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UNIVERSALITY OF THE HOMOTOPY INTERLEAVING DISTANCE

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Abstract. As a step towards establishing homotopy-theoretic foundations for topological data analysis (TDA), we introduce and study *homotopy interleavings* between filtered topological spaces. These are homotopy-invariant analogues of *interleavings*, objects commonly used in TDA to articulate stability and inference theorems. Intuitively, whereas a strict interleaving between filtered spaces X and Y certifies that X and Y are approximately isomorphic, a homotopy interleaving between X and Y certifies that X and Y are approximately weakly equivalent.

The main results of this paper are that homotopy interleavings induce an extended pseudometric d_{HI} on filtered spaces, and that this is the universal pseudometric satisfying natural stability and homotopy invariance axioms. To motivate these axioms, we also observe that d_{HI} (or more generally, any pseudometric satisfying these two axioms and an additional "homology bounding" axiom) can be used to formulate lifts of several fundamental TDA theorems from the algebraic (homological) level to the level of filtered spaces.

Finally, we consider the problem of establishing a persistent Whitehead theorem in terms of homotopy interleavings. We provide a counterexample to a naive formulation of the result.

Contents

1. Introduction

Topological data analysis (TDA) is a branch of statistics whose goal is to apply topology to develop tools for analyzing the global, non-linear, geometric features of data. At a high level of abstraction, the basic TDA workflow is easy to describe: Given a data set P, for example a finite metric space, we associate to P a diagram of topological spaces $F(P)$ whose topological structure encodes information about the geometric features of P, and then study $F(P)$ using familiar tools from algebraic topology. The prime example of this workflow is *persistent homology*, which takes $F(P)$ to be a filtered topological space and then applies homology with coefficients in a field to obtain simple, easily computed invariants of P called *barcodes*.

Of course, diagrams of spaces have been studied extensively in modern algebraic topology; what distinguishes the TDA theory from classical work is an emphasis on *approximate relations* between diagrams of spaces and their invariants. For example, we are typically more interested in whether the barcodes of a pair of data sets are close in some suitably defined metric than in whether they are exactly equal.

From this vantage point, the well-known stability theory for persistent homology and its applications to TDA can be seen as a rudimentary form of "approximate algebraic topology" for filtered spaces, in which a central role is played by such approximate relations. In analogy with classical algebraic topology, one imagines that there should exist an "approximate homotopy theory" for filtered spaces which serves as the foundation for this approximate algebraic topology; indeed, the beginnings of such a theory are already implicit in the proofs of well-known TDA results.

The main goal of this paper is to develop the formal language needed to start fleshing out this approximate homotopy theory. The essential first problem is to select a suitable notion of *approximate weak equivalence* of filtered spaces; we focus primarily on this. Our results establish that our *homotopy interleavings* and the metric they induce, the *homotopy interleaving distance*, provide a notion of approximate weak equivalence that is very well behaved, relative to the needs of TDA.

In the remainder of the introduction, we state our main results and motivate them by explaining how the homotopy interleaving distance can be used to obtain a homotopy-theoretic strengthening of a fundamental stability result for the persistent homology of finite metric spaces.

1.1. **Persistent Homology.** As mentioned above, persistent homology provides invariants of data called *barcodes*. A barcode is a multiset of intervals in $\mathbb{R}$. Intuitively, each interval in the barcode represents a topological feature of our data, and the length of the interval is a measure of the robustness of that feature to perturbations of the data. In the last fifteen years, these invariants have been widely applied to the study of scientific data [21, 5].

For categories $\mathbf{C}$ and $\mathbf{D}$ with $\mathbf{C}$ small, let $\mathbf{D}^{\mathbf{C}}$ denote the category of functors $\mathbf{C} \to \mathbf{D}$ with morphisms the natural transformations. Let **Top** denote the category of compactly-generated weakly Hausdorff (CGWH) topological spaces [30, 35, 39]. Regarding $\mathbb{R}$, together with its total order, as a category in the usual way, we define an $\mathbb{R}$-*space* to be an object X of $\mathbf{Top}^{\mathbb{R}}$. If for each $r \leq s \in \mathbb{R}$, the internal map $X_{r,s} \colon X_r \to X_s$ is an *inclusion* (i.e., a homeomorphism onto its image), we call X a *filtration*.

Let **Vec** denote the category of k-vector spaces over some fixed field k. A *persistence module* is an object M of $\mathbf{Vec}^{\mathbb{R}}$. We say M is *pointwise finite dimensional (p.f.d.)* if $\dim M_r < \infty$ for all $r \in \mathbb{R}$. Let $H_i \colon \mathbf{Top} \to \mathbf{Vec}$ denote the i^{th} homology functor with coefficients in k.

Persistent Homology Pipeline. Here is the standard TDA pipeline for constructing barcode invariants of finite metric spaces:

(1) Given a finite metric space (P, d), we construct $\mathrm{Rips}(P)$, the *Vietoris-Rips filtration of* P. For $r \geq 0$, $\mathrm{Rips}(P)_r$ is the clique complex on the graph with vertex set P and edge set $\{[p, q] \mid d(p, q) \leq 2r\}$; for $r < 0$, $\mathrm{Rips}(P)_r := \emptyset$.
(2) For any $i \geq 0$, we obtain a p.f.d. persistence module $H_i \,\mathrm{Rips}(P)$.
(3) The structure theorem for persistence modules [42, 16] yields a barcode $\mathcal{B}_M$ as a complete invariant of any p.f.d. persistence module M; this barcode specifies the decomposition of M into indecomposable summands. Thus, we obtain a barcode invariant $\mathcal{B}_{H_i \,\mathrm{Rips}(P)}$ of P.

This construction of barcodes of finite metric spaces in fact generalizes to compact metric spaces, using a slightly different definition of barcode [10, 12].[1] Moveover, by altering the first step of this pipeline, we can adapt the pipeline to provide invariants of other kinds of data as well, or to provide other kinds of barcode invariants of finite metric spaces.

Bottleneck Distance on Barcodes. In contrast to classical algebraic topology, metrics on collections of topological invariants play a key role in TDA. One standard choice of metric on barcodes is the *bottleneck distance*, denoted d_B. Roughly, for two barcodes $\mathcal{C}$ and $\mathcal{D}$, $d_B(\mathcal{C}, \mathcal{D})$ is the maximum amount we need to perturb any interval in $\mathcal{C}$ to transform $\mathcal{C}$ into $\mathcal{D}$. We now give the precise definition.

[1] In fact, almost everything we say in this paper about finite metric spaces generalizes to compact metric spaces. However, in order to avoid defining barcodes of compact metric spaces, which requires some explanation [10, 1], we will restrict attention to finite metric spaces here.

Define a *matching* between sets S and T simply to be a bijection between subsets of S and T; this definition extends in the expected way to multisets. For $I \subseteq \mathbb{R}$ a nonempty interval and $\delta \geq 0$, define the interval

$$\mathrm{Ex}(I,\delta) := \{r \in \mathbb{R} \mid \exists\, s \in I \text{ with } |r-s| \leq \delta\}.$$

For $\mathcal{D}$ a barcode and $\delta \geq 0$, let $\mathcal{D}^\delta$ denote the set of intervals in $\mathcal{D}$ which contain an interval of the form $[r, r+\delta]$ for some $r \in \mathbb{R}$. Define a *δ-matching* between barcodes $\mathcal{C}$ and $\mathcal{D}$ to be a matching σ between these barcodes such that

(1) σ matches each interval in $\mathcal{C}^{2\delta}$ and $\mathcal{D}^{2\delta}$,
(2) if $\sigma(I) = J$ then $I \subseteq \mathrm{Ex}(J,\delta)$ and $J \subseteq \mathrm{Ex}(I,\delta)$.

Finally, we define the bottleneck distance d_B by

$$d_B(\mathcal{C},\mathcal{D}) := \inf\{\delta \mid \exists \text{ a } \delta\text{-matching between } \mathcal{C} \text{ and } \mathcal{D}\}.$$

The bottleneck distance is easily seen to be an extended pseudometric on barcodes, restricting to a genuine metric on barcodes arising as the persistent homology of finite metric spaces. Recall that an extended pseudometric on S is a function

$$d\colon S \times S \to [0,\infty]$$

satisfying the axioms of a metric, except that we may have $d(x,y) = 0$ for $x \neq y$, and the triangle inequality only holds for finite values of d. In this paper, by a *distance* we will always mean an extended pseudometric.

Stability of Persistent Homology of Metric Spaces. A metric on the set of barcodes allows us to quantify how persistent homology changes when the input data is perturbed. For this, we also need a distance on finite metric spaces. We use the well-known Gromov-Hausdorff distance, denoted d_{GH}; see Section 6 for the definition.

The following is one of the fundamental results of TDA:

Theorem 1.1 (Rips Stability [9, 12])**.** *For finite metric spaces P and Q,*

$$d_B(\mathcal{B}_{H_i\,\mathrm{Rips}(P)}, \mathcal{B}_{H_i\,\mathrm{Rips}(Q)}) \leq 2\, d_{GH}(P,Q).$$

As we discuss in Section 3.1, one can give a more general, purely algebraic formulation of the stability of persistent homology; this is called the *algebraic stability theorem*.

1.2. **Filtration-Level Refinement of the Rips Stability Theorem.** Given our goal of developing homotopy-theoretic foundations of TDA, a natural question is whether Theorem 1.1 can be regarded as the consequence of some purely topological (homotopy-theoretic) result about the filtrations $\mathrm{Rips}(P)$ and $\mathrm{Rips}(Q)$. More generally, many TDA theorems—for example, those in [14, 9, 13, 7]—tell us that the barcodes of two filtrations are close in the bottleneck distance, and it is reasonable to ask whether these theorems can be understood as corollaries of purely topological results about the filtrations themselves.

The natural way to strengthen Theorem 1.1 to a purely topological result is to introduce a distance d on $\mathbb{R}$-spaces such that

(i) For all finite metric spaces P and Q,

$$d(\mathrm{Rips}(P), \mathrm{Rips}(Q)) \leq 2\, d_{GH}(P,Q). \tag{1.2}$$

(ii) [*Homology bounding axiom*] For all $i \geq 0$ and $\mathbb{R}$-spaces X, Y with H_iX and H_iY pointwise finite dimensional,

$$d_B(\mathcal{B}_{H_iX}, \mathcal{B}_{H_iY}) \leq d(X, Y).$$

Axioms for a Distance on $\mathbb{R}$-Spaces. We next define stability and homotopy invariance axioms for a distance d on $\mathbb{R}$-spaces. Together these axioms imply Eq. (1.2).

Given a (not necessarily continuous) function $\gamma\colon T \to \mathbb{R}$ with $T \in \operatorname{ob} \mathbf{Top}$, we define the *sublevelset filtration* $\mathcal{S}(\gamma)\colon \mathbb{R} \to \mathbf{Top}$ by taking

$$\mathcal{S}(\gamma)_r = \gamma^{-1}(-\infty, r], \quad r \in \mathbb{R}.$$

where each $\mathcal{S}_r$ is given the subspace topology. (In the CGWH context, the subspace topology is understood to be that of [39, Definition 2.25]; this coincides with the standard subspace topology on subsets that are already CGWH in the standard topology.)

We define a distance d_∞ on the collection of real-valued functions on T by

$$d_\infty(\gamma, \kappa) = \sup_{x \in T} |\gamma(x) - \kappa(x)|$$

for $\gamma, \kappa\colon T \to \mathbb{R}$.

