INTRODUCTION TO MODERN ANALYSIS I

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Abstract

This is a draft of the lecture notes. Enevitably, there will be mistakes, typos and omissions. Please use with care and inform me of any errors. I will update these notes as we go.

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1 Introduction: What is Analysis?

Very simply the field of mathematical analysis is the study of functions with particular emphasis on approximation and limiting behaviour. This course focuses on 'real analysis', which means our functions will take in real numbers and output real numbers. We will generalise in places to functions that take in elements of 'metric spaces' (which can be thought of as spaces on which there is a appropriate notion of distance).

To study functions on the real numbers we need good knowledge of properties of the reals, i.e. how are they constructed, what characterises them, how and when do sequences and series of real numbers converge, etc. This will be the first part of the course.

We will then come to analyze functions on the reals. We will define continuity, differentiability and integrability. Many of the topics we will study will (and should) be familiar to you from calculus. However, the emphasis will be very different. We are not approaching the study of functions from the point of view of computation but rather we approach from the point of view of understanding the theoretical underpinnings of calculus. We want to understand where the usual rules come from and when they break down. In the next subsection we will visit some problems which can lead to nonsensical answers if one does not know when and why rules apply.

There are many other areas of analysis:

- complex analysis: functions now map the complex numbers to themselves.
- functional analysis: the analysis of spaces of functions with particular focus on their vector space structure.
- harmonic analysis: the study of waves and the Fourier transform.
- analysis of partial differential equations: the study of equations which govern how quantities change from place to place. These often arise in physics, chemistry, biology and economics. The methods of functional analysis turn out to be very useful here.

This course provides the first stepping stone to the other areas listed above and the methods and rigour developed here will be rehashed in many guises in these other areas of analysis.

1.1 Examples

Stating/proving/understanding a theorem can be essential to getting/predicting correct results when applied in practise. The ability to apply a theorem, or to extend it to other settings, comes only from knowing what ingredients are essential to make it work, i.e. from getting to grips with its proof. Otherwise one can end up with mathematical fallacies.

Example 1.1. *A common mistake in first year calculus is the following. Let $a, b, c \in \mathbb{R}$ and suppose*

$$a \cdot c = b \cdot c.$$

Then $a = b$, right?

One can produce the same fallacy in a different guise. Suppose $a = b$ *then*

$$a^2 = ab \Longleftrightarrow a^2 - b^2 = ab - b^2 \Longleftrightarrow (a-b)(a+b) = b(a-b).$$

Cancelling $a - b$ *and using* $a = b$ *gives*

$$a + b = b + b = b \implies 2b = b \implies 2 = 1.$$

Example 1.2. *Let's write* 1 *in the following way:*

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1.$$

The problem here is that the square root operation only distributes over a product if at least one of the real numbers involved is positive.

Example 1.3. *Suppose* $\varepsilon > 0$ *and we want to find the derivative of the function*

$$f(x) = \frac{x^3}{x^2 + \varepsilon^2}.$$

Using your standard toolbox of differentiation rules gives

$$\frac{df}{dx} = \frac{x^2(x^2 + 3\varepsilon^2)}{(x^2 + \varepsilon^2)^2} \implies \frac{df}{dx}(x = 0) = 0$$

for all $\varepsilon > 0$. *Therefore,*

$$\frac{d}{dx} \lim_{\varepsilon \to 0} f(x)\Big|_{x=0} = \lim_{\varepsilon \to 0} \frac{d}{dx} f(x)\Big|_{x=0} = 0.$$

On the other hand,

$$\lim_{\varepsilon \to 0} f(x) = x \implies \frac{d}{dx} \lim_{\varepsilon \to 0} f(x)\Big|_{x=0} = 1.$$

What we have assumed (incorrectly) is that the limit and the derivative always commute. Now, the derivative is defined by a limiting process itself (which depends on the point x*)! So what we are asking is: when can we interchange limits?*

Example 1.4. *Consider the following sum:*

$$s = \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots. \tag{1}$$

We note the following manipulation

$$2s = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 + s \implies s = 2.$$

The manipulation we have done here does in fact give the correct answer. We will see why when we discuss convergence of geometric series.

