-spectrum (see [**23**, Lemma 15.5]). This observation helps explain why the general form of the proposition is reasonable, even though the analogous statement for spaces is much too strong. We now give the proof of Proposition 5.8 under the hypotheses enumerated above.

Proof. A cofibrant C-spectrum is a retract of a cell C-spectrum, and so we can assume without loss of generality that X is a cell C-spectrum. The argument for Proposition 5.4 implies that cell objects can be described as $X = \operatorname{colim}_n X_n$, where $X_0 = C(*)$ and X_{n+1} is obtained from X_n via a pushout (in C-algebras) of the form

$$\begin{array}{ccc} CA & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ CB & \longrightarrow & X_{n+1} \end{array}$$

where $A \to B$ is a wedge of generating cofibrations of spectra. Furthermore, by the proof of Proposition 5.4 (specifically, the Cofibration Hypothesis), the map $X_n \to X_{n+1}$ is a Hurewicz cofibration of spectra. The hypotheses on C and the fact that A and B are CW-spectra imply that CA and CB have the homotopy type of CW-spectra, and thus inductively so does X_n. Therefore, since $f_!$ is a left adjoint, it suffices to show that $X_n \to f^* f_! X_n$ is a weak equivalence for each X_n; under these circumstances, a sequential colimit of weak equivalences is a weak equivalence.

We proceed by induction on the number of stages required to build the C-spectrum. The base case follows from the remarks preceding the proof. For the induction hypothesis, assume that $f_!$ is a weak equivalence for all cell C-algebras that can be built in n or fewer stages. The spectrum X_{n+1} is a pushout $CB \coprod_{CA} X_n$ in C-algebras, and this pushout is homeomorphic to a bar construction $B(CB, CA, X_n)$, which is the geometric realization of a simplicial spectrum where the mth space is the coproduct $CB \coprod^m CA \coprod X_n$. Since $f_!$ is a continuous left adjoint, it commutes with geometric realization and coproducts in C-algebras, and so $f_!(B(CB, CA, X_n))$ is homeomorphic to $B(DB, DA, f_! X_n)$.

The bar constructions we are working with are proper simplicial spectra by the hypothesis that $C(1)$ and $D(1)$ have non-degenerate basepoints, and thus it suffices to show that, at each level in the bar construction

$$B_q(CB, CA, X_n) \longrightarrow B_q(DB, DA, f_! X_n),$$

we have a weak equivalence of tame spectra. This follows from the inductive hypothesis: we have already shown that the spectra are tame, and $CB \coprod^q CA \coprod X_n$ can be built in n stages, since X_n can be built in n stages and the free algebras can be built and added in a single stage. □

The idea of the following corollary goes all the way back to [**24**].

Corollary 5.10. *If C and D are any two E_∞ operads over the linear isometries operad, then the categories of C-algebras and D-algebras are connected by a zig-zag of continuous Quillen equivalences.*

Proof. Proposition 5.8 allow us to compare each of the categories of algebras to algebras over the linear isometries operad. □

Backed by this result, we adopt the following convention.

Definition 5.11. We write $\mathrm{ho}\,\mathscr{S}[E_\infty]$ for the homotopy category of E_∞ ring spectra. By this, we mean the homotopy category $\mathrm{ho}\,\mathscr{S}[C]$ for any E_∞ operad C over the linear isometries operad.

5.2. E_∞ *spaces*

We adopt notation for operad actions on spaces analogous to our notation for spectra in § 5.1. Let C be an operad in topological spaces. The free C-algebra on a space X is

$$CX = \coprod_{k \geqslant 0} C(k) \times_{\Sigma_k} X^k. \tag{5.7}$$

We set $C(\emptyset) = C(0)$. The category of C-algebras in spaces, or C-spaces, will be denoted by $\mathscr{T}[C]$.