Definition 1.3. For $\mathbf{I}$ any small category and functors $X, Y : \mathbf{I} \to \mathbf{Top}$, we say a natural transformation $f\colon X \to Y$ is an *(objectwise) weak equivalence* if $f_a\colon X_a \to Y_a$ is a weak homotopy equivalence for all $a \in \operatorname{ob} \mathbf{I}$. We say X and Y are *weakly equivalent*, and write $X \simeq Y$ if there exists a functor $W\colon \mathbf{I} \to \mathbf{Top}$ and a diagram of objectwise weak equivalences

$$X \xleftarrow{\simeq} W \xrightarrow{\simeq} Y.$$

It can be shown that $\simeq$ is an equivalence relation; see Section 2 and the references given there.

Definition 1.4 (Stability and homotopy invariance axioms)**.** We a say distance d on $\mathbb{R}$-spaces is

(1) *stable* if for any $T \in \operatorname{ob} \mathbf{Top}$ and functions $\gamma, \kappa\colon T \to \mathbb{R}$,

$$d(\mathcal{S}(\gamma), \mathcal{S}(\kappa)) \leq d_\infty(\gamma, \kappa),$$

(2) *homotopy invariant* if $d(X, Y) = d(X', Y)$ whenever X, X', Y are $\mathbb{R}$-spaces with $X' \simeq X$.

The following result is implicit in the original proof of the Rips stability theorem:

Proposition 1.5. *Any stable, homotopy invariant distance on $\mathbb{R}$-spaces satisfies* (1.2)*, i.e., strengthens the Rips stability theorem to a filtration-level result.*

We give a proof of Proposition 1.5 in Section 6, following a proof of the Rips stability theorem due to Memoli [33].

Refinements of Other TDA Results. In fact, any stable, homotopy invariant, and homology bounding distance distance on $\mathbb{R}$-spaces can be used to formulate filtration-level strengthenings of several other fundamental TDA theorems, analogous to (1.2). For example, one can give filtration-level strengthenings of results of Sheehy et al. on the sparse approximation of Rips and Cech filtrations [36, 7]. Similarly, one can give a filtration-level formulation of a result, implicit in the work of Chazal et al. [13] and made explicit in [26, Theorem 4.5.2], about the consistency of an estimator of the superlevelset persistent homology of a probability density function.

1.3. **Properties of the Homotopy Interleaving Distance.** The *interleaving distance* d_I, the standard pseudometric on filtrations in the TDA literature, is stable and homology bounding, in the above senses, but is not homotopy-invariant, and does not satisfy (1.2); see Remark 3.3. In Section 3.3, we define *homotopy interleavings* and the *homotopy interleaving distance* d_{HI} on $\mathbb{R}$-spaces by modifying the definition of d_I to enforce the homotopy invariance axiom. Our first main result is the following:

Theorem 1.6. *d_{HI} is a distance on $\mathbb{R}$-spaces satisfying the stability, homotopy invariance, and homology bounding axioms.*

Proposition 1.5 and Theorem 1.6 together then tell us in particular that d_{HI} satisfies (1.2). Whereas it is trivial to show that d_I satisfies the triangle inequality, establishing the triangle inequality for d_{HI} requires some work. Given the triangle inequality for d_{HI} and the algebraic stability theorem, the rest of the proof of Theorem 1.6 is trivial.

Universality of the Homotopy Interleaving Distance. There are several pseudometrics on $\mathbb{R}$-spaces, besides d_{HI}, that satisfy the stability, homotopy invariance, and homology bounding axioms, and comparing these different choices can be difficult; see Section 1.5 and Section 7. This raises the question of whether one can make a canonical choice of such a distance. The second main result of this paper, a simple axiomatic characterization of d_{HI}, provides an affirmative answer to this question. The statement is as follows:

Theorem 1.7 (Universality). *If d is any stable, homotopy-invariant distance on $\mathbb{R}$-spaces, then $d \leq d_{HI}$.*

This result is a homotopy-theoretic analogue of a universality result for the interleaving distance on multidimensional persistence modules over prime fields, established in [27]. In fact, the proof of Theorem 1.7 also extends to the multidimensional setting. For simplicity, however, we will restrict attention to the 1-D case in this paper.

1.4. **Persistent Whitehead Conjectures.** With a good definition of "approximate weak equivalence" of $\mathbb{R}$-spaces in hand, we are led to ask how other aspects of homotopy theory might extend to the approximate setting. For example, one has a Whitehead theorem for $\mathbb{R}$-spaces, which says that an objectwise weak equivalence of cofibrant $\mathbb{R}$-spaces is a homotopy equivalence. It is natural to wonder whether one can use the language of interleavings to formulate a persistent analogue of this result. We explore this problem in Section 8, considering along the way the question of how to define persistent homotopy groups. We present an example showing that in its most naive formulation, the persistent Whitehead theorem does not hold, even up to a constant, and offer a persistent Whitehead conjecture for cofibrant diagrams of CW-complexes of bounded dimension.

1.5. **Related Work.** This work is, in part, an outgrowth of a chapter in the second author's Ph.D. thesis, which introduced a pseudometric d_{WI} on filtrations satisfying the stability, homotopy invariance, and homology bounding axioms considered in this paper [26, Chapter 3]. This chapter showed that the strict interleaving distance on $\mathbb{R}$-spaces satisfies a universality property, and raised but did not answer the question of whether d_{WI} is universal. It is clear that $d_{WI} \leq d_{HI}$, but the problem of determining whether $d_{HI} = d_{WI}$ is non-trivial, due to technical issues related to homotopy coherence; see Section 7 and Remark 8.4.

Around the same time [26] was completed, Mémoli [32, 31, 33] and Chazal et al. [12] each introduced (different) definitions of a pseudometric on simplicial filtrations, both of which can be used to provide a refinement of the Rips Stability theorem analogous to the one we give using d_{HI}. However, neither definition extends in a naive way to arbitrary **Top**-valued filtrations, and universality of these pseudometrics has not been considered. Our proof strategy of the triangle inequality for d_{HI} was inspired by Mémoli's proof of the triangle inequality for his pseudometric on simplicial filtrations [33]: Mémoli's proof hinges on a pullback construction, and our proof arose out of an effort to adapt that construction to our setting.

Aside from the above, there has been relatively little work so far with an explicit emphasis on persistent homotopy theory. One exception is the theory of the contiguity complex developed in [3]. Another is the work of Letscher, which studies a definition of persistent homotopy groups in the context of knot theory [28]. In addition, a recent result of Frosini et al. [22] strengthens the well known stability result for the persistent homology of $\mathbb{R}$-valued functions [14] to pairs of functions with homotopy equivalent (but not necessarily homeomorphic) domains.

1.6. **Outline of the Paper.** Section 2 provides a brief review of the tools from homotopy theory needed in our proofs. Section 3 reviews the ordinary interleaving distance and introduces the homotopy interleaving distance d_{HI}. Section 4 gives the proof of the triangle inequality for d_{HI}, thereby establishing Theorem 1.6, and Section 5 gives the proof of Theorem 1.7, our universality result for d_{HI}. Section 6 gives the proof of Proposition 1.5, which uses d_{HI} to strengthen the Rips stability theorem to the level of filtrations. Section 7 gives a characterization of d_{HI} in terms of homotopy coherent diagrams of spaces, and explains the difficulties of using homotopy commutative rather than homotopy coherent diagrams for this. Section 8 studies the persistent Whitehead problem.

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2. Background on Homotopy Theory

Interleavings are diagrams of topological spaces, and so to study their homotopy theory, we use standard ideas from the modern homotopy theory of diagram categories. In this section, we briefly review some of these ideas; specifically, we review model categories (which provide an abstract axiomatic framework for homotopy theory), homotopy colimits, and homotopy Kan extensions. While we have tried to make our exposition accessible to readers unfamiliar with model categories and homotopy colimits, this section is no substitute for a systematic introduction. For a more detailed introduction to model categories we recommend the survey article [20]; the books [24, 23] provide comprehensive treatments. Thorough discussions of homotopy colimits can be found in [35, 37, 17].

Recall that in Section 1.1 we defined **Top** to be the category of compactly-generated weak Hausdorff spaces (CGWH spaces). It is standard in modern homotopy theory to restrict attention to this category because it contains most spaces that arise in practice and its mapping spaces behave well. In this paper, all topological constructions are carried out in the context of CGWH spaces; except when necessary, we will not comment on this point further.

2.1. Model Categories.

The basic object of study in the homotopy theory of topological spaces is the *homotopy category* $\mathrm{Ho}(\mathbf{Top})$, obtained from **Top** by formally inverting the continuous maps that are weak homotopy equivalences. However, it turns out that many constructions are difficult to carry out directly in $\mathrm{Ho}(\mathbf{Top})$, and so it is convenient to work with constructions in **Top** and study their interaction with weak equivalences. Moreover, in order to construct homotopy-invariant notions of limit and colimit (known as homotopy limits and colimits, see below), additional scaffolding is employed, in the form of distinguished maps called *cofibrations* (which generalize closed inclusions and are intended to have "nice" quotients) and *fibrations* (which generalize bundles and are intended to have "nice" fibers).

A model category is an abstraction of this structure. Specifically, a *model category* is a complete and cocomplete category $\mathbf{C}$, together with three distinguished collections of morphisms in $\mathrm{Ho}(\mathbf{C})$, called the *weak equivalences*, *fibrations*, and *cofibrations*, satisfying several axioms. We say a (co)fibration is *acyclic* if it is also a weak equivalence. We will not list the model category axioms here, except for one: Any morphism in $\mathbf{C}$ factors functorially as a composite of a cofibration followed by an acyclic fibration [24, Definition 1.1.3]. (It is sometimes convenient to drop the requirement that the factorization be functorial, but we will work in situations where this does hold.)

Any model category $\mathbf{C}$ has an initial object $\emptyset$ and a final object $*$. We say $X \in \operatorname{ob}\mathbf{C}$ is *cofibrant* if the unique morphism $\emptyset \to X$ is a cofibration; dually, an object is *fibrant* if the unique morphism $X \to *$ is a fibration. Applying the functorial factorization axiom above to morphisms $\emptyset \to X$ yields a *cofibrant replacement functor* $Q\colon \mathbf{C} \to \mathbf{C}$, with each QX a cofibrant replacement of X, and a natural transformation $Q \to \mathrm{Id}_{\mathbf{C}}$ which is an acyclic fibration on each object of $\mathbf{C}$ [24].

For any model category $\mathbf{C}$, we construct an associated homotopy category $\mathrm{Ho}(\mathbf{C})$, which formally inverts the weak equivalences. The homotopy category $\mathrm{Ho}(\mathbf{C})$ has the same collection of objects as $\mathbf{C}$ but a different collection of morphisms, and is equipped with a localization functor $\Pi^{\mathbf{C}}\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ which is the identity on objects. Up to equivalence of categories, $\mathrm{Ho}(\mathbf{C})$ depends only on the weak equivalences of $\mathbf{C}$, not on the (co)fibrations. We write $c \simeq d$ if $c, d \in \mathrm{ob}\,\mathbf{C}$ are isomorphic in $\mathrm{Ho}(\mathbf{C})$. This is true if and only if c and d are weakly equivalent in the sense of Definition 1.3; see [20, Definition 5.6].

Given model categories $\mathbf{C}$ and $\mathbf{D}$ and a functor $F\colon \mathbf{C} \to \mathbf{D}$, the *total left derived functor of* F is the functor $\Lambda^F\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ given by the right Kan extension of $\Pi^{\mathbf{D}} F\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{D})$ along $\Pi^{\mathbf{C}}\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$. We say a functor $\tilde{F}\colon \mathbf{C} \to \mathbf{D}$ *computes* Λ^F if $\Pi^{\mathbf{D}}\tilde{F} \cong \Lambda^F \Pi^{\mathbf{C}}$. If F preserves weak equivalences between cofibrant objects, then the total left derived functor of F exists, and for $Q\colon \mathbf{C} \to \mathbf{C}$ a cofibrant replacement functor, FQ computes Λ^F [20, Section 9].