Example 1.5. *Consider the sum of all positive natural numbers:*

$$s \doteq \sum_{n=1}^{\infty} n$$

Obviously, this sum diverges. However, this can be manipulated in the following way:

$$\begin{aligned}
s &= 1 + 2 + 3 + 4 + 5 + 6 + ... \\
4s &= 4 + 8 + 12 + 16 + 20 + 24 + ... \\
-3s &= s - 4s = 1 + (2 - 4) + 3 + (4 - 8) + 5 + (6 - 12) + ... = 1 - 2 + 3 - 4 + 5 - 6 + ...
\end{aligned}$$

Note, that formally, (doing a Taylor/Maclaurin expansion)

$$\frac{1}{1+x} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = 1 - x + x^2 - x^3 + x^4 - ...,$$

which implies

$$\frac{d}{dx}\frac{1}{1+x} = -\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 + ... \implies \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3.$$

So,

$$\frac{1}{(1+1)^2} = 1 - 2 + 3 - 4 + 5 - 6 + ... \implies -3s = \frac{1}{4} \implies s = -\frac{1}{12}.$$

What are the (many) issues with these manipulations?

1. *We have manipulated infinite sums as if they were finite sums.*
2. *We have assumed that the Maclaurin expansion of* $1/1+x$ *is valid for* $x = 1$*.*
3. *To add insult to injury, we differentiated this Maclaurin expansion.*

Remarkably, it turns out that this manipulation and sum can be made sense of in a rigorous sense if one reinterprets what one means by summation (Ramanujun summation). See a course in complex analysis, in particular, see the Riemann-Zeta function. This actually sees practical application in quantum field theory and string theory.

Example 1.6. *Let's consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0. \end{cases}$$

What does is mean to take the limit of this function towards the origin $(0,0)$*? We could mean*

$$L_1 \doteq \lim_{x\to 0}\Big(\lim_{y\to 0} f(x,y)\Big) \quad \text{or} \quad L_2 \doteq \lim_{y\to 0}\Big(\lim_{x\to 0} f(x,y)\Big).$$

As you probably know neither of these (iterated) limits is what we mean by a multivariable limit but if the limit did exist these objects would compute it. We have $L_1 = 1$ *and* $L_2 = 0$*, which is inconsistent and, therefore, the limit does not exist. What this illustrates is that one has to be careful with swapping limits.*

Example 1.7. *Suppose we are presented with a matrix*

$$M = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

Let's sum rows and the columns to produce two vectors

$$v_1 = \begin{pmatrix} a+b \\ c+d \\ e+f \end{pmatrix}, \qquad v_2 = \begin{pmatrix} a+c+e \\ b+d+f \end{pmatrix}.$$

If one sums the entries in each of these vectors we get the same result. In other words, if we have a matrix $A = (a_{ij})$ $i = 1, ..., m$ and $j = 1, ..., n$ and we construct the double sums

$$s_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}, \qquad s_2 = \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}, \tag{2}$$

we have that $s_1 = s_2$ for finite m and n. What about for $m, n = \infty$? Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 1.8. *Consider the functions*

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}, \qquad g(x,y) = \frac{\sin(x^2+y^2)}{x^2-y^2}.$$

You can attempt a naive application of L'Hôpital's rule to find the limit at $(x,y) \to (0,0)$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\partial_x \sin(x^2+y^2)}{\partial_x (x^2+y^2)} = \lim_{(x,y)\to(0,0)} \cos(x^2+y^2) = 1$$
$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{\partial_x \sin(x^2+y^2)}{\partial_x (x^2-y^2)} = \lim_{(x,y)\to(0,0)} \cos(x^2+y^2) = 1$$

The problem is there is no naive multivariable generalisation of L'Hôpital's rule. It turns out that $\lim_{(x,y)\to(0,0)} g(x,y)$ is undefined but $\lim_{(x,y)\to(0,0)} f(x,y) = 1$ (you can apply L'Hôpital's successfully here, set $z = x^2 + y^2$).

2 Set Theory

Much of the language of mathematics is based upon set theorectic notation. We briefly review it here.

2.1 Definitions and Operations

Definition 2.1. *A set A is any unordered collection of objects. If x is an object in the set A, we say that x is an element of A and write $x \in A$. If x is an object not in the set A, we write instead $x \notin A$.*

Example 2.2. *The collection of objects (animals) $A = \{dog, cat, fish, horse\}$ is a set. In this case, $horse \in A$ but $whale \notin A$.*

Example 2.3. *The collection of objects (numbers) $A = \{1, 7, 3, 4, 2, 101\}$ is a set. Note that we have not ordered these in any way. Indeed, $A = \{101, 7, 3, 4, 2, 1\}$ or $A = \{1, 2, 3, 4, 7, 101\}$.*

Remark 2.4. Sets are themselves objects. In particular, if A and B are sets, it is meaningful to ask whether A is an element of B. For example $B = \{1, \{4, 5\}, 10\}$ is a set with the object/set $A = \{4, 5\}$ as an element.