Note that the sequence of spaces given by

$$\begin{aligned} P(0) &= * = P(1), \\ P(k) &= \emptyset \quad \text{for } k > 1 \end{aligned}$$

has a unique operad structure, whose associated monad is

$$PX = X_+,$$

so

$$\mathscr{T}[P] \cong \mathscr{T}_*.$$

If C is a unital operad and if Y is a pointed space, let C_*Y be the pushout in the category of C-algebras

$$\begin{array}{ccc} C* & \longrightarrow & C(\emptyset) = * \\ \downarrow & & \downarrow \\ CY & \longrightarrow & C_*Y. \end{array} \tag{5.8}$$

Then C_* participates in a monad on the category of pointed spaces. Indeed C_* is isomorphic to the monad C_{May} introduced in [**24**], since, for a test C-space T,

$$\mathscr{T}[C](C_*Y, T) \cong \mathscr{T}_*(Y, T) \cong \mathscr{T}[C](C_{\mathrm{May}}Y, T).$$

There is a natural isomorphism

$$CX \cong C_*(X_+),$$

and an equivalence of categories

$$\mathscr{T}[C] \cong \mathscr{T}_*[C_*]. \tag{5.9}$$

Part of this equivalence is the observation that if X is a C-algebra, then it is a C_*-algebra via

$$C_*X \longrightarrow C_*(X_+) \cong CX \longrightarrow X.$$

We have the following analog of Proposition 5.4.

Proposition 5.12. (1) *The category $\mathscr{T}[C]$ has the structure of a cofibrantly generated topological closed model category, in which the forgetful functor to $\mathscr{T}$ creates fibrations and weak equivalences. If $\{A \to B\}$ is a set of generating (trivial) cofibrations of $\mathscr{T}$, then $\{CA \to CB\}$ is a set of generating (trivial) cofibrations of $\mathscr{T}[C]$.*

(2) *The analogous statements hold for C_* and $\mathscr{T}_*[C_*]$.*

(3) *Taking* $C = P$*, the resulting model category structure on the category* $\mathscr{T}[P] \cong \mathscr{T}_*$ *is determined by the forgetful functor to* $\mathscr{T}$*.*

(4) *The equivalence* $\mathscr{T}[C] \cong \mathscr{T}_*[C_*]$ (5.9) *carries the model structure arising from part* (1) *to the model structure arising from part* (2).

Proof. The statements about the model structure on $\mathscr{T}[C]$ or on $\mathscr{T}_*[C_*]$ can be proved, for example, by adapting the argument in [**12**, **23**]. The third part is standard, and together the first three parts imply the last. □

We conclude this subsection with two results that will be useful in § 5.5. For the first, note that a point of $C(0)$ determines a map of operads

$$P \longrightarrow C,$$

and so we have a forgetful functor

$$\mathscr{T}[C] \longrightarrow \mathscr{T}[P] \cong \mathscr{T}_*.$$

We say that a point of Y is *non-degenerate* if the inclusion $* \to Y$ is a Neighborhood Deformation Retract (NDR), that is, a Hurewicz cofibration.

Proposition 5.13. *Suppose that* C *is a unital operad in topological spaces* (*or more generally, an operad in which the basepoint of* $C(0)$ *is non-degenerate*)*. If* X *is a cofibrant object of* $\mathscr{T}_*[C_*]$*, then its basepoint is non-degenerate.*

Note that Rezk [**31**] and Berger and Moerdijk [**5**] have proved a similar result, for algebras in a general model category over a *cofibrant* operad. In our case, we need only assume that the zero space $C(0)$ of our operad has a non-degenerate basepoint.

Proof. In the model structure described in Proposition 5.12, a cofibrant object is a retract of a cell object, and so we can assume without loss of generality that X is a cell C-space. That is,

$$X = \operatorname*{colim}_n X_n, \tag{5.10}$$

where $X_0 = C(\emptyset)$ and X_{n+1} is obtained from X_n as a pushout in C-spaces

$$\begin{array}{ccc} CA & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ CB & \longrightarrow & X_{n+1} \end{array} \tag{5.11}$$

where $A \to B$ is a disjoint union of generating cofibrations of $\mathscr{T}$.

Our argument relies on a form of the Cofibration Hypothesis [**12**, § VII]. The key points are the following.

(1) By assumption $X_0 = C(\emptyset) = C(0)$ is non-degenerately based.
(2) The space underlying the C-algebra colimit X in (5.10) is just the space-level colimit.
(i) In the pushout above,

$$X_n \longrightarrow X_{n+1}$$

is a based map and an unbased Hurewicz cofibration.