The category **Top** of compactly-generated weak Hausdorff spaces is complete and cocomplete (although note that the colimits are not always the same as those in the category of all topological spaces; e.g., see [39]). The *standard model structure* on **Top** is specified by taking the weak equivalences to be the weak homotopy equivalences and the fibrations to be the Serre fibrations; the cofibrations are then determined from the acyclic fibrations by the model category axioms. (The cofibrations in the standard model structure also admit a concrete description, as *retracts of generalized CW inclusions*; see [20]). Henceforth, weak equivalences, fibrations, and cofibrations of topological spaces will be understood to be those in the standard model structure.

For any small category $\mathbf{I}$, there exists a model category structure on $\mathbf{Top}^{\mathbf{I}}$, the *projective model structure*, for which the weak equivalences are the objectwise weak equivalence and the fibrations are the objectwise fibrations. For most choices of $\mathbf{I}$, objectwise cofibrations are not necessarily cofibrations; it takes some care to concretely specify the cofibrations in this model structure. However, it is straightforward to check that if $X \in \mathbf{Top}^{\mathbf{I}}$ is cofibrant, then each object in X is cofibrant, and each internal map in X is a cofibration.

2.2. **Homotopy Colimits and Homotopy Left Kan extensions.** Homotopy colimits are analogues of colimits which are invariant under weak equivalences of diagrams. There are several (weakly equivalent) ways to define these. We will work with definitions given using the derived functor formalism and the Bousfield-Kan formula. Alternatively, one can define these via homotopy coherent analogues of the universal properties of colimits [41], or using the language of homotopical categories and homotopy initial objects [19].

Let $\mathrm{colim}\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ denote the colimit functor . We define

$$\mathrm{hocolim}\colon \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}) \to \mathrm{Ho}(\mathbf{Top}),$$

the *homotopy colimit functor*, to be the total left derived functor of colim. Since the functor colim sends weak equivalences between cofibrant diagrams in $\mathbf{Top}^{\mathbf{I}}$ to weak equivalences [20, Remark 9.8, Lemma 9.9 and Proposition 10.7], $\mathrm{colim} \circ Q$ computes hocolim.

Similarly, homotopy Kan extensions are analogues of Kan extensions which are invariant under weak equivalences of diagrams: For a functor $F\colon \mathbf{I} \to \mathbf{J}$, let $L_F\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{J}}$ denote the left Kan extension along F. (Later, when F is clear from context, we will sometimes write L_F as $L_{\mathbf{J}}$.) We define the *homotopy left Kan extension* to be the total left derived functor of L_F. As above, $L_F Q$ computes the homotopy left Kan extension.

It is often useful to have an explicit formula for the homotopy colimit. In fact, early work took the following such formula as the definition of homotopy colimit: Define

$$\mathrm{Hocolim}\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$$

by

$$\operatorname{Hocolim} X := \operatorname{coeq}\left(\coprod_{a\to b} \mathcal{N}(b \downarrow \mathbf{I}) \times X_a \rightrightarrows \coprod_{a\in \mathbf{I}} \mathcal{N}(a \downarrow \mathbf{I}) \times X_a \right), \tag{2.1}$$

where for a small category $\mathbf{C}$, $\mathcal{N}(\mathbf{C})$ denotes the classifying space of $\mathbf{C}$, i.e., the geometric realization of the nerve, and $a \downarrow \mathbf{I}$ denotes a comma category. To parse (2.1), note that any morphism $a \to b$ in $\mathbf{I}$ induces a map

$$\mathcal{N}(b \downarrow \mathbf{I}) \times X_a \to \mathcal{N}(b \downarrow \mathbf{I}) \times X_b$$

and also a map

$$\mathcal{N}(b \downarrow \mathbf{I}) \times X_a \to \mathcal{N}(a \downarrow \mathbf{I}) \times X_a.$$

As the morphism $a \to b$ varies, the collection of all such pairs of maps induces the two maps whose coequalizer we take in (2.1).

A natural transformation $X \to Y$ induces a map $\operatorname{Hocolim} X \to \operatorname{Hocolim} Y$, giving Hocolim the structure of a functor. In addition, Hocolim is functorial under restriction, in the sense that a functor $\iota\colon \mathbf{J} \to \mathbf{I}$ between small categories induces a map $\operatorname{Hocolim} \iota^* X \to \operatorname{Hocolim} X$. Hocolim in fact computes hocolim; see [35, Corollary 5.1.3 and Remark 6.3.4], or [37, Theorem 9.1] and [18, Appendix A].

Remark 2.2. If X is objectwise cofibrant, then the coequalizer in **Top** which appears in the formula (2.1) for $\operatorname{Hocolim} X$ coincides with the coequalizer in the category of all topological spaces; we use this tacitly in Section 5. We can deduce this fact by recalling that $\operatorname{Hocolim} X$ is homeomorphic to the geometric realization of a simplicial space [35, Theorem 6.6.1]. It is a standard fact that geometric realization can be described as the sequential colimit of a sequence of pushouts; for example, see [35, Chapter 14]. Moreover, these pushouts are along closed inclusions [35, Lemma 5.2.1], and so the claim then follows from [39, Proposition 2.35 and Lemma 3.3].

3. The Homotopy Interleaving Distance

In this section, we define homotopy interleavings and the homotopy interleaving distance. We begin by recalling the definition of ordinary interleavings.

3.1. Interleavings.

A category $\mathbf{C}$ is said to be *thin* if for every $a, b \in \operatorname{ob} \mathbf{C}$, there is at most one non-zero morphism in $\mathbf{C}$ from a to b. If $\mathbf{C}$ is thin, $F\colon \mathbf{C} \to \mathbf{D}$ is any functor, and $g\colon a \to b$ is a morphism in $\mathbf{C}$, we denote $F(g)$ as $F_{a,b}$.

For $\delta \geq 0$, let the δ*-interleaving category*, denoted $\mathbf{I}^\delta$, be the thin category with object set $\mathbb{R} \times \{0, 1\}$ and a morphism $(r, i) \to (s, j)$ if and only if either

(1) $r + \delta \leq s$, or
(2) $i = j$ and $r \leq s$;

There are evident functors

$$E^0, E^1 \colon \mathbb{R} \to \mathbf{I}^\delta$$

mapping $r \in \mathbb{R}$ to $(r, 0)$ and $(r, 1)$, respectively.

Definition 3.1. For $\mathbf{C}$ any category, we define a *δ-interleaving* between functors $X, Y \colon \mathbb{R} \to \mathbf{C}$ to be a functor

$$Z \colon \mathbf{I}^\delta \to \mathbf{C}$$

such that $Z \circ E^0 = X$ and $Z \circ E^1 = Y$.

Let $X(\delta) \colon \mathbb{R} \to \mathbf{C}$, be the functor obtained by shifting each object and morphism of X downward by δ, i.e., $X(\delta)_r := X_{r+\delta}$ and $X(\delta)_{r,s} := X_{r+\delta,s+\delta}$ for all $r \leq s \in \mathbb{R}$. Note that Z restricts to a pair of morphisms $X \to Y(\delta)$ and $Y \to X(\delta)$ and that, conversely, these morphisms fully determine Z; we call these morphisms *δ-interleaving morphisms*. In the case δ=0, these are simply an inverse pair of natural isomorphisms between X and Y.

Definition 3.2. We define

$$d_I \colon \text{ ob}\, \mathbf{C}^\mathbb{R} \times \text{ob}\, \mathbf{C}^\mathbb{R} \to [0, \infty],$$

the *interleaving distance*, by taking

$$d_I(X, Y) = \inf \{\delta \mid X \text{ and } Y \text{ are } \delta\text{-interleaved}\}.$$

It easy to show that if W and X are δ-interleaved, and X and Y are ϵ-interleaved, then W and Y are $(\delta + \epsilon)$-interleaved; it follows easily that d_I is a distance on $\text{ob}\, \mathbf{C}^\mathbb{R}$. Moreover, if we have $X, X', Y \in \text{ob}\, \mathbf{C}^\mathbb{R}$ with $X \cong X'$, then $d_I(X, Y) = d_I(X', Y)$, so d_I descends to a distance on isomorphism classes of objects in $\mathbf{C}^\mathbb{R}$.

Remark 3.3. As noted earlier, it's easy to see that the interleaving distance d_I on $\mathbb{R}$-spaces is stable and homology bounding. However, d_I is not homotopy invariant: Consider $\mathbb{R}$-spaces X and Y with $X_r = 0$ and $Y_r = \mathbb{R}$ for all $r \in \mathbb{R}$. The inclusion $0 \hookrightarrow \mathbb{R}$ induces an objectwise homotopy equivalence $X \hookrightarrow Y$ is but $d_I(X, Y) = \infty$. More generally, it is easy to check that $d_I(X, Y) = \infty$ for any two filtrations X and Y with $\operatorname{colim} X$ not homeomorphic to $\operatorname{colim} Y$.

3.2. **Algebraic Stability.** The *algebraic stability theorem*, a generalization of the Rips stability theorem (Theorem 1.1), is arguably the central result in the theory of persistent homology. It was first introduced in [8], building on earlier work on the stability of persistent homology of $\mathbb{R}$-valued functions by Cohen-Steiner et al. [14]. Since then, the result has been revisited by several papers, which have provided simpler proofs and more general formulations [27, 11, 1, 2, 4]. In particular, it has been observed that the converse to the algebraic stability theorem also holds [27]; this is an easy consequence of the structure theorem for persistence modules [16].

We state the algebraic stability theorem in its sharp form for pointwise finite dimensional (p.f.d.) persistence modules [1]:

Theorem 3.4 (Forward and converse algebraic stability)**.** *A pair of p.f.d. persistence modules M and N are δ-interleaved if and only if there exists a δ-matching between $\mathcal{B}_M$ and $\mathcal{B}_N$. In particular,*

$$d_B(\mathcal{B}_M, \mathcal{B}_N) = d_I(M, N).$$

3.3. **Homotopy Interleavings.** We now introduce our homotopical generalization of interleavings.

Definition 3.5. For $\delta \geq 0$, we say $\mathbb{R}$-spaces X and Y are *δ-homotopy-interleaved* if there exist $\mathbb{R}$-spaces $X' \simeq X$ and $Y' \simeq Y$ such that X' and Y' are δ-interleaved.

Remark 3.6. An equivalent definition of homotopy interleavings can be given using the language of *homotopy coherent diagrams*; see Section 7.4. Though this alternative definition is more technical, some readers may find it helpful for building intuition about homotopy interleavings.

Definition 3.7. The homotopy interleaving distance between $\mathbb{R}$-spaces X and Y is given by

$$d_{HI}(X,Y) := \inf\{\delta \mid X, Y \text{ are } \delta\text{-homotopy-interleaved}\}.$$

Partial Proof of Theorem 1.6. It is clear that d_{HI} is symmetric and non-negative, and that for any $\mathbb{R}$-space X, $d_{HI}(X,X) = 0$. To establish that d_{HI} is a distance, then, it suffices to check that d_{HI} satisfies the triangle inequality; we verify this in Section 4 below.

It is easy to check that d_{HI} is stable and homotopy invariant. A δ-interleaving between $\mathbb{R}$-spaces X, Y induces a δ-interleaving between H_iX, H_iY, and a weak equivalence between $\mathbb{R}$-spaces X, Y induces a 0-interleaving between H_iX, H_iY. From these observations, the triangle inequality for d_I on persistence modules, and Theorem 3.4, we have that d_{HI} is homology bounding. □

4. The Triangle Inequality for d_{HI}

In this section, we prove the triangle inequality for d_{HI}. Our proof hinges on an interpretation of the triangle inequality as a statement about a certain homotopy left Kan extension.

4.1. **Preliminaries on Left Kan Extensions.** As preparation for the proof, we recall some facts about left Kan extensions which we need for the argument.