Let us state some definitions for terminology/axioms for sets:

Definition 2.5 (Equality of Sets)**.** *Two sets A and B are equal, written $A = B$, if every element $x \in A$ is also an element of B and vice versa.*

Axiom 2.6. *There exists a set called the empty set, denoted $\emptyset$, which contains no objects, i.e. $x \notin \emptyset$ for any object x.*

Definition 2.7 (Subsets)**.** *We say that A is a subset of B, written $A \subseteq B$, if all elements of A are in B. (If A is not a subset of B, we may write $A \nsubseteq B$.)*

Example 2.8. *Let's return to our set $A = \{dog, cat, fish, horse\}$: a subset is $\{dog, cat, horse\}$. The set $\{dog, cat, fish, horse\}$ is also a subset of A.*

Example 2.9. *The sets $A = \{1, 4, 5\}$ and $B = \{1, 1, 4, 5, 1\}$ are equal.*

Remark 2.10. Note that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 2.11 (Properties of Sets)**.** *Suppose A is a set and P is a property concerning $x \in A$, i.e. for each $x \in A$, $P(x)$ is either true or false. We write*

$$\{x \in A : P(x)\}$$

for the subset of A containing all elements $x \in A$ such that $P(x)$ is true.

Example 2.12. *Suppose $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. We could ask for the set $B = \{x \in A : x \text{ is prime}\}$. This would then be the subset $B = \{2, 3, 5, 7, 11\}$.*

Definition 2.13 (Set Operations)**.** *Let A and B be two sets. We define the following operations:*

- *Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.*

- *Union:* $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- *Set difference:* $A \setminus B = \{x \in A : x \notin B\}$.
- *Compliment: suppose* $A \subseteq B$*, then the complement of* A *in* B*, denoted* A^c*, is given by* $A^c = B \setminus A$.
- *Symmetric difference:* $A \Delta B = \{x : x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$.
- *Power set:* $\mathcal{P}(A) = \{B : B \subseteq A\}$.

Example 2.14. *Let* $A = \{1, 3, 4\}$*,* $B = \{1, 2, 5\}$ *and* $C = \{1, 3, 4, 7\}$*. We have:*

$$A \cap B = \{1\}, \qquad A \cup B = \{1, 2, 3, 4, 5\}, \qquad A \setminus B = \{3, 4\} \qquad A \Delta B = \{2, 3, 4, 5\}$$

and

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}.$$

Finally, since $A \subseteq C$*, we have* $A^c = C \setminus A = \{7\}$.

Remark 2.15. There is always $2^{\text{No. of elements in } A}$ in the power set. Therefore, sometimes it is denoted 2^A.

Definition 2.16 (Arbitrary Intersection and Union)**.** *Let* I *be an index set for a family of sets* A_α*,* $\alpha \in I$*. Then we define*

$$\bigcap_{\alpha \in I} A_\alpha \doteq \{x : \forall \alpha \in I, x \in A_\alpha\}, \qquad \bigcup_{\alpha \in I} A_\alpha \doteq \{x : \exists \alpha \in I \text{ such that } x \in A_\alpha\}$$

Finally, we introduce ordered pairs and the Cartesian product for sets:

Definition 2.17 (Ordered Pairs/Cartesian Product)**.** *An ordered pair* (a, b) *is a pair of two elements where the order in which they are written matters, i.e.* $(a, b) = (c, d)$ *if and only if* $a = c$ *and* $b = d$.

The Cartesian product of two sets A *and* B*, denoted* $A \times B$ *is the set of ordered pairs constructed from* A *and* B*, i.e.*

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Let's now prove some properties of the operations we've introduced but first a note on logic.

Remark 2.18 (A note on Logic). The statement 'if and only if' is a statement of logical equivalence. Suppose you have two statements A and B. One often comes across the statement A if B, this means assume B is true then A follows. Similarly, one often comes across A only if B. This means that assuming that A is true then B follows. Combining them gives the 'if and only if' terminology. So, if one wants to prove said statement, then you need to show that B implies A and that $A \implies B$.

Proposition 2.19. *Let* A, B *and* C *be sets. Then,*

i) $(A \cup B) \cup C = A \cup (B \cup C)$

ii) $(A \cap B) \cap C = A \cap (B \cap C)$

iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

The laws *i), ii)* are called *associative laws*, and *iii), iv)* are called *distributive laws*.

