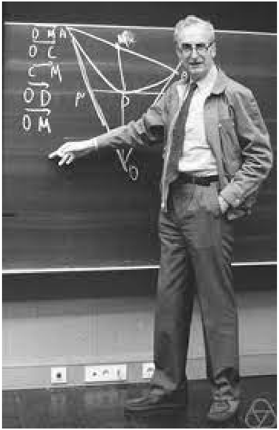
MINIMAL SURFACES IN THE THREE SPHERE

In fond remembrance of Gene Calabi



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It is also equivalent to a certain **differential equation**.

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$$\boxed{4F^2(1-K) \;=\; B_{12}^2 - B_{11}B_{22}}$$

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Proposition. *Let* $\omega = \varphi dz^2$ *where*

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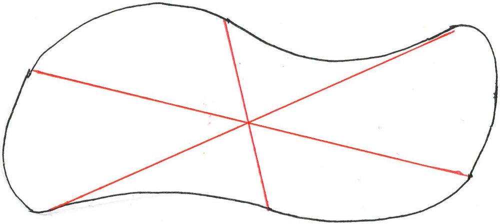
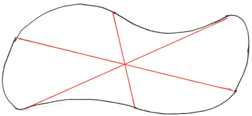
(2) If $g \ge 1$, then

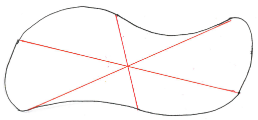
$$4g - 4 \;=\; \sum_{p \in \mathcal{R}} d_p$$

$d_p + 1$ = the degree of contact at p of the surface with a tangent geodesic 2-sphere.





These local lines of intersection **must propagate to the boundary**



These local lines of intersection **must propagate to the boundary** since any minimal surface in a hemisphere H with boundary on ∂H must lie completely in ∂H by a maximum principle.

THE CLIFFORD TORUS

Write $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and consider

$$\mathbb{T} \;=\; S^1\left(\tfrac{1}{\sqrt{2}}\right) \times S^1\left(\tfrac{1}{\sqrt{2}}\right) \;\subset\; S^3$$

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This is the intersection of S^3 with the algebraic variety

$$X_1^2 + X_2^2 \;=\; X_3^2 + X_4^2$$

or by a linear change of coordinates

$$Y_1 Y_2 + Y_3 Y_4 \;=\; 0$$

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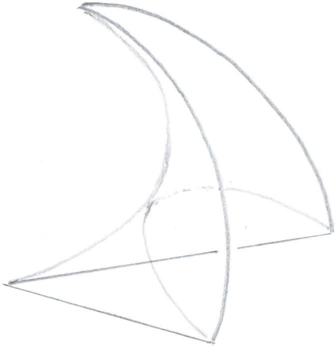
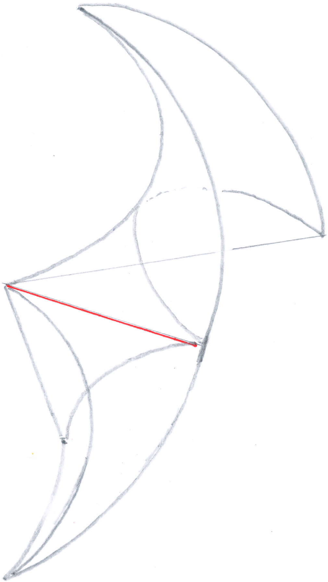
Proposition. Let $\varphi : S^3 \to S^3$ be the isometry of order 2 which fixes γ. Then

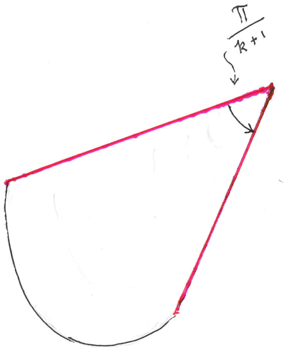
$$\Sigma \cup \varphi(\Sigma)$$

is a real analytic extension of Σ across γ.

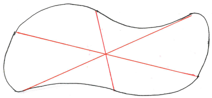




TWO GEODESIC PIECES OF THE BOUNDARY MEETING IN INTERIOR ANGLE $\frac{\pi}{k+1}$, $k \geq 1$



REFLECT 2K+1 TIMES

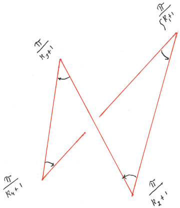


WE GET A REGULAR SURFACE

with a possible singularity at the center

which can be shown not to exist.