For $\mathbf{C}$ a full subcategory of a thin category $\mathbf{D}$ and $d \in \operatorname{ob}\mathbf{D}$, the comma category $\mathbf{C}\downarrow d$ is (up to canonical isomorphism) the full subcategory of $\mathbf{C}$ with object set

$$\{c \in \operatorname{ob}\mathbf{C} \mid \exists \text{ a morphism } c \to d \text{ in } \mathbf{D}\}.$$

$d \downarrow \mathbf{C}$ is defined analogously.

Consider a functor $F\colon \mathbf{C} \to \mathbf{E}$ with $\mathbf{E}$ cocomplete. By the usual pointwise formula for left Kan extensions, $L_{\mathbf{D}}F$ is given at any $d \in \operatorname{ob}\mathbf{D}$ by

$$(L_{\mathbf{D}}F)_d = \operatorname{colim}_{\mathbf{C}\downarrow d} F, \tag{4.1}$$

and moreover, the internal morphisms in $L_{\mathbf{D}}F$ are given by the universality of the colimit [29, Theorem X.3.1]. In particular, we have the following:

Lemma 4.2. *For $d \in \operatorname{ob}\mathbf{D} \setminus \operatorname{ob}\mathbf{C}$, let $\hat{\mathbf{C}}$ denote the full subcategory of $\mathbf{D}$ with object set $\operatorname{ob}(\mathbf{C}\downarrow d) \cup \{d\}$. The restriction of F to $\hat{\mathbf{C}}$ is naturally isomorphic to the colimit cocone of $F|_{\mathbf{C}\downarrow d}$.*

4.2. **Generalized Interleaving Categories.** For our proof of the triangle inequality for d_{HI}, it will be convenient to introduce a generalization of our definition of an interleaving category from Section 3.1. For a finite thin category $\mathbf{I}$, we define a *marking* of $\mathbf{I}$ to be a map $\Delta\colon S \to [0,\infty)$, where S is a some subset of the set of unordered pairs of isomorphic objects in $\mathbf{I}$. We denote a pair $(a,b) \in S$ with $\Delta(a,b) = \delta$ as follows:

$$a \underset{\delta}{\rightleftarrows} b.$$

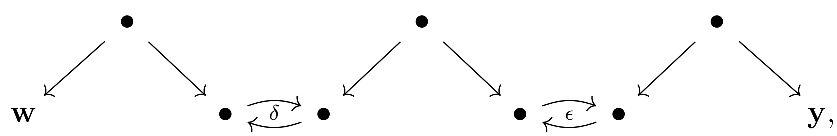
Define $\bar{\mathbf{I}}$, the *interleaving category of the marked category* $\mathbf{I}$, to be the thin category with $\operatorname{obj}\bar{\mathbf{I}} = \operatorname{obj}\mathbf{I} \times \mathbb{R}$ and $\hom \bar{\mathbf{I}}$ generated by the set of arrows

$$\{(a,r) \to (b,r) \mid r \in \mathbb{R},\ a \to b \in \hom(\mathbf{I}) \setminus S\}$$
$$\cup\ \{(a,r) \to (b, r + \Delta(a,b)) \mid r \in \mathbb{R},\ (a,b) \in S\}.$$

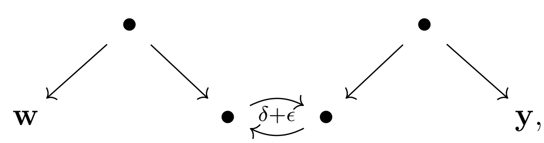
We define a *diagram of* $\mathbb{R}$*-spaces indexed by* $\mathbf{I}$ to be a functor $D\colon \bar{\mathbf{I}} \to \mathbf{Top}$. D restricts to an $\mathbb{R}$-space D_a for each $a \in \operatorname{ob}\mathbf{I}$, to a natural transformation $D_{a,b}\colon D_a \to D_b$ for each $a \to b \in \hom(\mathbf{I}) \setminus S$, and to a $\Delta(a,b)$-interleaving between D_a and D_b for each $(a,b) \in S$.

4.3. **Proof of the Triangle Inequality.** It suffices to show that if W and X are δ-homotopy-interleaved and X and Y are ϵ-homotopy-interleaved, then W and Y are $(\delta+\epsilon)$-homotopy-interleaved.

Let $\mathbf{I}'$ be the marked category:

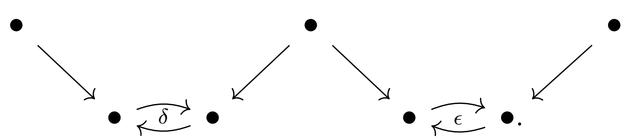


where objects we don't (yet) wish to name explicitly are denoted by $\bullet$. If W and X are δ-homotopy-interleaved and X and Y are ϵ-homotopy-interleaved, then using the fact that $\simeq$ is an equivalence relation, we obtain a diagram D' of $\mathbb{R}$-spaces indexed by $\mathbf{I}'$ such that $D'_{\mathbf{w}} = W$, $D'_{\mathbf{y}} = Y$, and all diagonal arrows in D' are weak equivalences. We need to show that there exists a diagram E of $\mathbb{R}$-spaces indexed by the following marked category



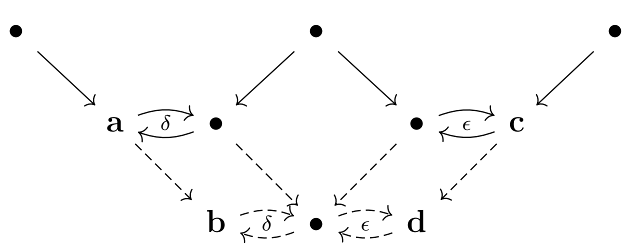
such that $E_{\mathbf{w}} = W$, $E_{\mathbf{y}} = Y$, and all diagonal arrows in E are weak equivalences.

Let $\mathbf{I}$ be the marked subcategory of $\mathbf{I}'$ obtained by removing the two extremal objects:



Let D denote the restriction of D' to $\bar{\mathbf{I}}$. By taking a cofibrant replacement QD of D, we may assume that D' has been chosen so that D is cofibrant.

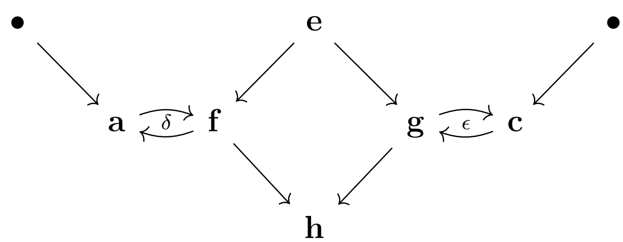
Now we introduce notation for some of the objects of $\mathbf{I}$. Let $\mathbf{J}$ be the following marked extension of $\mathbf{I}$:



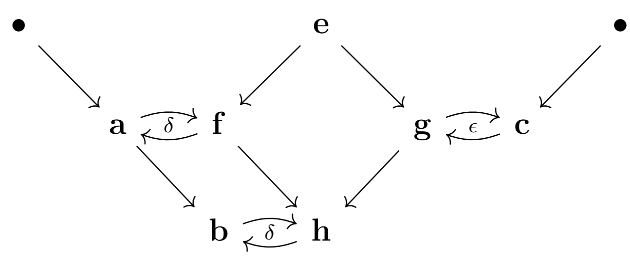
and let $\iota\colon \mathbf{I} \hookrightarrow \mathbf{J}$ denote the inclusion functor. The inclusion ι induces an inclusion functor $\bar{\iota}\colon \bar{\mathbf{I}} \hookrightarrow \bar{\mathbf{J}}$. Since we assume D to be cofibrant, $L_{\bar{\mathbf{J}}}D$ in fact computes the homotopy left Kan extension of D along $\bar{\iota}$.

Since $\bar{\iota}$ is fully faithful, we have that $L_{\bar{\mathbf{J}}}D \circ \bar{\iota} \cong D$ [29, Corollary X.3.3]. Therefore, writing $L := L_{\bar{\mathbf{J}}}D$, to establish that X and Z are $(\delta+\epsilon)$-homotopy-interleaved, it suffices to show that the morphisms of $\mathbb{R}$-spaces $L_{\mathbf{a},\mathbf{b}}\colon L_{\mathbf{a}} \to L_{\mathbf{b}}$ and $L_{\mathbf{c},\mathbf{d}}\colon L_{\mathbf{c}} \to L_{\mathbf{d}}$ are both weak equivalences; the desired $(\delta+\epsilon)$-homotopy-interleaving is then given by composition. We will show that $L_{\mathbf{a},\mathbf{b}}$ is a weak equivalence; by symmetry, the argument for $L_{\mathbf{c},\mathbf{d}}$ is exactly the same.

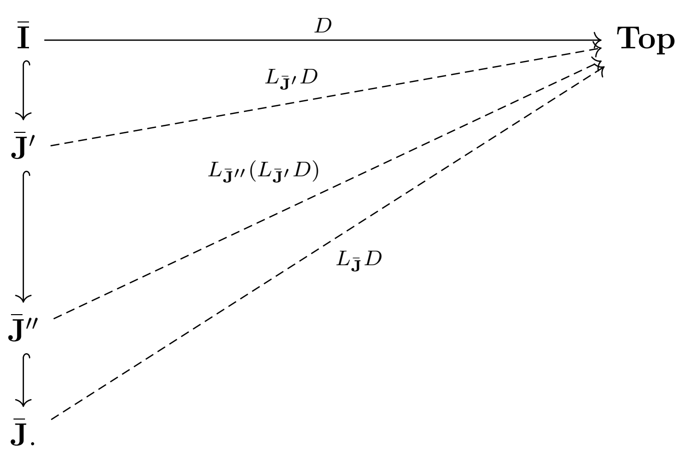
Let $\mathbf{J}'$ denote the marked category:



and let $\mathbf{J}''$ denote the marked category:



Note that we have inclusions $\mathbf{I} \hookrightarrow \mathbf{J}' \hookrightarrow \mathbf{J}'' \hookrightarrow \mathbf{J}$ factoring $\iota\colon \mathbf{I} \hookrightarrow \mathbf{J}$; these induce inclusions $\bar{\mathbf{I}} \hookrightarrow \bar{\mathbf{J}}' \hookrightarrow \bar{\mathbf{J}}'' \hookrightarrow \bar{\mathbf{J}}$ factoring $\bar{\iota}\colon \bar{\mathbf{I}} \hookrightarrow \bar{\mathbf{J}}$. By universality, $L_{\bar{\mathbf{J}}}(L_{\bar{\mathbf{J}}''}(L_{\bar{\mathbf{J}}'}D)) \cong L_{\bar{\mathbf{J}}}D$. We thus obtain the following diagram of left Kan extensions, commuting up to natural isomorphism:



To show that $L_{\mathbf{a},\mathbf{b}}$ is a weak equivalence, we first show that $L_{\mathbf{f},\mathbf{h}} \cong (L_{\bar{\mathbf{J}}'}D)_{\mathbf{f},\mathbf{h}}$ is an objectwise acyclic cofibration. The key step in the argument is to show that for each $r \in \mathbb{R}$, the restriction of $L' := L_{\bar{\mathbf{J}}'} D$ to the full subcategory of $\bar{\mathbf{J}}'$ with the four objects

$$\{(\mathbf{e}, r), (\mathbf{f}, r), (\mathbf{g}, r), (\mathbf{h}, r)\}$$

is a pushout square. Let $\mathbf{K} := \bar{\mathbf{I}} \downarrow (\mathbf{h}, r)$, and let $\hat{\mathbf{K}}$ be the full subcategory of $\mathbf{J}'$ with object set $\mathrm{ob}(\mathbf{K}) \cup \{(\mathbf{h}, r)\}$. By Lemma 4.2, the restriction of L' to $\hat{\mathbf{K}}$ is naturally isomorphic to the colimit cocone of $D|_{\mathbf{K}}$. In addition, it is easy to see that

$$\mathbf{K}' := (\mathbf{f}, r) \leftarrow (\mathbf{e}, r) \rightarrow (\mathbf{g}, r)$$

is a final subcategory of $\mathbf{K}$. Hence, the natural map $\operatorname{colim} D|_{\mathbf{K}'} \to \operatorname{colim} D|_{\mathbf{K}}$ is an isomorphism [35, Section 8.3], and by construction, this isomorphism commutes with the maps in the colimit cocones. It follows that the square in question is a pushout, as claimed.