The second point is easily checked (and is the space-level analog of [**12**, Lemma 3.10]). For the last part, the argument in [**12**, Proposition 3.9 of § VII] (see also [**23**, Lemma 15.9]) shows that the pushout (5.11) is isomorphic to a two-sided bar construction $B(CB, CA, X_n)$: this is the geometric realization of a simplicial space where the k-simplices are given as

$$CB \coprod_C (CA)^{\coprod k} \coprod_C X_n,$$

and the simplicial structure maps are induced by the folding map and the maps $CA \to CB$ and $CA \to X_n$. Note that by $\coprod_C$ we mean the coproduct in the category of C-spaces. Recall that coproducts (and more generally all colimits) in C-spaces admit a description as certain coequalizers in $\mathscr{T}$. Specifically, for C-spaces X and Y the coproduct $X \coprod_C Y$ can be described as the coequalizer in $\mathscr{T}$

$$C(CX \coprod CY) \rightrightarrows C(X \coprod Y) \longrightarrow X \coprod_C Y,$$

where the unmarked coproducts are taken in $\mathscr{T}$ and the maps are induced from the action maps and the monadic structure map. Following an argument along the lines of [**12**, § VII.6], we can show that, for any C-algebra A and space B, the map $A \to A \coprod_C CB$ is an inclusion of a component in a disjoint union.

This implies that the simplicial degeneracy maps in the bar construction are unbased Hurewicz cofibrations and hence that the simplicial space is proper, that is, Reedy cofibrant in the Hurewicz/Strøm model structure. Thus, the inclusion of the zero simplices $CB \coprod_C X_n$ in the realization is an unbased Hurewicz cofibration, and hence the map $X_n \to X_{n+1}$ is itself a unbased Hurewicz cofibration. As a map of C-algebras, it is also a based map. □

The second result we need is the following.

Proposition 5.14. *Let C be an operad and suppose that each $C(n)$ has the homotopy type of a Σ_n-CW complex. Let X be a C-space with the homotopy type of a cofibrant C-space. Then CX has the homotopy type of a cofibrant C-space and the underlying space of X has the homotopy type of a CW-complex.*

Proof. The first statement is an easy consequence of the fact that C preserves homotopies and cofibrant objects. To see the second, observe that the forgetful functor preserves homotopies, so it suffices to suppose that X is a cofibrant C-space. Under our hypotheses on C, if A has the homotopy type of a CW-complex, then so does the underlying space of CA (see, for instance, [**18**, p. 372] for a proof). The result now follows from an inductive argument along the lines of the preceding proposition. □

5.3. *E_∞ spaces and E_∞ spectra*

Suppose that $C \to \mathscr{L}$ is an operad over $\mathscr{L}$. In this section, we recall the proof of the following result.

Proposition 5.15 (**18**, p. 366; **27**). *The continuous Quillen pair*

$$\Sigma^\infty_+ : \mathscr{T} \leftrightarrows \mathscr{S} : \Omega^\infty \tag{5.12}$$

induces by restriction a continuous Quillen adjunction

$$\Sigma^\infty_+ : \mathscr{T}_*[C_*] \cong \mathscr{T}[C] \leftrightarrows \mathscr{S}[C] : \Omega^\infty, \tag{5.13}$$

between topological model categories.

The first thing to observe is that C and Σ^∞_+ satisfy a strong compatibility condition.

LEMMA 5.16. *There is a natural isomorphism*

$$C\Sigma^\infty_+ X \cong \Sigma^\infty_+ CX. \tag{5.14}$$

Proof. It follows from [18, § VI, Proposition 1.5] that if X is a space, then

$$C(k) \ltimes_{\Sigma_k} (\Sigma^\infty_+ X)^{\wedge k} \cong \Sigma^\infty_+ (C(k) \times_{\Sigma_k} X^k),$$

and so

$$\begin{aligned} C\Sigma^\infty_+ X = \bigvee_{k\geqslant 0} C(k) \ltimes_{\Sigma_k} (\Sigma^\infty_+ X)^{\wedge k} &\cong \bigvee_{k \geqslant 0} \Sigma^\infty_+ (C(k) \times X^k) \\ &\cong \Sigma^\infty_+ \left(\coprod_{k\geqslant 0} C(k) \times X^k \right) = \Sigma^\infty_+ CX. \end{aligned}$$ □

Next we have the following from [18, p. 366].