SUPPOSE WE HAVE A GEODESIC QUADILATERAL



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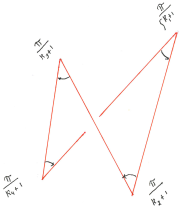

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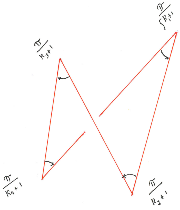

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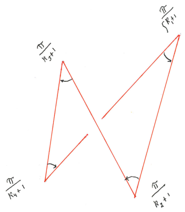

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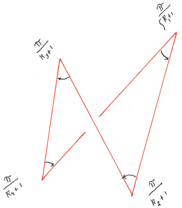

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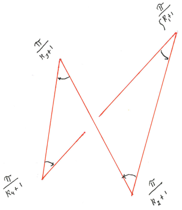

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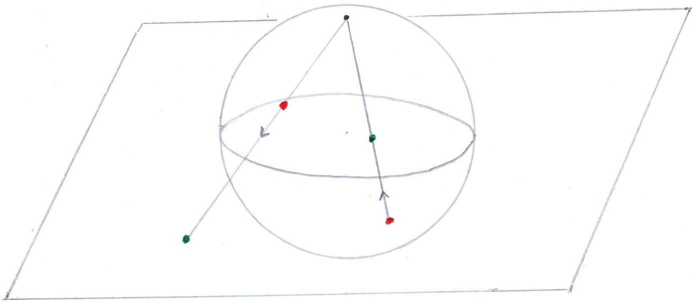
IF G IS FINITE, THE SURFACE IS COMPACT

LET US REVIEW STEREOGRAPHIC PROJECTION

For S^2



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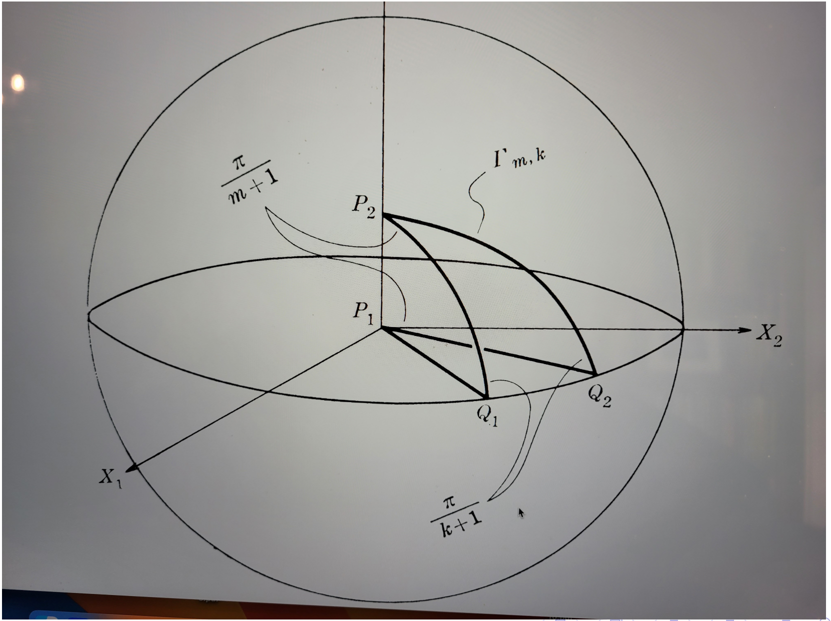
We transfer all this to stereographic projection

$$S^3 - \{N\} \longrightarrow \mathbb{R}^3$$

THE SURFACES $\xi_{m,k}$



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One can show that :

The surface is regularly embedded in the interior of the simplex.

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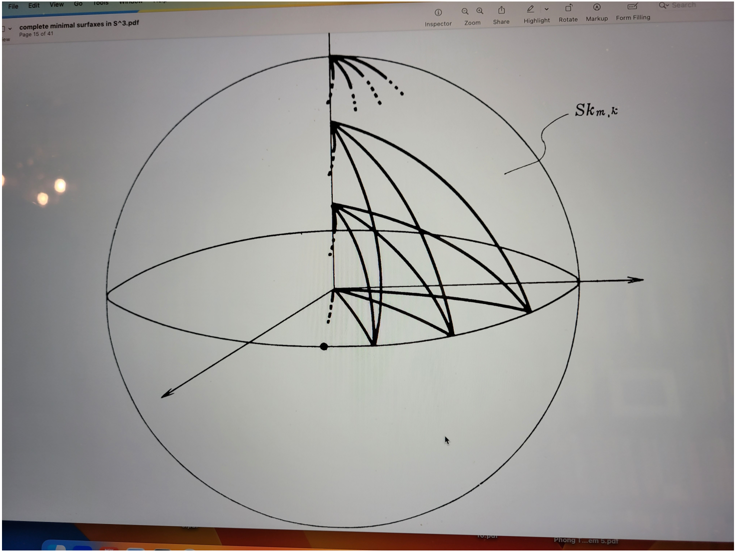
This gives a compact minimal surface

$$\xi_{m,k} \subset S^3$$

We have a triangulation of S^3 into $4(m+1)(k+1)$ congruent spherical simplicies.