Since D is cofibrant, the map $D_{(\mathbf{e},r),(\mathbf{g},r)}$ is a cofibration; by assumption it is in fact an acyclic cofibration. In any model category, acyclic cofibrations are stable under cobase change [20, Proposition 3.14 (ii)], so since the square in question is a pushout, $L'_{(\mathbf{f},r),(\mathbf{h},r)}$ is an acyclic cofibration as well. This holds for all r, so $L'_{\mathbf{f},\mathbf{h}}$ is an objectwise acyclic cofibration.

Essentially the same argument applied to $L_{\bar{\mathbf{J}}''}(L_{\bar{\mathbf{J}}'} D)$ shows that for any $r \in \mathbb{R}$, the restriction of $L_{\bar{\mathbf{J}}''}(L_{\bar{\mathbf{J}}'} D)$ to the full subcategory of $\bar{\mathbf{J}}''$ with the four objects

$$\{(\mathbf{a}, r), (\mathbf{f}, r - \delta), (\mathbf{b}, r), (\mathbf{h}, r - \delta)\}$$

is a pushout square. Then as above, since $L'_{\mathbf{f},\mathbf{h}}$ is an objectwise acyclic cofibration, so is $L_{\mathbf{a},\mathbf{b}} \cong L_{\bar{\mathbf{J}}''}(L_{\bar{\mathbf{J}}'} D)_{\mathbf{a},\mathbf{b}}$.

Remark 4.3. Notice that in the argument above, we have not used the full strength of the hypothesis that D is cofibrant, only that each element of D is cofibrant and each internal map of D is a cofibration.

5. Universality of the Homotopy Interleaving Distance

In this section, we prove our universality result for d_{HI}, Theorem 1.7.

5.1. A Choice of Cofibrant Replacement Functor.

To establish the universality of d_{HI}, we will work with a specific choice of cofibrant replacement functor $\mathbf{Q}\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{I}}$ for the projective model structure on $\mathbf{Top}^{\mathbf{I}}$. For now, we take $\mathbf{I}$ to be an arbitrary small category, though later we will need to consider only the cases where $\mathbf{I}$ is either $\mathbb{R}$ or an interleaving category. We consider $\mathbf{Q}$ because, as we show below, it has a special property, which we call *1-criticality*, that is essential to our universality argument.

We define $\mathbf{Q}$ using Hocolim, the formula for the homotopy colimit given in Section 2.2. $\mathbf{Q}$ is a standard construction in homotopy theory: It is used to establish that Hocolim in fact computes the left derived functor of colim [35, Section 5.1, Remark 6.3.4], and it also arises naturally in the study of homotopy coherent maps of diagrams of spaces; see [35, Equation 7.7.5] and [17, Section 9], for example.

To give the definition of $\mathbf{Q}$, let $q\colon \mathbf{Top} \to \mathbf{Top}$ be any cofibrant replacement functor. We define $\mathbf{Q}\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{I}}$ by

$$\mathbf{Q}X_a := \mathrm{Hocolim}_{(\mathbf{I}\downarrow a)}\, qX\pi^a,$$

where $\pi^a\colon (\mathbf{I} \downarrow a) \to \mathbf{I}$ is the forgetful functor. The internal maps in $\mathbf{Q}X$ are given by the functoriality of Hocolim under restriction, and the functoriality of $\mathbf{Q}$ follows from the functoriality of Hocolim.

Proposition 5.1. $\mathbf{Q}$ *is a cofibrant replacement functor for the projective model structure on* $\mathbf{Top}^{\mathbf{I}}$.

Proof. In any model category, the cofibrations are exactly those morphisms satisfying the *left lifting property* with respect to all acyclic fibrations [20, Proposition 3.13]. Thus, we need that for any diagram $X\colon \mathbf{I} \to \mathbf{Top}$, the map $\emptyset \to \mathbf{Q}X$ satisfies the left lifting property with respect to any acyclic fibration. The required lift can be constructed directly; Dugger sketches of a proof of this in his notes on homotopy colimits [17, Section 9]. The details are easy to fill in, using the fact that the standard model structure on $\mathbf{Top}$ is a simplicial model structure, as defined in [23]. (Dugger's argument appeals to a correspondence between natural transformations $QX \to Y$ and homotopy coherent natural transformations $X \to Y$, but the lift can be constructed directly, without considering homotopy coherent natural transformations.)

By the definition of a cofibrant replacement for $\mathbf{Top}$, we have a weak equivalence $qX \to X$. Thus, to complete the proof, it suffices to observe that we have a weak equivalence $\mathbf{Q}X \to qX$. For a proof of this, see for example [35, Lemma 5.1.5] or [17, Section 9]. We outline the argument: For any small category $\mathbf{J}$ and diagram $Y\colon \mathbf{J} \to \mathbf{Top}$, we have a natural map $\operatorname{Hocolim}(Y) \to \operatorname{colim} Y$, given by collapsing each nerve appearing in the definition of Hocolim to a point. When $\mathbf{J}$ has a terminal object, this map is a homotopy equivalence. Since in this case we also have $\operatorname{colim} Y = Y$, for each $a \in \operatorname{ob} \mathbf{I}$, we obtain a homotopy equivalence $\mathbf{Q}X_a \to qX_a$. These maps are natural, so they define an objectwise homotopy equivalence $\mathbf{Q}X \to qX$. □

If $\mathbf{I}$ is a poset, then $\mathbf{I} \downarrow c = \{a \in \mathbf{I} \mid a \leq c\}$ and for any $a \leq c \in \mathbf{I}$,

$$a \downarrow (\mathbf{I} \downarrow c) := a \downarrow \mathbf{I} \downarrow c = \{b \in \mathbf{I} \mid a \leq b \leq c\}.$$

Then

$$\mathbf{Q}X_c := \operatorname{Hocolim}_{(\mathbf{I}\downarrow c)} qX = \operatorname{coeq}\left(\coprod_{a \leq b \leq c} \mathcal{N}(b \downarrow \mathbf{I} \downarrow c) \times qX_a \rightrightarrows \coprod_{a \leq c} \mathcal{N}(a \downarrow \mathbf{I} \downarrow c) \times qX_a \right).$$

5.2. **1-Critical Filtrations.** For any small category $\mathbf{I}$, we say a functor $X\colon \mathbf{I} \to \mathbf{Top}$ is a *closed filtration* if each of the internal maps $X_{a,b}\colon X_a \to X_b$ is a closed inclusion.

Proposition 5.2. *For any* $X\colon \mathbf{I} \to \mathbf{Top}$*,* $\mathbf{Q}X$ *is a closed filtration.*

Proof. By Proposition 5.1, $\mathbf{Q}X$ is cofibrant. As noted in Section 2, each internal map in a cofibrant diagram in $\mathbf{Top}$ is a cofibration, and hence a closed inclusion [24, Lemma 2.4.6, Theorem 2.4.23, Lemma 2.4.25]. □

A *directed set* is a poset $\mathbf{I}$ such that for all $a, b \in \mathbf{I}$, there exists $c \in \mathbf{I}$ with $a, b \leq c$. Given a directed set $\mathbf{I}$, a functor $X\colon \mathbf{I} \to \mathbf{Top}$, and $a \in \mathbf{I}$, let $\mu_a^X\colon X_a \to \operatorname{colim} X$ denote the canonical map.

Lemma 5.3 ([39, Lemma 3.3])**.** *If* $\mathbf{I}$ *is a directed set and* $X\colon \mathbf{I} \to \mathbf{Top}$ *is a closed filtration, then* $\operatorname{colim} X$ *is the colimit in the category of all topological spaces, and each map* μ_a^X *is a closed inclusion.*

Adapting terminology introduced in [6], for $\mathbf{I}$ a directed set, we say a functor $X : \mathbf{I} \to \mathbf{Top}$ is *1-critical* if X is a closed filtration and for each $x \in \operatorname{colim} X$, the set

$$\{a \in \mathbf{I} \mid x \in \operatorname{im} \mu_a^X\}$$

has a minimum element. We then have a function $\zeta^X \colon \operatorname{colim} X \to \mathbf{I}$ sending each $x \in \operatorname{colim} X$ to this minimum element.

Proposition 5.4. *For any directed set $\mathbf{I}$ and functor $X \colon \mathbf{I} \to \mathbf{Top}$, $\mathbf{Q}X$ is 1-critical.*

Proof. Proposition 5.2 gives that $\mathbf{Q}X$ is a closed filtration. As explained in Section 5.1, we have

$$\mathbf{Q}X_c = \operatorname{coeq}\left(\coprod_{a\le b\le c} \mathcal{N}(b \downarrow \mathbf{I} \downarrow c) \times qX_a \rightrightarrows \coprod_{a\le c} \mathcal{N}(a \downarrow \mathbf{I} \downarrow c) \times qX_a\right).$$

Note that

$$\operatorname{colim} \mathbf{Q}X = \operatorname{Hocolim} qX = \operatorname{coeq}\left(\coprod_{a\to b} \mathcal{N}(b \downarrow \mathbf{I}) \times qX_a \rightrightarrows \coprod_{a\in \mathbf{I}} \mathcal{N}(a \downarrow \mathbf{I}) \times qX_a\right),$$

with the natural maps $\mathbf{Q}X_c \to \operatorname{colim} \mathbf{Q}X$ induced by the inclusions

$$a \downarrow \mathbf{I} \downarrow c \hookrightarrow a \downarrow \mathbf{I}.$$

Recall from Remark 2.2 that the coequalizers appearing here are the usual ones in the category of all topological spaces.

The projections

$$\mathcal{N}(a \downarrow \mathbf{I}) \times qX_a \to \mathcal{N}(a \downarrow \mathbf{I})$$

induce a surjection $\alpha \colon \operatorname{colim} \mathbf{Q}X \to \mathcal{N}(\mathbf{I})$. We also define a function $\beta \colon \mathcal{N}(\mathbf{I}) \to \mathbf{I}$ in the following way: Since $\mathcal{N}(\mathbf{I})$ is the geometric realization of a simplicial set, each point $y \in \mathcal{N}(\mathbf{I})$ can be associated to a unique nondegenerate simplex; set $\beta(y) = b_n$ when y is associated to

$$b_1 < b_2 < \cdots < b_n.$$

It is easy to check that for $x \in \operatorname{colim} \mathbf{Q}X$,

$$\beta \circ \alpha(x) = \min\{a \in \mathbf{I} \mid x \in \operatorname{im} \mu_a^{\mathbf{Q}X}\}.$$

Hence $\mathbf{Q}X$ is 1-critical, with $\zeta^{\mathbf{Q}X} = \beta \circ \alpha$. □

We next define a category $\mathbf{Fns}$, whose objects are functions $\gamma_T \colon T \to \mathbf{I}$, where $T \in \operatorname{ob} \mathbf{Top}$, and $\hom_{\mathbf{Fns}}(\gamma_S, \gamma_T)$ is the set of continuous functions $f \colon S \to T$ such that $\gamma_T \circ f \le \gamma_S$. We emphasize that when $\mathbf{I}$ happens to carry a topology (e.g., when $\mathbf{I} = \mathbb{R}$), we do not require $\gamma_T \in \operatorname{ob} \mathbf{Fns}$ to be continuous, but we do require morphisms in $\mathbf{Fns}$ to be continuous.