Proof. For *i)* and *ii)* we will show the equivalence of both sides:
i) Let $x \in (A \cup B) \cup C$. This holds if and only if $x \in A \cup B$ or $x \in C$, which in turn means that $x \in A$ or $x \in B$ or $x \in C$. Consequently, this holds if and only if $x \in A$ or $x \in (B \cup C)$, which is the same as $x \in A \cup (B \cup C)$.
ii) Problem Sheet 1.

iii) To prove this law we will show two inclusions (which is a useful technique for showing equality of two sets):
"$\subseteq$" Let $x \in A \cap (B \cup C)$. This means that $x \in A$ and $x \in B \cup C$ and hence $x \in A$, and $x \in B$ or $x \in C$. This in turn means that $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$, which means that $x \in (A \cap B) \cup (A \cap C)$ and hence we showed that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
"$\supseteq$" Now assume that $x \in (A \cap B) \cup (A \cap C)$. This implies that $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$ then this means that $x \in A$ and $x \in B$, hence also $x \in B \cup C$. Therefore, $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$, hence also $x \in C \cup B = B \cup C$. As a result, we get that $x \in A \cap (B \cup C)$.

iv) Problem Sheet 1. $\square$

Remark 2.20. In logic, we often we denote “or” by $\vee$ and “and” by $\wedge$, and we write $\iff$ for “in and only if”. In this language, the above proof can be written somewhat more concisely, e.g. to prove *iii)* we write the following equivalences: $x \in A \cap (B \cup C) \iff x \in A \wedge (x \in B \vee x \in C) \iff (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \iff x \in (A \cap B) \cup (A \cap C)$.

Remark 2.21. Try demonstrating the above laws using Venn diagrams.

Proposition 2.22 (De Morgans' Laws)**.** *Suppose X is a set and $A, B \subseteq X$. Then*

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

Proof. In both cases will show two inclusions.
"$\subseteq$" Assume that $x \in (A \cup B)^c$. This is true if and only if $x \notin A \cup B$, hence $x \notin A$ and $x \notin B$. This means that x belongs to A^c and to B^c, i.e. $x \in A^c \cap B^c$. This shows that $(A \cup B)^c \subseteq A^c \cap B^c$.
"$\supseteq$" Let $x \in A^c \cap B^c$. This means that $x \in A^c$ and $x \in B^c$, i.e. $x \notin A$ and $x \notin B$. Putting this together we get that $x \notin A \cup B$, which in turn means that $x \in (A \cup B)^c$. Hence $A^c \cap B^c \subseteq (A \cup B)^c$, which finishes the proof.

For the second assertion, we will start with the inclusion "$\subseteq$". Let $x \in (A \cap B)^c$, i.e. $x \notin A \cap B$. Because $A \cap B = \{y : y \in A \text{ and } y \in B\}$, this means that $x \notin A$ or $x \notin B$. If $x \notin A$ then $x \in A^c$, and thus $x \in A^c \cap B^c$. If $x \notin B$ then $x \in B^c$ and hence $x \in B^c \cup A^c$. Hence, $x \in A^c \cup B^c$, and we showed that $(A \cap B)^c \subseteq A^c \cup B^c$.
To show the opposite inclusion "$\supseteq$" we will argue towards a contradiction: assume that $x \in A^c \cup B^c$ but that $x \notin (A \cap B)^c$. From this follows that $x \in A \cap B$, and hence $x \in A$ and $x \in B$, which implies that $x \notin A^c$ and $x \notin B^c$. We conclude that $x \notin A^c \cup B^c$, which contradicts our hypothesis and finishes the proof. $\square$

Proposition 2.23. *Let X be a set, and $A, B \subseteq X$. Then $A \subseteq B$ implies that $B^c \subseteq A^c$.*

Proof. Assume that $A \subseteq B$ and let $x \in B^c$. This means that $x \notin B$, which by our assumption implies that $x \notin A$ since every element of A also belongs to B. Therefore, $x \in A^c$, which finishes the proof. $\square$

Definition 2.24 (Partition of a Set)**.** *A partition of a set X is a collection of subsets A_α of X such that each element of X is contained in exactly one A_α.*

2.2 Functions/Mappings on Sets

Definition 2.25 (Function/Map)**.** *A function or map f from a set A to a set B is an assignment of precisely one element $f(a) \in B$ to each $a \in A$. Typical notations are*

$$f : A \to B, \qquad a \mapsto f(a).$$

In this context the set A is called the domain and the set B the codomain.