LEMMA 5.17. *The adjoint pair*

$$\Sigma^\infty_+ \colon \mathscr{T} \leftrightarrows \mathscr{S} \colon \Omega^\infty \tag{5.15}$$

induces an adjunction

$$\Sigma^\infty_+ \colon \mathscr{T}[C] \leftrightarrows \mathscr{S}[C] \colon \Omega^\infty, \tag{5.16}$$

and so also

$$\Sigma^\infty_+ \colon \mathscr{T}_*[C_*] \cong \mathscr{T}[C] \leftrightarrows \mathscr{S}[C] \colon \Omega^\infty.$$

Proof. We show that adjunction (5.15) restricts to adjunction (5.16). If X is a C-space with structure map $\mu\colon CX \to X$, then, using isomorphism (5.14), $\Sigma^\infty_+ X$ is a C-algebra via

$$C\Sigma^\infty_+ X \cong \Sigma^\infty_+ CX \xrightarrow{\Sigma^\infty_+ \mu} \Sigma^\infty_+ X.$$

If A is a C-spectrum, then $\Omega^\infty A$ is a C-space via

$$C\Omega^\infty A \longrightarrow \Omega^\infty CA \longrightarrow \Omega^\infty A.$$

The second map is just Ω^∞ applied to the C-structure on A; the first map is the adjoint of the map

$$\Sigma^\infty_+ C\Omega^\infty A \cong C\Sigma^\infty_+ \Omega^\infty A \longrightarrow CA$$

obtained using the counit of the adjunction. □

This adjunction allows us to prove the pointed analog of Lemma 5.16.

LEMMA 5.18 [18, § VII, Proposition 3.5]. *If C is a unital operad over $\mathscr{L}$, then there is a natural isomorphism*

$$\Sigma^\infty_+ C_* Y \cong C_* \Sigma^\infty_+ Y \cong C\Sigma^\infty Y. \tag{5.17}$$

Proof. Let Y be a pointed space. By Lemma 5.17 and the isomorphism (5.14), applying the left adjoint Σ^∞_+ to the pushout diagram (5.8) defining $C_* Y$ identifies $\Sigma^\infty_+ C_* Y$ with the pushout of diagram (5.3) defining $C_* \Sigma^\infty_+ Y$. The second isomorphism is just the isomorphism (5.4) together with the isomorphism (for pointed spaces) Y

$$\Sigma^\infty_+ Y \cong \Sigma^\infty(S \vee Y).$$ □

Proof of Proposition 5.15. It remains to show that the adjoint pair $(\Sigma^\infty_+, \Omega^\infty)$ induces a Quillen adjunction. For this, it suffices to show that the right adjoint Ω^∞ preserves fibrations and weak equivalences (see, for example, [**15**, Lemma 1.3.4]). Now recall that the forgetful functor $\mathscr{S}[C] \to \mathscr{S}$ creates fibrations and weak equivalences, and similarly for $\mathscr{T}$ (see [**12**, **23**]). It follows that the functor

$$\Omega^\infty \colon \mathscr{S}[C] \longrightarrow \mathscr{T}[C]$$

preserves fibrations and weak equivalences, since

$$\Omega^\infty \colon \mathscr{S} \longrightarrow \mathscr{T}$$

does. □

Remark 5.19. Note that if A is an E_∞ ring spectrum, then $\Omega^\infty A$ is an E_∞ space in two ways: one is described above, and arises from the multiplication on A. The other arises from the additive structure of A, that is, the fact that $\Omega^\infty A$ is an infinite loop space. Together these two E_∞ structures give an E_∞ ring space in the sense of [**27**] (see also [**28**]).

5.4. *E_∞ spaces and group-like E_∞ spaces*

Suppose that C is a unital E_∞ operad, and let X be a C-algebra in spaces. The structure maps

$$* \longrightarrow C(0) \longrightarrow X,$$
$$C(2) \times X \times X \longrightarrow X$$

correspond to a family of H-space structures on X and give to $\pi_0 X$ the structure of a monoid.