Let $\mathbf{Top}^{\mathbf{I}}_{\text{crit}}$ denote the full subcategory of $\mathbf{Top}^{\mathbf{I}}$ whose objects are the 1-critical diagrams. The functoriality of colimits tells us that for diagrams $X, Y \colon \mathbf{I} \to \mathbf{Top}$, a natural transformation $f \colon X \to Y$ induces a map

$$\operatorname{colim} f \colon \operatorname{colim} X \to \operatorname{colim} Y.$$

If X, Y are 1-critical, then $\zeta^Y \circ \operatorname{colim} f \le \zeta^X$. We thus have a functor

$$\operatorname{fcolim} \colon \mathbf{Top}^{\mathbf{I}}_{\text{crit}} \to \mathbf{Fns}$$

which sends each 1-critical diagram X to $\zeta^X \colon \operatorname{colim} X \to \mathbf{I}$.

We also have an obvious functor $\mathcal{S}\colon \mathbf{Fns} \to \mathbf{Top}^{\mathbf{I}}$ with

$$\mathcal{S}(\gamma_T)_a := \{y \in T \mid \gamma_T(y) \le a\}.$$

This generalizes the sublevelset filtration construction introduced in Section 1.2.

Proposition 5.5. *If* $\mathbf{I}$ *is a directed set, then* $\mathcal{S} \circ \operatorname{fcolim} \cong \operatorname{Id}_{\mathbf{Top}^{\mathbf{I}}_{\text{crit}}}$.

Proof. Consider a diagram $X \in \operatorname{ob} \mathbf{Top}^{\mathbf{I}}_{\text{crit}}$. For $a \in \mathbf{I}$,

$$(\mathcal{S} \circ \operatorname{fcolim} X)_a = \operatorname{im} \mu^X_a.$$

By Lemma 5.3, μ^X_a is a homeomorphism onto its image. For $a \le b \in \mathbf{I}$ we have $\mu^X_b \circ X_{a,b} = \mu^X_a$ so these homeomorphisms define a natural isomorphism $\mu^X\colon X \to \mathcal{S} \circ \operatorname{fcolim} X$. Further, the natural isomorphisms $\{\mu^X\}_{X \in \mathbf{Top}^{\mathbf{I}}_{\text{crit}}}$ are natural in X, so this collection assembles into a natural isomorphism $\operatorname{Id}_{\mathbf{Top}^{\mathbf{I}}_{\text{crit}}} \to \mathcal{S} \circ \operatorname{fcolim}$. □

5.3. **Proof of Universality.** The main step in our proof that d_{HI} is universal is the following:

Proposition 5.6. *For any δ-interleaved $\mathbb{R}$-spaces X, Y, there exists a topological space T and functions $\gamma^X, \gamma^Y : T \to \mathbb{R}$ such that $\mathcal{S}(\gamma^X) \simeq X$, $\mathcal{S}(\gamma^Y) \simeq Y$, and $d_\infty(\gamma^X, \gamma^Y) \le \delta$.*

Proof. It will be convenient for us to treat the cases $\delta = 0$ and $\delta > 0$ separately. First, let $\delta = 0$, so that we have an isomorphism $X \to Y$. We take $T = \operatorname{colim} \mathbf{Q}X$. By Proposition 5.4, $\mathbf{Q}X$ is 1-critical. We let $\gamma^X = \gamma^Y = \zeta^{\mathbf{Q}X}$. Since $\mathbb{R}$, together with its total order, is a directed set, by Proposition 5.4 and Proposition 5.5, $\mathcal{S}(\gamma^Y) = \mathcal{S}(\gamma^X) \cong \mathbf{Q}X$. We also have a weak equivalence $\mathbf{Q}X \to X$. Composing, we thus obtain weak equivalences $\mathcal{S}(\gamma^X) \to X$, $\mathcal{S}(\gamma^Y) \to Y$, as desired. This completes the proof in the case $\delta = 0$.

Now assume $\delta > 0$. Recall the definitions of the interleaving category $\mathbf{I}^\delta$ and the functors $E^0, E^1 \colon \mathbb{R} \to \mathbf{I}^\delta$ from Section 3.1, and note that when $\delta > 0$, $\mathbf{I}^\delta$ is a poset category; in fact the underlying poset is a directed set.

Since X and Y are δ-interleaved, there exists a functor $Z\colon \mathbf{I}^\delta \to \mathbf{Top}$ such that $Z \circ E^0 = X$ and $Z \circ E^1 = Y$. We define $T := \operatorname{colim} \mathbf{Q}Z$.

E^0 and E^1 are both final functors [35, Section 8.3]. Hence, we have canonical identifications of $\operatorname{colim}(\mathbf{Q}Z \circ E^0)$ and $\operatorname{colim}(\mathbf{Q}Z \circ E^1)$ with T such that for each $r \in \mathbb{R}$,

$$\mu^{\mathbf{Q}Z}_{(r,0)} = \mu^{\mathbf{Q}Z \circ E^0}_r, \qquad \mu^{\mathbf{Q}Z}_{(r,1)} = \mu^{\mathbf{Q}Z \circ E^1}_r.$$

We claim that $\mathbf{Q}Z \circ E^0$ and $\mathbf{Q}Z \circ E^1$ are each 1-critical. We show this for $\mathbf{Q}Z \circ E^0$; the proof for $\mathbf{Q}Z \circ E^1$ is the same. First note that $\mathbf{Q}Z$ is a 1-critical by Proposition 5.4. In particular, for each $a \in \mathbf{I}^\delta$, $\mu^{\mathbf{Q}Z}_a \colon Z_a \to T$ is a closed inclusion. Therefore, for each $r \in \mathbb{R}$, $\mu^{\mathbf{Q}Z \circ E^0}_r \colon (\mathbf{Q}Z \circ E^0)_r \to T$ is a closed inclusion.

Since $\mathbf{Q}Z$ is 1-critical, for each $z \in T$ there is a minimum element $(r, j) \in \operatorname{ob} \mathbf{I}^\delta$ such that $z \in \operatorname{im} \mu^{\mathbf{Q}Z}_{(r,j)}$. We then have

$$r + j\delta = \min \{s \in \mathbb{R} \mid z \in \operatorname{im} \mu^{\mathbf{Q}Z \circ E^0}_s\}.$$

It follows that $\mathbf{Q}Z \circ E^0$ is 1-critical.

Since $\mathbf{Q}Z \circ E^0$ and $\mathbf{Q}Z \circ E^1$ are each 1-critical, we may define $\gamma^X, \gamma^Y : T \to \mathbb{R}$ by

$$\gamma^X = \zeta^{\mathbf{Q}Z \circ E^0},$$
$$\gamma^Y = \zeta^{\mathbf{Q}Z \circ E^1}.$$

By Proposition 5.5, we have

$$\mathcal{S}(\gamma^X) \cong \mathbf{Q}Z \circ E^0, \qquad \mathcal{S}(\gamma^Y) \cong \mathbf{Q}Z \circ E^1.$$

By construction, there exists a weak equivalence $\mathbf{Q}Z \to Z$; restricting, we obtain weak equivalences $\mathbf{Q}Z \circ E^0 \to X$ and $\mathbf{Q}Z \circ E^1 \to Y$. Hence, there exist weak equivalences $\mathcal{S}(\gamma^X) \to X$ and $\mathcal{S}(\gamma^Y) \to Y$.

It remains to check that $d_\infty(\gamma^X, \gamma^Y) \le \delta$. Consider $z \in T = \operatorname{colim} \mathbf{Q}Z$. There is a minimum index $(r, j) \in \operatorname{ob} \mathbf{I}^\delta$ such that $z \in \operatorname{im} \mu^{\mathbf{Q}Z}_{(r,j)}$. If $j = 0$ then $\gamma^X(z) = r$ and $\gamma^Y(z) = r + \delta$. If on the other hand $j = 1$, then $\gamma^Y(z) = r$ and $\gamma^X(z) = r + \delta$. Clearly, in either case, we have $\|\gamma^X(z) - \gamma^Y(z))\|_\infty = \delta$. Since this holds for all $z \in T$ we have $d_\infty(\gamma^X, \gamma^Y) \le \delta$ as desired (with strict equality unless $\operatorname{colim} \mathbf{Q}Z = \emptyset$). □

Proof of Theorem 1.7. Let X and Y be $\mathbb{R}$-spaces with $d_{HI}(X, Y) = \delta$. Then for all $\delta' > \delta$, X and Y are homotopy δ'-interleaved, i.e., there exist δ'-interleaved $\mathbb{R}$-spaces X', Y' with $X' \simeq X$ and $Y' \simeq Y$. Proposition 5.6 gives us a topological space T and functions $\gamma^{X'}, \gamma^{Y'} : T \to \mathbb{R}$ with $\mathcal{S}(\gamma^{X'}) \simeq X$, $\mathcal{S}(\gamma^{Y'}) \simeq Y$, and $d_\infty(\gamma^{X'}, \gamma^{Y'}) \le \delta'$.

Suppose d is a stable, homotopy invariant distance on $\mathbb{R}$-spaces. Then by stability, $d(\mathcal{S}(\gamma^{X'}), \mathcal{S}(\gamma^{Y'})) \le \delta'$. Therefore, by homotopy invariance and the triangle inequality for d, we have $d(X, Y) \le \delta'$. Since this holds for arbitrary $\delta' > \delta$ we have that $d(X, Y) \le \delta = d_{HI}(X, Y)$. □

6. Rips Stability for Filtrations

In this section, we use homotopy interleavings to strengthen the Rips stability theorem (Theorem 1.1) to a purely homotopy-theoretic result (Proposition 1.5). Together with the similar results mentioned at the end of Section 1.2, this refinement justifies our perspective that homotopy interleavings are the fundamental homotopical concept in topological data analysis. Recall that Proposition 1.5 says that if d is any distance on filtrations satisfying the stability and homotopy invariance axioms introduced in Definition 1.4, then for all finite metric spaces P and Q,

$$d(\operatorname{Rips}(P), \operatorname{Rips}(Q)) \le 2\, d_{GH}(P, Q).$$

We are aware of three different proofs of the Rips stability theorem. The original proof [9] embeds the metric spaces into a Euclidean space endowed with the ℓ^∞ norm and applies the nerve theorem. A proof of Proposition 1.5 is already implicit in this proof. A later proof [12] of the Rips stability theorem avoids use of embeddings and the nerve theorem, and instead considers multi-valued maps between simplicial complexes. An elegant third proof, due to Facundo Mémoli [33], relies on Quillen's Theorem A for simplicial complexes [34, Page 93]. Using a suitable definition of barcode [10, 1], each of these three proofs extends readily to arbitrary compact metric spaces.

We verify Proposition 1.5 by following Mémoli's proof of the Rips stability theorem. To prepare for the proof, we first review the definition of the Gromov-Hausdorff distance. Given sets S, T, a *correspondence* between S and T is a set $C \subset S \times T$ such that the coordinate projections $\mathrm{proj}_S \colon C \to S$ and $\mathrm{proj}_T \colon C \to T$ are surjections. Let $\Gamma(S,T)$ denote the set of all correspondences between S and T.

Definition 6.1. The Gromov-Hausdorff distance between compact metric spaces P and Q is given by

$$d_{GH}(P,Q) = \frac{1}{2} \inf_{C \in \Gamma(P,Q)} \sup_{(p,q),(p',q') \in C} |d(p,p') - d(q,q')|.$$

Proof of Proposition 1.5. If P and Q are finite metric spaces with $d_{GH}(P,Q) = \frac{\delta}{2}$, then there exists a correspondence $C \subset P \times Q$ with $|d(p,p') - d(q,q')| \leq \delta$ for all $((p,q),(p',q')) \in C$.

