MAPS BETWEEN LOCAL PICARD GROUPS

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Let X be a scheme and $x \in X$ a point. The *local Picard group* of X at x, denoted by $\mathrm{Pic}^{\mathrm{loc}}(x, X)$, is the Picard group of the *punctured neighborhood* $\mathrm{Spec}_X \, \mathcal{O}_{x,X} \setminus \{x\}$. For an isolated, normal singularity (x, X) the local Picard group is isomorphic to the divisor class group of the local ring $\mathcal{O}_{x,X}$.

Our aim is to study the pull-back map on the local Picard group in two situations.

Question 1 (Normalization). Let $\pi : \bar{X} \to X$ denote the normalization and $\bar{x}_i \in \bar{X}$ the reduced preimages of x. What is the kernel of the pull-back map

$$\pi^* : \mathrm{Pic}^{\mathrm{loc}}(x, X) \to \textstyle\sum_i \mathrm{Pic}^{\mathrm{loc}}(\bar{x}_i, \bar{X})?$$

Question 2 (Restriction to a divisor). Let $x \in D \subset X$ be an effective Cartier divisor. What is the kernel of the restriction map

$$r_D^X : \mathrm{Pic}^{\mathrm{loc}}(x, X) \to \mathrm{Pic}^{\mathrm{loc}}(x, D)?$$