Definition 5.20. The C-algebra X is said to be *group-like* if $\pi_0 X$ is a group. We write $\mathscr{T}[C]^\times$ for the full subcategory of $\mathscr{T}[C]$ consisting of group-like C-spaces.

Note that if $f \colon X \to Y$ is a weak equivalence of C-spaces, then X is group-like if and only if Y is.

Definition 5.21. We write $\mathrm{ho}\,\mathscr{T}[C]^\times$ for the image of $\mathscr{T}[C]^\times$ in $\mathrm{ho}\,\mathscr{T}[C]$. It is the full subcategory of homotopy types represented by group-like spaces.

If X is a C-space, then note that $GL_1 X$ defined as in Definition 2.1 is a group-like C-space.

Proposition 5.22. *The functor GL_1 is the right adjoint of the inclusion*

$$\mathscr{T}[C]^\times \longrightarrow \mathscr{T}[C].$$

Proof. If X is a group-like C-space, and Y is a C-space, then

$$\mathscr{T}[C](X, Y) \cong \mathscr{T}[C]^\times(X, GL_1 Y);$$

just as if G is a group and M is a monoid, then

$$(\text{monoids})(G, M) = (\text{groups})(G, GL_1 M). \qquad \square$$

5.5. *Group-like E_∞ spaces and connective spectra*

A guiding result of infinite loop space theory is that group-like E_∞ spaces provide a model for connective spectra. We take a few pages to show how the primary sources (in particular [**8**, **24**, **25**]) may be used to prove a formulation of this result in the language of model categories.

To begin, suppose that C is a *unital* E_∞ operad, and f is a map of monads (on pointed spaces)

$$f\colon C_* \longrightarrow Q \overset{\text{def}}{=} \Omega^\infty \Sigma^\infty.$$

For example, we can take C to be a unital E_∞ operad over the infinite little cubes operad, but it is interesting to note that any map of monads will do. If V is a spectrum, then $\Omega^\infty V$ is a group-like C-algebra, via the map

$$C_* \Omega^\infty V \xrightarrow{f} \Omega^\infty \Sigma^\infty \Omega^\infty V \longrightarrow \Omega^\infty V.$$

Thus, we have a factorization

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\Omega^f} & \mathscr{T}[C]^\times \\ & {\scriptstyle \Omega^\infty}\searrow & \downarrow \\ & & \mathscr{T}_*. \end{array} \tag{5.18}$$

We next show that the functor Ω^f has a left adjoint Σ^f. By regarding a C-space X as a pointed space via $* \to C(0) \to X$, we may form the spectrum $\Sigma^\infty X$. Let $\Sigma^f X$ be the coequalizer in the diagram of spectra

$$\begin{array}{ccccc} \Sigma^\infty C_* X & \overset{\Sigma^\infty \mu}{\rightrightarrows} & \Sigma^\infty X & \longrightarrow & \Sigma^f X \\ {\scriptstyle \Sigma^\infty f}\searrow & & \nearrow & & \\ & \Sigma^\infty \Omega^\infty \Sigma^\infty X. & & & \end{array}$$

Then we have the following lemma.

LEMMA 5.23. *The pair*

$$\Sigma^f\colon \mathscr{T}[C] \leftrightarrows \mathscr{S} : \Omega^f \tag{5.19}$$

is a Quillen pair. Moreover, the natural transformation

$$\Sigma^f C_* \longrightarrow \Sigma^\infty$$

is an isomorphism.