Our surface lies in half of these in a checkerboard array.

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$$\int_{\Sigma} K\, dA \;=\; \pi\left[2 - 2\frac{k}{k+1} - 2\frac{m}{m+1}\right]$$

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$\xi_{1,1}$ = the Clifford Torus

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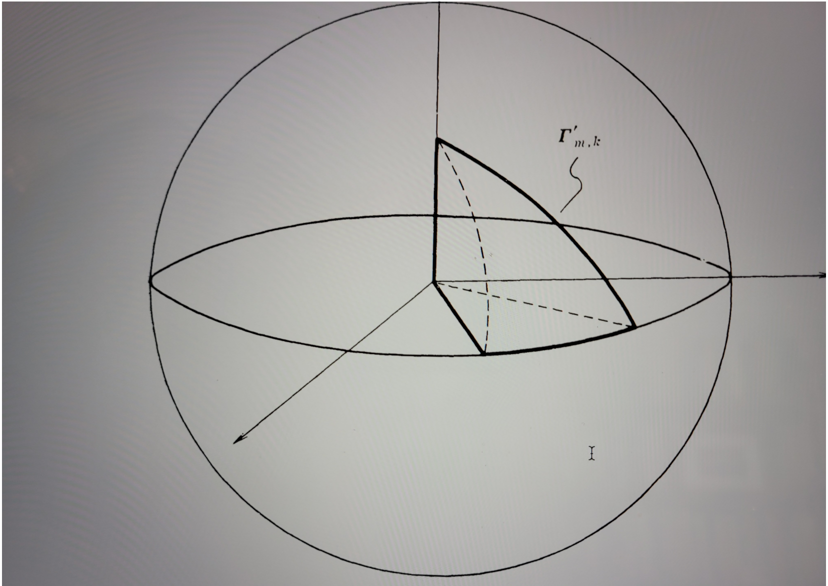

Figure 4.

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However, in this case there is an explicit formula.

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- $\text{Area}(\tau_{m,k}) \ \geq \ \min\{m,k\}$

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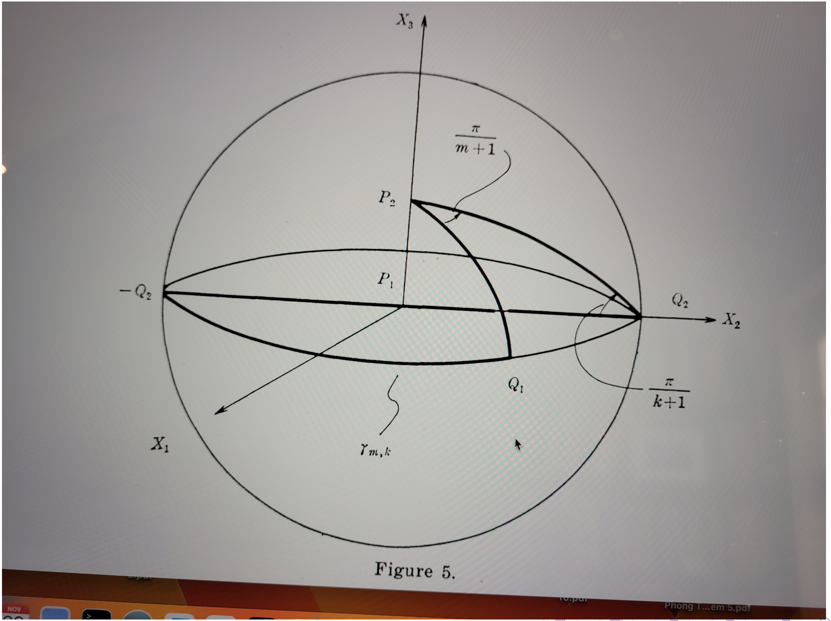
After many failed attempts over the years:

Theorem (Simon Brendle, 2012)

The conjecture is true

THE SURFACES $\eta_{m,k}$



Theorem:

To each ordered pair of positive integers (m, k), where k is odd, there corresponds a compact, non-orientable minimal surface $\eta_{m,k}$ containing $\gamma_{m,k}$ and having Euler characteristic $1 - mk$.

THEOREM

Every compact orientable surface can be minimally embedded into S^3.

Every non-orientable surface can be minimally immersed into S^3

except for the real projective plane which is prohibited by Almgren's theorem.

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there is an associated **polar map**

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$\Psi(x)$ is just the unit normal to Ψ at x.

$$\Psi^{**} \ = \ \Psi$$