Example 2.26. *The identity function is a function defined for every set A. It is the map $\mathrm{id}_A : A \to A$ defined by $a \mapsto a$.*

Example 2.27. *The cube function $(\cdot)^3$ defined by taking $x \in \mathbb{R}$ to $x^3 \in \mathbb{R}$ is a function.*

Example 2.28. *The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not well-defined since $f(0)$ isn't defined. However, $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is well-defined.*

Example 2.29. *Suppose one writes the rule $f : \{0, 1\} \to \{0, 1\}$ such that $f(0) = 0$, $f(0) = 1$ and $f(1) = 0$. This is not a function since we have two outputs for 0 in the domain.*

Definition 2.30 (Injective/Surjective/Bijective Functions)**.** *Suppose $f : X \to Y$ is a function. We say that f is*

- *injective/one-to-one, if for all $x_1, x_2 \in X$ we have that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.*
- *surjective/onto, if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.*
- *bijective, if it is both injective and surjective.*

Injectivity, surjectivity and, therefore, bijectivity are heavily dependent on the domain and codomain.

Example 2.31. *Consider the function f $x \mapsto x^2$.*

- *If $f : \mathbb{R} \to \mathbb{R}$ it is neither bijective or surjective.*
- *If $f : [0, \infty) \to \mathbb{R}$ then the square function is injective but not surjective.*
- *If $f : \mathbb{R} \to [0, \infty)$ then the square function is surjective but not injective.*
- *If $f : [0, \infty) \to [0, \infty)$ then the square function is bijective.*

Definition 2.32 (Image/Preimage)**.** *If $f : A \to B$ and $U \subseteq A$ then we denote*

$$f(U) = \{f(u) : u \in U\} \subset B.$$

The set $f(A)$ is called the image of A.

Suppose $V \subseteq B$, then we define

$$f^{-1}(V) = \{a \in A : f(a) \in V\}.$$

A related but non-equivalent concept is the inverse of a **bijective function**.

Definition 2.33 (Inverse of a Function)**.** *Let $f : X \to Y$ be a bijection. Then the inverse function $f^{-1} : Y \to X$ is the map which assigns to each $y \in Y$ the element $x = f^{-1}(y)$ determined uniquely (by bijectivity) the element $x \in X$ satisfying $f(x) = y$.*

Definition 2.34 (Function Composition)**.** *Let $f : X \to Y$ and $g : Y \to Z$, then we define the composition of f and g, denoted $g \circ f : X \to Z$, as $(g \circ f)(x) = g(f(x))$.*

Remark 2.35. Note that if $f : X \to Y$ is bijective then $f \circ f^{-1} : Y \to Y$ so $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f : X \to X$ so $f^{-1} \circ f = \mathrm{id}_X$.

Definition 2.36 (Graph)**.** *Let $f : A \to B$, the graph of a function is the subset of $A \times B$:*

$$\mathrm{Graph}(f) \doteq \{(a, f(a)) : a \in A\}.$$

2.3 Relations

Definition 2.37 (Relation)**.** *Suppose X, Y are sets. A relation R on the set $X \times Y$ is a subset of the $X \times Y$. We write aRb if $(a, b) \in R$. If we have a relation on $X \times X$ we will say this is a relation on X.*

Example 2.38. *Let $X = Y = \{1, 2, 3, 4, 5, 6, 7\}$. We define the relation R by saying aRb if $|a - b| \le 2$. Then $1R2$ and $1R3$ but $1 \not R 4$.*

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 4), (3, 5), (4, 4), ...\}$$

Example 2.39. *Let $X = Y = \mathbb{Z}$ the integers. We define the relation R by saying aRb if there exists an integer k such that $a - b = 3k$. Then $0 \not R 2$ but $0R3$.*

$$R = \{..., (0, -3), (0, 3), (0, 6), (0, 9), ..., (1, 4), (1, 7), (1, 10), ..., (2, 5), (2, 8), (2, 11), ...\}$$

This is how one defines modular arithmetic (here $x \mod 3$).