Proof. As mentioned in the proof of Proposition 5.5, it is essentially a formal consequence of the construction that Σ^f is the left adjoint of Ω^f. Given the adjunction, we find that $\Sigma^f C_* \cong \Sigma^\infty$, since, for any pointed space X and any spectrum V, we have

$$\begin{aligned} \mathscr{S}(\Sigma^f C_* X, V) &\cong \mathscr{T}[C](C_* X, \Omega^f V) \\ &\cong \mathscr{T}_*(X, \Omega^\infty V) \\ &\cong \mathscr{S}(\Sigma^\infty X, V). \end{aligned}$$

To show that we have a Quillen pair, it suffices [**15**, Lemma 1.3.4] to show that Ω^f preserves weak equivalences and fibrations. This follows from the commutativity of diagram (5.18), the fact that Ω^∞ preserves weak equivalences and fibrations, and the fact that the forgetful functor

$$\mathscr{T}[C] \longrightarrow \mathscr{T}$$

creates fibrations and weak equivalences. □

Lemma 5.23 implies that the pair (Σ^f, Ω^f) induces a continuous Quillen adjunction

$$\Sigma^f\colon \mathscr{T}[C] \leftrightarrows \mathscr{S}\colon \Omega^f.$$

It is easy to see that this cannot be a Quillen equivalence. Instead, one expects it to induce an equivalence between the homotopy categories of *group-like* C-spaces and *connective* spectra. In [**23**, 0.10], this situation is called a 'connective Quillen equivalence'. The rest of this subsection is devoted to the proof of the following result along these lines.

Theorem 5.24. *Suppose that C is a unital E_∞ operad, equipped with a map of monads*

$$f\colon C \longrightarrow \Omega^\infty\Sigma^\infty.$$

Suppose moreover that

(1) *the basepoint $* \to C(1)$ is non-degenerate and*
(2) *for each n, the n-space $C(n)$ has the homotopy type of a Σ_n-CW-complex.*

Then the adjunction (Σ^f, Ω^f) induces an equivalence of categories

$$\Sigma^f\colon \operatorname{ho}\mathscr{T}[C]^\times \rightleftarrows \operatorname{ho}(\textit{connective spectra})\colon \Omega^f$$

enriched over $\operatorname{ho}\mathscr{T}$.

Remark 5.25. As observed in [**24**, A.8], adding a whisker to a degenerate basepoint produces a new operad C' from C. Also, if C is a unital E_∞ operad equipped with a map of monads $f\colon C \to \Omega^\infty\Sigma^\infty$, then taking the geometric realization of the singular complex of the spaces $C(n)$ produces an operad $|S(C)|$ with the properties we require.

The following lemma, easily checked, is implicit in [**23**]. Let

$$F\colon \mathcal{M} \leftrightarrows \mathcal{M}'\colon G$$

be a Quillen adjunction between topological closed model categories. Let $\mathcal{C} \subseteq \mathcal{M}$ and $\mathcal{C}' \subseteq \mathcal{M}'$ be full subcategories, stable under weak equivalence, so we have sensible subcategories $\operatorname{ho}\mathcal{C} \subseteq \operatorname{ho}\mathcal{M}$ and $\operatorname{ho}\mathcal{C}' \subseteq \operatorname{ho}\mathcal{M}'$. Suppose that F takes $\mathcal{C}$ to $\mathcal{C}'$, and G takes $\mathcal{C}'$ to $\mathcal{C}$.

Lemma 5.26. *If, for every cofibrant $X \in \mathcal{C}$ and every fibrant $Y \in \mathcal{C}'$, a map*

$$\phi\colon FX \longrightarrow Y$$

is a weak equivalence if and only if its adjoint

$$\psi\colon X \longrightarrow GY$$

is, then F and G induce equivalences

$$F\colon \operatorname{ho}\mathcal{C} \leftrightarrows \operatorname{ho}\mathcal{C}'\colon G$$

of categories enriched over $\operatorname{ho}\mathscr{T}$.

The key result in our setting is the following classical proposition; we recall the argument from [**24**, **25**].

Proposition 5.27. *Let C be a unital E_∞ operad, equipped with a map of monads*

$$f\colon C \longrightarrow \Omega^\infty\Sigma^\infty.$$

Suppose that the basepoint $* \to C(1)$ is non-degenerate, and that each $C(n)$ has the homotopy type of a Σ_n-CW-complex. If X is a cofibrant C-space, then the unit of the adjunction

$$X \longrightarrow \Omega^f\Sigma^f X$$

is group completion, and so a weak equivalence if X is group-like.