Let $[C]$ denote the simplex with vertices C. We define a simplicial filtration X^P on $[C]$ by taking $\sigma \in X^P_r$ if and only if

$$d^P(\mathrm{proj}_P(u), \mathrm{proj}_P(v)) \leq 2r$$

for all $u, v \in \sigma$. X^P induces a function $\gamma^P \colon [C] \to \mathbb{R}$ which sends a simplex σ to the minimum $r \in \mathbb{R}$ such that $\sigma \in X^P_r$. Note that X^P is equal to the simplicial sublevelset filtration $\mathcal{S}(\gamma^P)$. Define a simplicial filtration X^Q and function $\gamma^Q \colon [C] \to \mathbb{R}$ analogously.

By the way we chose C, we have that

$$d_\infty(\gamma^P, \gamma^Q) \leq \delta.$$

Thus, by the stability axiom for d and the fact that

$$\mathcal{S}(\gamma^P) = X^P, \quad \mathcal{S}(\gamma^Q) = X^Q,$$

we have that $d(X^P, X^Q) \leq \delta$.

Quillen's theorem A for simplicial complexes [34] says that if $f \colon S \to T$ is a simplicial map of simplicial complexes such that $f^{-1}(\sigma)$ is contractible for each (closed) simplex $\sigma \in T$, then f is a homotopy equivalence. Note that $\mathrm{proj}_P \colon C \to P$ induces a morphism $g \colon X^P \to \mathrm{Rips}(P)$. By Quillen's theorem A for simplicial complexes, g is an objectwise homotopy equivalence. Symmetrically, $\mathrm{proj}_Q \colon C \to Q$ induces an objectwise homotopy equivalence $X^Q \to \mathrm{Rips}(Q)$. Thus by the homotopy invariance of d,

$$d(\mathrm{Rips}(P), X^P) = d(\mathrm{Rips}(Q), X^Q) = 0.$$

By the triangle inequality for d, we have

$$d(\mathrm{Rips}(P), \mathrm{Rips}(Q)) \leq \delta. \qquad \square$$

7. Homotopy Commutative and Homotopy Coherent Interleavings

A simpler candidate definition of the homotopy interleaving distance can be formulated in terms of *homotopy commutative* interleaving diagrams, i.e., interleaving diagrams taking values in Ho(**Top**), the homotopy category of spaces. In this section, we explore this definition and explain why we expect that it is not equal to d_{HI}, hence not universal.

In homotopy theory, one typically avoids working with homotopy commutative diagrams, and works instead with richer objects called *homotopy coherent diagrams*, which are homotopically better behaved. Roughly speaking, a homotopy coherent diagram is a homotopy commutative diagram together with explicit choices of all homotopies, homotopies between the homotopies, and so on. At the end of this section, we observe that d_{HI} admits an equivalent definition in terms of homotopy coherent interleaving diagrams.

7.1. **Homotopy Commutative Interleavings.** As in Section 2, let $\Pi\colon \mathbf{Top} \to \mathrm{Ho}(\mathbf{Top})$ denote the functor that takes a space to its representative in the homotopy category (i.e., the localization with respect to the standard weak equivalences in **Top**).

Definition 7.1. A *homotopy commutative δ-interleaving* between $\mathbb{R}$-spaces X and Y is a δ-interleaving in $\mathrm{Ho}(\mathbf{Top})$ between ΠX and ΠY.

This definition induces a definition of an interleaving distance d_{HC} on $\mathbb{R}$-spaces, in the usual way. d_{HC} is easily seen to be stable, homotopy invariant, and homology bounding.

Remark 7.2. For many choices of small category $\mathbf{I}$, the natural functor $\mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}) \to (\mathrm{Ho}(\mathbf{Top}))^{\mathbf{I}}$ discards some higher order homotopy theoretic information, and this can make it difficult to work directly in the category $(\mathrm{Ho}(\mathbf{Top}))^{\mathbf{I}}$. As a rule, the constructions one considers in homotopy theory are well defined on $\mathrm{Ho}(\mathbf{Top}^{I})$ but not necessarily on $(\mathrm{Ho}(\mathbf{Top}))^{I}$. For example, whereas we define hocolim as a functor from $\mathrm{Ho}(\mathbf{Top}^{\mathbf{I}})$ to $\mathrm{Ho}(\mathbf{Top})$, $\mathrm{Ho}(\mathbf{Top})$ is not cocomplete (it does not even have all pushouts), and so homotopy colimits generally cannot be defined as functors out of $(\mathrm{Ho}(\mathbf{Top}))^{\mathbf{I}}$.

In view of the above remark, one does not expect homotopy commutative δ-interleavings to be homotopically well-behaved objects. Thus, the definition of d_{HC}, while especially simple, is somewhat unnatural. Nevertheless, one might wonder about the relationship between d_{HC} and d_{HI}. By the universality of d_{HI}, we have that $d_{HC} \leq d_{HI}$. We expect that $d_{HC} \neq d_{HI}$. Though we do not have a proof of this, we will present an example which shows that in the category of based topological spaces, homotopy commutative interleavings needn't lift to homotopy interleavings; we imagine that a similar example can be found which shows that $d_{HC} < d_{HI}$.

7.2. **Rectification of Homotopy Commutative Diagrams.** In the next two subsections, it will be convenient for us to work with the category $\mathbf{Top}_*$ of *based* CGWH topological spaces and its associated homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$ [20, Remark 3.10].

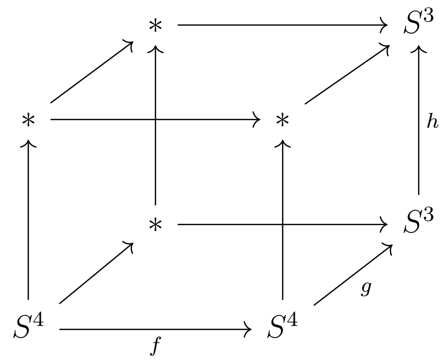
For $X\colon \mathbf{I} \to \mathrm{Ho}(\mathbf{Top}_*)$, a *rectification of* X is a functor $\tilde{X}\colon \mathbf{I} \to \mathbf{Top}_*$ such that $\Pi\tilde{X} \cong X$. Rectifications do not always exist. The following folklore example, brought to our attention by Tyler Lawson, demonstrates this:

Example 7.3. Consider the sequence of based maps

$$S^4 \xrightarrow{f} S^4 \xrightarrow{g} S^3 \xrightarrow{h} S^3, \tag{7.4}$$

where f and h are degree 2 maps, and g is the suspension of the Hopf map. The maps $g \circ f$ and $h \circ g$ are null homotopic.

Let $[\mathbf{1}]$ denote the category with object set $\{0, 1\}$ and a single non-identity morphism $0 \to 1$. We extend the sequence (7.4) above to a diagram indexed by the cube $[\mathbf{1}]^3$ which commutes up to homotopy, as follows:



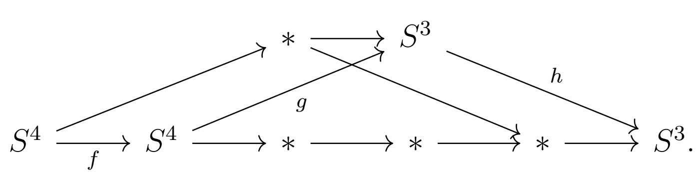
(7.5)

A homotopy commutative diagram of this form can be rectified if and only if the Toda bracket $\langle f, g, h\rangle$ contains the trivial map [38]. In this case, the Toda bracket consists of the non-zero element of $\pi_5(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ [40], so the diagram cannot be rectified.

7.3. **Rectification of Interleavings.** To establish that $d_{HC} = d_{HI}$ for based spaces, it would suffice to show that for any $X, Y : \mathbb{R} \to \mathbf{Top}_*$ and homotopy commutative δ-interleaving $W : \mathbf{I}^\delta \to \mathrm{Ho}(\mathbf{Top}_*)$ between X and Y, there exists a rectification $\tilde{W} : \mathbf{I}^\delta \to \mathbf{Top}_*$ of W such that $\tilde{W} \circ E^1 \simeq X$ and $\tilde{Z} \circ E^2 \simeq Y$. However, the next example shows that such a rectification does not always exist, even if we ignore the conditions on the restrictions $\tilde{W} \circ E^i$.

Example 7.6. We can define homotopy commutative interleavings between functors $\mathbb{Z} \to \mathbf{Top}_*$ in the same way as for $\mathbb{R}$-spaces. Leveraging Example 7.3, we give an example of functors $X, Y : \mathbb{Z} \to \mathbf{Top}_*$ and a homotopy commutative 2-interleaving between X and Y which cannot be rectified. It's easy to see that that this example extends to yield an unrectifiable homotopy commutative interleaving in the $\mathbb{R}$-indexed case, as well.

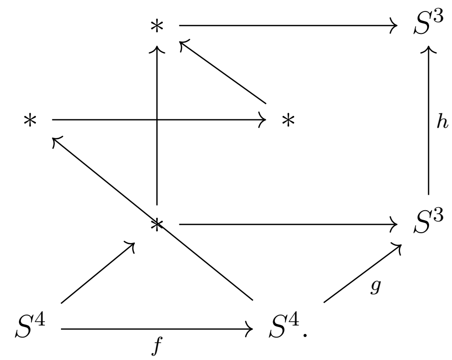
Letting f, g, and h be as in Example 7.3, consider the following homotopy commutative diagram:



(7.7)

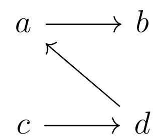
This diagram clearly extends to a homotopy commutative 2-interleaving between a pair of functors $X, Y : \mathbb{Z} \to \mathbf{Top}_*$, by taking the remaining spaces in X and Y to be points. To check that this cannot be rectified, it suffices to check that (7.7) cannot be rectified. To do so, we will observe that (7.7) can be rectified only if (7.5) can be rectified. Since (7.5) cannot be rectified, this establishes that (7.7) cannot be rectified.

Let us draw (7.7) in a different way, placing each object at a vertex of the cube:

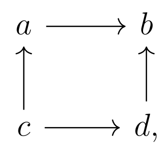


(7.8)

Noting that by composition, a commutative diagram of the form



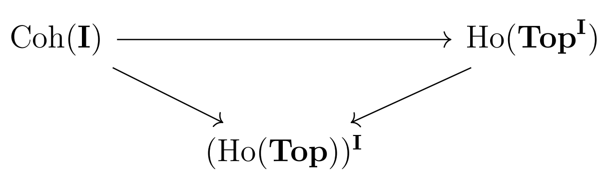
determines one of the form



it's clear that if a rectification of (7.8) were to exist, it would yield a rectification of (7.5). Thus (7.7) cannot be rectified, as we wanted to show.

Note however that there does exist a homotopy 1-interleaving between X and Y, obtained by taking all maps between spaces in X and Y to be trivial. Thus, $d_{HI}(X, Y) = d_{HC}(X, Y) = 1$. Consequently, this example does not establish that $d_{HC} \neq d_{HI}$ for based spaces.

7.4. **Homotopy Coherent Interleavings.** Though homotopy commutative diagrams cannot always be rectified, a theorem of Vogt tells us that a functorial rectification does exist for homotopy coherent diagrams [41, 15]. Specifically, let $\mathrm{Coh}(\mathbf{I})$ denote the homotopy category of homotopy coherent diagrams in **Top** indexed by **I**, as defined for example in [15]. Vogt's theorem gives an equivalence $\mathrm{Coh}(\mathbf{I}) \to \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}})$ making the following diagram commute up to natural isomorphism:



where the arrows into $(\mathrm{Ho}(\mathbf{Top}))^{\mathbf{I}}$ are the evident ones. This result in fact generalizes to diagrams taking values in the cofibrant-fibrant objects $\mathcal{C}^{\mathrm{cf}}$ of an arbitrary simplicial model category $\mathcal{C}$. The theorem tells us that model categories provide a formalism for studying homotopy coherent diagrams, using strict diagrams and zig-zags of objectwise weak equivalences.