Definition 2.40 (Classification of Relations)**.** *We call a relation R on X:*

- *reflexive: for all $x \in X$, xRx.*
- *symmetric: for all $x, y \in X$, if xRy, then yRx.*
- *antisymmetric: for all $x, y \in X$, if xRy and yRx, then $x = y$.*
- *transitive: for all $x, y, z \in X$, if xRy and yRz then xRz.*
- *an equivalence relation if it is reflexive, symmetric and transitive. In this case, it is common notation to use $\sim$ instead of R. The equivalence class of x, denoted $[x]$, is the set:*

  $$[x] = \{y \in X : y \sim x\}.$$

- *a partial order on a set X is a reflexive, antisymmetric, transitive relation R on X. Traditionally, a partial order is denoted $\le$.*
- *a total order on a set X is a partial order with the extra condition that for all $x, y \in X$ either $x \le y$ or $y \le x$ holds.*

Example 2.41. *Let $X = \{a, b, c\}$ and let the relation on X be $R = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$. Clearly R is reflexive. We have that R is symmetric also since bRc and cRb. The relation is also transitive since $(b, c) \in R$ and $(c, b) \in R$ and $(b, b) \in R$, $(c, c) \in R$. So R is an equivalence relation. We need the equivalence classes:*

$$[a] = \{a\}, \qquad [b] = \{b, c\}, \qquad [c] = \{b, c\} = [b].$$

Example 2.42. *Let $X = \{a, b, c\}$ and consider its power set:*

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

Define a relation R on $\mathcal{P}(X)$ by set inclusion, i.e. let A and B be sets then $A \leq B$ if $A \subseteq B$. This is a partial order but not a total order since $\{a\} \not\subseteq \{b\}$ for example.

2.4 Infimum and Supremum

Definition 2.43 (inf and sup)**.** *Let X be a partially ordered set. An element $s \in X$ is a supremum (or least upper bound) for the set $S \subset X$, denoted $s = \sup S$, if*

- *s is an upper bound: for every $x \in S$, $x \leq s$.*
- *s' is an upper bound for S then $s \leq s'$.*

An element $s \in X$ is an infimum (or greatest lower bound) for the set $S \subset X$, denoted $s = \inf S$, if

- *s is an lower bound: for every $x \in S$, $x \geq s$.*
- *s' is an lower bound for S then $s \geq s'$.*

We also will say that S is bounded from above (below) if it has an upper (lower) bound;

Definition 2.44 (Max/Min)**.** *An element $s \in S$ (if it exists) is a maximum for S if s is a supremum such that $s \in S$.*

An element $s \in S$ (if it exists) is a minimum for S if s is an infimum such that $s \in S$.

Example 2.45. *Let's return to our example above with $X = \{a, b, c\}$ and a partial order $\leq$ on*

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

by set inclusion. Let $S = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. The infimum of this set is $\inf S = \emptyset$ and the supremum of this set is $\sup S = X$. The maximum and minimum do not exist.

However, take $S = \{\emptyset, \{a\}, \{a, b\}\}$ then $\inf S = \min S = \emptyset$ and $\sup S = \max S = \{a, b\}$.

3 The Real Numbers

There are a few ways to define the real numbers. We will take a slightly hybrid approach. Effectively, we will be defining them axiomatically after sketching their construction starting with the natural numbers. The reason for this is that (hopefully) this is the most satisfying from a philosophical point of view; you could state a bunch of axioms to define a set but whether there exists a set satisfying those axioms might remain to be seen. One may also wonder about the uniqueness of such a set.

3.1 The Naturals, Integers and Rationals

You, of course, know intuitively what the natural numbers $\mathbb{N}$ are:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, ...\}.$$

You start at 0 (or 1) and you count indefinitely. For most practical purposes this is fine. However, mathematically, this is slightly unsatisfactory. For example, how do you know that counting indefinitely doesn't loop and come back to 0? Also, I've not stated how to add or multiply numbers, things you have been doing since you were a child. We will discuss briefly how to define the natural numbers via the Peano Axioms:

Definition 3.1 (Natural Numbers)**.** *A set $\mathbb{N}$ is the natural numbers if it contains a special element, which we will denote 0, and a map $S : \mathbb{N} \to \mathbb{N}$, called the succession (or increment) map, that maps $n \in \mathbb{N}$ to its successor $S(n) \in \mathbb{N}$ (intuitively $S(n) = n + 1$) such that*

1. *$S(n) \neq 0$ for all $n \in \mathbb{N}$*
2. *for all $n, m \in \mathbb{N}$, if $S(n) = S(m)$, then $n = m$.*
3. *For any subset $A \subseteq \mathbb{N}$, if $0 \in A$ and if $n \in A$ implies that $S(n) \in A$, then $A = \mathbb{N}$.*

We define $1 \doteq S(0)$, $2 \doteq S(1)$, $3 \doteq S(2)$ and so on.