The proof of the proposition follows from analysis of the following commutative diagram of simplicial C-spaces:

$$\begin{array}{ccc} B_\bullet(C_*, C_*, X) & \longrightarrow & \Omega^f\Sigma^f B_\bullet(C_*, C_*, X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \Omega^f\Sigma^f X. \end{array} \tag{5.20}$$

Specifically, we will show that under the hypotheses, on passage to realization the vertical maps are weak equivalences and the top horizontal map is group completion.

We begin by studying the left-hand vertical map; the usual simplicial contraction argument shows the underlying map of spaces is a homotopy equivalence, and so on passage to realization we have a weak equivalence of C-spaces.

Lemma 5.28. *For any operad C and any C-space X, the left vertical arrow is a map of simplicial C-spaces and a homotopy equivalence of simplicial spaces, and so induces a weak equivalence of C-spaces*

$$B(C_*, C_*, X) \longrightarrow X$$

upon geometric realization.

The right vertical map is more difficult to analyze, because we do not know that Σ^f preserves homotopy equivalences of spaces. May [**24**, 12.3] shows that, for suitable simplicial pointed spaces $Y_\bullet$, the natural map

$$|\Omega Y_\bullet| \longrightarrow \Omega|Y_\bullet| \tag{5.21}$$

is a weak equivalence, and he explains in [**28**, § 8] how this weak equivalence gives rise to a weak equivalence of C-spaces

$$|\Omega^f\Sigma^f B_\bullet(C_*, C_*, X)| \longrightarrow \Omega^f|\Sigma^f B_\bullet(C_*, C_*, X)| \cong \Omega^f\Sigma^f B(C_*, C_*, X),$$

by passage to colimits. (We note that in [**28**], May describes proving that (5.21) is a weak equivalence as the hardest thing in [**24**].) Therefore, to show that the map

$$|\Omega^f\Sigma^f B_\bullet(C_*, C_*, X)| \longrightarrow \Omega^f\Sigma^f X$$

is a weak equivalence, it suffices to show that, for cofibrant X, the map

$$\Sigma^f B(C_*, C_*, X) \longrightarrow \Sigma^f X$$

is a weak equivalence. As it is straightforward to check from the definition that Σ^f does preserve weak equivalences between C-spaces with the homotopy type of cofibrant C-spaces, the desired result will follow once we show that $B(C_*, C_*, X)$ has the homotopy type of a cofibrant C-space if X is cofibrant.

LEMMA 5.29. *Suppose that C is a unital operad, such that the basepoint $* \to C(1)$ is non-degenerate and each $C(n)$ has the homotopy type of a Σ_n-CW-complex. Let X be a cofibrant C-space. Then $B(C_*, C_*, X)$ has the homotopy type of a cofibrant C-space.*

Proof. With our hypotheses, it follows from Proposition 5.14 that the spaces $C_*^n X$ have the homotopy type of cofibrant C-spaces. By Proposition 5.13, the simplicial space $B_\bullet(C_*, C_*, X)$ is proper. Finally, we apply an argument analogous to that of [**12**, Theorem X.2.7] to show that if $Y_\bullet$ is a proper C-space in which each level has the homotopy type of a cofibrant C-space, then $|Y_\bullet|$ has the homotopy type of a cofibrant C-space. □

Finally, we consider the top horizontal map in (5.20). We have *isomorphisms* of simplicial C-spaces

$$\Omega^f \Sigma^f B_\bullet(C_*, C_*, X) \cong B_\bullet(\Omega^f \Sigma^f C_*, C_*, X) \cong B_\bullet(\Omega^f \Sigma^\infty, C_*, X) \cong B_\bullet(Q, C_*, X)$$

(we used the isomorphism $\Sigma^f C_* \cong \Sigma^\infty$ of Lemma 5.23), and so an isomorphism of C-spaces

$$B(Q, C_*, X) \cong |\Omega^f \Sigma^f B_\bullet(C_*, C_*, X)|.$$

We then apply the following result from [**25**, § 2].