We now formulate a homotopy-coherent notion of interleavings.

Definition 7.9. We define a *homotopy coherent δ-interleaving* between functors $X, Y : \mathbb{R} \to \mathbf{Top}$ to be a homotopy coherent diagram $Z \in \mathrm{Coh}(\mathbf{I}^\delta)$ such that $Z \circ E^0 \cong X$ and $Z \circ E^1 \cong Y$ in $\mathrm{Coh}(\mathbb{R})$.

Using the equivalence $\mathrm{Coh}(\mathbf{I}^\delta) \to \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}^\delta})$ provided by Vogt's theorem, it is straighforward to prove the following comparison.

Proposition 7.10. *There exists a homotopy coherent δ-interleaving between $\mathbb{R}$-spaces X and Y if and only if there exists a δ-homotopy-interleaving between X and Y.*

As a consequence, we can give an equivalent definition of d_{HI} in terms of homotopy coherent interleavings.

8. Towards a Persistent Whitehead Theorem

To conclude the paper, we explore of the problem, mentioned in Section 1.4, of formulating a persistent Whitehead theorem.

8.1. Persistent Homotopy Groups.

We first need to define persistent homotopy groups. In the setting of based spaces, this is straightforward.

Definition 8.1. For a functor $X : \mathbb{R} \to \mathbf{Top}_*$ and $i \geq 0$, we call the composite functor $\pi_i X$ the i^{th} *based persistent homotopy group* of X. For $i > 0$, this is a functor $\mathbb{R} \to \mathbf{Grp}$, while π_0 takes values in $\mathbf{Set}$.

A related definition of persistent homotopy group appears in [28], but concerns only pairs of indices in $\mathbb{R}$. We have seen that interleavings are defined on objects of $\mathbf{C}^{\mathbb{R}}$, for arbitrary categories $\mathbf{C}$. Thus, interleavings can be defined in the usual way for based persistent homotopy groups.

We next briefly consider the definition of persistent homotopy groups in the unbased setting. Let us say $X : \mathbb{R} \to \mathbf{Top}_*$ is *(path) connected* if for each $r \in \mathbb{R}$, X_r (path) connected. For X path connected, our definition of $\pi_i X$ is the natural one. However, $\mathbb{R}$-spaces arising in TDA are rarely connected. For X not connected, the isomorphism type of $\pi_i X$ depends on the choice of basepoints in X, and may miss important topological information in components of the spaces X_r not containing the basepoint. Thus, we wish to define an unbased version of the persistent homotopy group, which keeps track of information at all components. We also wish to give a definition of interleavings between these objects.

One way to proceed is to use the definition of a *local system*, as given in [25, Section 4]. This definition is functorial, so one can associate a *persistent local system* $\Pi_i X$ to an $\mathbb{R}$-space X for each $i \geq 0$. Moreover, one has a natural notion of equivalence of local systems, and using this, one can give a definition of interleavings between persistent local systems similar to our definition of the homotopy interleaving distance. Our triangle inequality argument for the homotopy interleaving distance adapts to give the triangle inequality for this interleaving distance.

That said, a careful study of unbased persistent homotopy groups and their interleavings is beyond the scope of this work; below we restrict attention to connected $\mathbb{R}$-spaces and work with based persistent homotopy groups.

8.2. **Persistent Whitehead Conjectures.** In a model category $\mathbf{C}$, for cofibrant-fibrant objects $X, Y \in \mathbf{C}$ there is a well-behaved abstract notion of homotopy of morphisms $f, g\colon X \to Y$, generalizing the usual definition of homotopy for maps of topological spaces; see for example [20, Section 4]. As in the case of spaces, we write $f \simeq g$. This in turn yields a definition of homotopy equivalence. The axioms of a model category imply an abstract Whitehead theorem [24, Proposition 1.2.8]:

Theorem 8.2 (Whitehead theorem for model categories)**.** *For any model category* $\mathbf{C}$*, a weak equivalence between cofibrant-fibrant objects in* $\mathbf{C}$ *is a homotopy equivalence.*

The situation in **Top** is somewhat simpler: We have a well-behaved homotopy relation (the familiar one) for maps between arbitrary spaces X and Y. Moreover, in the standard model structure, a cofibrant object is a retract of a CW-complex and all objects are fibrant. Thus in this setting, Theorem 8.2 recovers the classical result that a weak equivalence between CW-complexes is a homotopy equivalence.

These good properties carry over to the category of $\mathbb{R}$-spaces; we have a canonical homotopy relation for morphisms between arbitrary $\mathbb{R}$-spaces, which is an equivalence relation, and all $\mathbb{R}$-spaces are fibrant in the projective model structure.

In light of this, we can introduce a persistent generalization of homotopy equivalence for arbitrary $\mathbb{R}$-spaces. First, note that for X an $\mathbb{R}$-space and $\delta \geq 0$, the internal maps $\{X_{r,r+\delta}\}_{r\in\mathbb{R}}$ assemble into a morphism $\varphi^{X,\delta}\colon X \to X(\delta)$.

Definition 8.3. Given $\mathbb{R}$-spaces X and Y, we will say a pair of morphisms $f\colon X \to Y(\delta)$ and $g\colon Y \to X(\delta)$ are *(inverse) δ-homotopy equivalences* if

$$g(\delta) \circ f \simeq \varphi^{X,2\delta} \quad \text{and} \quad f(\delta) \circ g \simeq \varphi^{Y,2\delta},$$

where $f(\delta)\colon X(\delta) \to Y(2\delta)$ is the map induced by f, and is $g(\delta)$ defined analogously.

Remark 8.4. It is easy to check that if cofibrant $\mathbb{R}$-spaces X and Y are δ-homotopy-interleaved, then X and Y are δ-homotopy equivalent. However, in view of homotopy coherence and rectification issues similar to those discussed in Section 7, it is not clear to us whether the converse is true.

It is natural to wonder whether for $\mathbb{R}$-spaces, the Whitehead theorem extends to a persistent version as follows:

Naive Persistent Whitehead Conjecture 1. For X and Y connected cofibrant $\mathbb{R}$-spaces, $\delta \geq 0$, and morphism $f\colon X \to Y(\delta)$ with $\pi_i f\colon \pi_i X \to \pi_i Y(\delta)$ a δ-interleaving morphism for all i, f is a δ-homotopy equivalence.

In view of Remark 8.4 and the universality of the homotopy interleaving distance, one might also wonder whether the following is true,

Naive Persistent Whitehead Conjecture 2. Given X, Y and f as in the previous conjecture, X and Y are δ-homotopy-interleaved.

However, the following example makes clear that both conjectures are far from true.

Example 8.5. We specify $X\colon \mathbb{R} \to \mathbf{CW}$, $Y\colon \mathbb{R} \to \mathbf{CW}$, and $f\colon X \to Y(1)$ satisfying the hypotheses of the above conjectures for $\delta = 1$, with X and Y not ϵ-homotopy equivalent, and hence also not ϵ-homotopy-interleaved, for any ϵ.

Let X be the trivial filtration, i.e., $X_r = *$ for all r. X is cofibrant. As a first step towards defining Y, for each $n \in \{1, 2, \ldots\}$, we define a functor $Y^n\colon \mathbb{R} \to \mathbf{CW}$ as follows:

$$Y^n_r := \begin{cases} \underbrace{S^{2^i} \times S^{2^i} \times \cdots \times S^{2^i}}_{2^{n-i} \text{ copies}} & \text{for } r \in [2i, 2i+2),\ i \in \{0, 1, \ldots n\} \\ * & \text{for } r \in (-\infty, 0) \cup [2n+2, \infty), \end{cases}$$

For $i \geq 0$, we have a map

$$S^{2^i} \times S^{2^i} \to S^{2^{i+1}} = S^{2^i} \wedge S^{2^i},$$

given by collapsing $S^{2^i} \vee S^{2^i} \subset S^{2^i} \times S^{2^i}$ to a point; here $\vee$ and $\wedge$ denote the wedge product and smash product, respectively. For $i \in \{0, 1, \ldots n-1\}$, $r \in [2i, 2i+2)$, and $s \in [2i+2, 2i+4)$, we take the internal map $Y^n_{r,s}$ to be the product of 2^{n-i-1} copies of this map. The remaining internal maps in Y^n are specified by composition.

For example, regarding the torus $S^1 \times S^1$ as a quotient of a square in the usual way, the map

$$Y^1_{0,2}\colon S^1 \times S^1 \to S^2$$

is the one induced by sending the whole boundary of the square to a single point, and the map

$$Y^2_{0,2}\colon S^1 \times S^1 \times S^1 \times S^1 \to S^2 \times S^2$$

is equal to $Y^1_{0,2} \times Y^1_{0,2}$.

For all i, the map $S^{2^i} \vee S^{2^i} \hookrightarrow S^{2^i} \times S^{2^i}$ induces a surjection on all homotopy groups. Thus, $\pi_i Y^n_{r,r+2}$ is trivial for all $r \in \mathbb{R}$ and $i \geq 0$. It follows that the trivial morphisms $X \to Y^n(1)$ and $Y^n \to X(1)$ induce 1-interleavings on all based persistent homotopy groups.

However, X and Y^n are not δ-homotopy equivalent for any $\delta < n+1$. To see this, assume that all spheres in the definition of Y^n are given the usual minimal CW-structure, and note that $Y^n_r = S^{2^n}$ for $r \in [2n, 2n+2)$. The map $Y^n_{0,r}$ acts by collapsing the $(2^n - 1)$-skeleton of Y^n_0 to a point, so it follows from an easy cellular homology computation that $H_{2^n}(Y^n)_{0,r} \neq 0$. Thus, $H_{2^n}(Y^n)$ and the trivial module $H_{2^n}(X)$ are not δ-interleaved. It is straightforward to check that a δ-homotopy equivalence between $\mathbb{R}$-spaces A and B induces a δ-interleaving between $H_i A$ and $H_i B$ for all i. Therefore, X and Y^n are not δ-homotopy equivalent, as claimed. On the other hand, X and Y^n are strictly $(n+1)$-interleaved, via trivial morphisms.

We next construct an $\mathbb{R}$-space Y' such that the trivial morphisms $X \to Y'(1)$ and $Y' \to X(1)$ induce 1-interleavings on all based persistent homotopy groups, but $H_i X$ and $H_i Y'$ are not δ-interleaved for any finite δ. To do so, we simply patch together the non-trivial portions of each Y^n, taking each morphism between spaces from two different Y^n to be trivial; that is, we take $Y'_r := Y^1_r$ for $r \in (-\infty, 4)$, $Y'_r := Y^2_{r-4}$ for $r \in [4, 10)$, and so on.

Finally, we obtain the desired cofibrant Y by taking a cofibrant replacement of Y'.

Example 8.5 motivates the following weaker pair of persistent Whitehead conjectures:

Conjecture 8.6 (Persistent Whitehead Conjectures)**.** *Suppose we are given connected cofibrant* $\mathbb{R}$*-spaces* $X, Y \colon \mathbb{R} \to \mathbf{CW}$ *with each* X_r *and* Y_r *of dimension at most* d*, and* $f \colon X \to Y(\delta)$ *with* $\pi_i f \colon \pi_i X \to \pi_i Y(\delta)$ *a* δ*-interleaving morphism for all* i*. Then there is a constant* c*, depending only on* d*, such that*

(i) f *is a* $c\delta$*-homotopy equivalence,*
(ii) X *and* Y *are* $c\delta$*-homotopy-interleaved.*

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