Sometimes the last axiom, known as the axiom of induction is formulated in the following equivalent way:

Theorem 3.2 ((Weak) Principle of Induction)**.** *Let $P(n)$ be a statement about the natural number n. Suppose that*

- *$P(0)$ is true,*
- *if $P(n)$ is true for some $n \in \mathbb{N}$ then this implies that $P(S(n))$ is true.*

Then $P(n)$ is true for all $n \geq 0$.

Proof. Let $E = \{n : P(n) \text{ is true}\} \subseteq \mathbb{N}$. We have assumed $P(0)$ is true, so $0 \in E$. We have also assumed that if $P(n)$ is true then $P(S(n))$ is true. This means that if $n \in E$ then $S(n) \in E$. Therefore, $E = \mathbb{N}$ by 3 above. $\square$

We now want to move on to define addition and multiplication. We do this recursively.

Definition 3.3 (Addition/Multiplication)**.** *We define addition* $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *on the natural numbers by*

$$n + 0 \doteq n, \qquad n + S(m) \doteq S(n + m),$$

for all $n, m \in \mathbb{N}$.

We define multiplication $\times : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *on the natural numbers by*

$$n \times 0 \doteq 0, \qquad n \times S(m) \doteq n \times m + n,$$

for all $n, m \in \mathbb{N}$.

This definition of addition and multiplication may look rather funny. However, let's put this into practice. Suppose I want to add 3 and 2:

$$3 + 2 = 3 + S(1) = S(3 + 1) = S(3 + S(0)) = S(S(3 + 0)) = S(S(3)) = S(4) = 5.$$

Let's try multiplication:

$$\begin{aligned} 3 \cdot 2 &= 3 \cdot S(1) = 3 \cdot 1 + 3 = 3 \cdot S(0) + 3 = (3 \cdot 0 + 3) + 3 = 3 + 3 \\ &= 3 + S(2) = S(3 + 2) = S(5) = 6. \end{aligned}$$

Remark 3.4. Notice at this point we can recover familiarity by noting that $S(n) = S(n+0) = n + S(0) = n + 1$.

Remark 3.5. One can totally order the naturals with a relation $\leq$ by stating that for any two $a, b \in \mathbb{N}$,

$$a \leq b$$

if there exists a $c \in \mathbb{N}$ such that $b = a + c$. We can also introduce $a < b$ if $a \leq b$ and $a \neq b$.

Proposition 3.6. *Addition and multiplication are commutative and associative, i.e.*

$$\begin{aligned} a + b &= b + a \qquad & (a + b) + c &= a + (b + c), \\ a \times b &= b \times a \qquad & (a \times b) \times c &= a \times (b \times c). \end{aligned}$$

Moreover, you can show that multiplication is distributive over addition:

$$a \times (b + c) = a \times b + a \times c$$

Finally, you have the cancellation laws:

- *if* $a + b = a + c$ *then* $b = c$.
- *if* $a \cdot b = a \cdot c$ *and* $a \neq 0$ *then* $b = c$.

Proof. We will only prove the commutativity of addition. The rest is an exercise for the reader. We will do this with a series of steps:

- *Claim*: for any natural number n, $0 + n = n$, i.e. adding zero is commutative.

  We use induction. The base case is $0+0 = 0$ which follows from the definition. Suppose we know that $0 + N = N$ for $N \in \mathbb{N}$. What we need to show is that

  $$0 + S(N) = S(N).$$

  By the definition of addition, we have

  $$0 + S(N) = S(0 + N) = S(N)$$

  since $0 + N = N$. This closes our induction argument.

- *Claim*: $S(n) + m = S(n + m)$ for any n, m natural, i.e. the second defining property of addition is commutative.

  We again use induction on m fixing an arbitrary n. So for $m = 0$ then

  $$S(n) + 0 = 0 + S(n) = S(0 + n) = S(n + 0)$$

  where the first equality follows from the claim above, the second from the definition and the third from the claim above. Suppose that $S(n) + M = S(n + M)$ for $M \in \mathbb{N}$. We now want to show

  $$S(n) + S(M) = S(n + S(M)).$$

  So,

  $$S(n) + S(M) = S(S(n) + M) = S(S(n + M)) = S(n + S(M))$$

  where the first equality is by definition, the second by our supposition, and the third by again definition.

- Finally we can show $n + m = m + n$. We again use induction on n with fixed m.

  The base case is with $n = 0$:

  $$m + 0 = m = 0 + m$$

  by our first claim.