LEMMA 5.30. *Let C be a unital E_∞ operad, equipped with a map of monads*

$$f \colon C_* \longrightarrow \Omega^\infty \Sigma^\infty.$$

Let X be a C-space (and so pointed via $C(0) \to X$). Suppose that the basepoint of $C(1)$ and the basepoint of X are non-degenerate. Then the map

$$B(C_*, C_*, X) \longrightarrow B(Q, C_*, X),$$

and so

$$B(C_*, C_*, X) \longrightarrow |\Omega^f \Sigma^f B_\bullet(C_*, C_*, X)|$$

is group completion.

Proof. The point is that, in general,

$$C_* Y \longrightarrow \Omega^\infty \Sigma^\infty Y$$

is group completion [**9**, **10**, **30**], and so we have the level-wise group completion

$$C_*(C_*)^n X \longrightarrow \Omega^\infty \Sigma^\infty (C_*)^n X$$

(see [**25**, 2.2]).

The argument requires the simplicial spaces involved to be 'proper,' that is, Reedy cofibrant with respect to the Hurewicz/Strøm model structure on topological spaces, so that the homology spectral sequences have the expected E_2-term. May proves that they are, provided that $(C(1), *)$ and $(X, *)$ are NDR-pairs. □

We can now complete the proof of Theorem 5.24.

Proof. It remains to show that if X is a group-like cofibrant C-algebra and V is a (fibrant) (-1)-connected spectrum, then a map

$$\phi\colon \Sigma^f X \longrightarrow V$$

is a weak equivalence if and only if its adjoint

$$\psi\colon X \longrightarrow \Omega^f V$$

is. These two maps are related by the factorization

$$\psi\colon X \longrightarrow \Omega^f \Sigma^f X \xrightarrow{\Omega^f \phi} \Omega^f V.$$

The unit of adjunction is a weak equivalence by Proposition 5.27. It follows that ψ is a weak equivalence if and only if $\Omega^f \phi$ is. Certainly, if ϕ is a weak equivalence, then so is $\Omega^f \phi$. Since both $\Sigma^f X$ and V are (-1)-connected, if $\Omega^f \phi$ is a weak equivalence, then so is ϕ. □

Remark 5.31. There is another perspective on Theorem 5.24 which elucidates the role of the 'group-like' condition on C-spaces. Define a map

$$\alpha\colon X \longrightarrow Y$$

of C-spaces to be a *stable* equivalence if the induced map

$$\Sigma^f \alpha'\colon \Sigma^f X' \longrightarrow \Sigma^f Y'$$

is a weak equivalence, where X' and Y' are cofibrant replacements of X and Y. The 'stable' model structure on C-spaces is the localization of the model structure we have been considering in which the weak equivalences are the stable equivalences, and the cofibrations are as before.

In this stable model structure a C-space is fibrant if and only if it is group-like; compare the model structure on Γ-spaces discussed in [**23**, § 18; **33**]. The homotopy category associated with the stable model structure is exactly $\mathrm{ho}\,\mathscr{T}[C]^\times$, and so this is a better encoding of the homotopy theory of $\mathcal{C}$-spaces.

5.6. *Proof of Theorem* 5.1

Let C be a unital E_∞ operad, equipped with a map of operads

$$C \longrightarrow \mathscr{L},$$

a map of monads on pointed spaces

$$f\colon C_* \longrightarrow \Omega^\infty \Sigma^\infty,$$

and satisfying the hypotheses of Theorem 5.24. For example, we can take C to be

$$C = |\,\mathrm{Sing}(\mathcal{C} \times \mathscr{L})|,$$

the geometric realization of the singular complex on the product operad $\mathcal{C} \times \mathscr{L}$, where $\mathcal{C}$ is the infinite little cubes operad of Boardman and Vogt [**8**].

Then we have a sequence of continuous adjunctions (the left adjoints are listed on top, and connective Quillen equivalence is indicated by $\approx$):

$$\Sigma^\infty_+ \Omega^\infty\colon ((-1)\text{-connected spectra}) \underset{\Omega^f}{\overset{\Sigma^f,\approx}{\rightleftarrows}} \mathscr{T}[C]^\times \underset{GL_1}{\rightleftarrows} \mathscr{T}[C] \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathscr{S}[C]\colon gl_1 .$$

By Proposition 5.7, $\mathscr{S}[C]$ is a model for the category of E_∞ spectra. This completes the proof of Theorem 5.1.

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