  Next, we suppose that $n + m = m + n$ up to some N with fixed arbitrary m. We want to show (to close the induction) that

  $$S(N) + m = m + S(N).$$

  So,

  $$S(N) + m = S(N + m) = S(m + N) = m + S(N)$$

  where the first equality is by our second claim, the second equality by our supposition and the third by definition.

$\square$

We want to be able to subtract numbers. This leads to us now defining the integers $\mathbb{Z}$ from the natural numbers as follows:

Definition 3.7 (Integers)**.** *The integers, denoted $\mathbb{Z}$, is defined to be the set of equivalence classes of $\mathbb{N}$ under the equivalence relation*

$$(a,b) \sim (c,d) \qquad \textit{if} \qquad a+d = b+c.$$

So,

$$\mathbb{Z} = \{[(a,b)] : (a,b) \sim (c,d) \qquad \textit{if} \qquad a+d = b+c\}.$$

We denote the equivalence class $[(a,b)]$ as $a-b \in \mathbb{Z}$ and if we have $[(a,0)]$ or $[(0,a)]$ we denote this as a or $-a$ respectively. We define addition and multiplication as follows:

$$\begin{aligned}
[(a,b)] + [(c,d)] &= [(a+c, b+d)] \\
[(a,b)] \times [(c,d)] &= [(ac+bd, bc+ad)]
\end{aligned}$$

Let's sanity check this: suppose we want to add $a = [(a,0)]$ and $-b = [(0,b)]$ then by definition

$$a + (-b) = [(a,b)] = a - b.$$

To multiply we find

$$a \times (-b) = [(a0 + 0b, 00 + ab)] = [(0, ab)] = -(ab)$$

and

$$(a-b) \times (c-d) = [(a,b)] \times [(c,d)] = [(ac+bd, bc+ad)] = ac + bd - bc - ad.$$

The 'usual' rules of addition, multiplication and cancellation follow from the same rules on naturals. We won't labour this point.

We want to be able to divide numbers. This leads us to the definition of the rationals.

Definition 3.8 (Rationals)**.** *The rationals, denoted $\mathbb{Q}$, is defined to be the set of equivalence classes of $\mathbb{Z}$ under the equivalence relation*

$$(a,b) \sim (c,d) \qquad \textit{if} \qquad ad = cb, \qquad b \neq 0 \neq d.$$

So,

$$\mathbb{Q} = \{[(a,b)] : (a,b) \sim (c,d) \qquad \textit{if} \qquad ad = cb, \qquad b \neq 0 \neq d\}.$$

We denote the equivalence class $[(a,b)]$ as $\frac{a}{b}$. We define addition and multiplication as follows:

$$\begin{aligned}
[(a,b)] + [(c,d)] &= [(ad+bc, bd)] \\
[(a,b)] \times [(c,d)] &= [(ac, bd)]
\end{aligned}$$

Let's do a sanity check:

$$\frac{a}{b} + \frac{c}{d} = [(a,b)] + [(c,d)] = [(ad+bc, bd)] = \frac{ad+bc}{bd}$$

The 'usual' rules of addition, multiplication, and cancellation follow from the same rules on integers. Again, we will not labour the details.

The rationals $\mathbb{Q}$ are what is known as a totally ordered field. We will now build to the definition of this object, which we will also need for the definition of the axiomatic definition of the reals.

Definition 3.9 (Group). *A group is a non-empty set G together with a binary operation $\diamond : G \times G \to G$ such that*

- *for all $a, b, c \in G$, one has $(a \diamond b) \diamond c = a \diamond (b \diamond c)$ (associativity),*
- *there exists an element $e \in G$ such that $a \diamond e = a = e \diamond a$ (identity element),*
- *for each $a \in G$, there exists an element a^{-1} such that $a \diamond a^{-1} = e = a^{-1} \diamond a$ (inverse).*

A group is commutative/abelian if $a \diamond b = b \diamond a$ for all $a, b \in G$.

Proposition 3.10. *The identity element and inverse are unique.*

Proof. Suppose e and e' are two identity elements. Then,

$$e = e \diamond e' = e'.$$

Suppose b and c are inverses for a. Then

$$e = a \diamond c \implies b \diamond e = b \diamond (a \diamond c) \implies b = (b \diamond a) \diamond c \implies b = e \diamond c \implies b = c.$$

□

Example 3.11. *The integers $\mathbb{Z}$ are a commutative group under addition $+$. Associativity and commutativity are trivial. The identity element is simply $0 = [(0, 0)]$, the (additive) inverse is $a = [(a, 0)]$ is $-a = [(0, a)]$ since $a + (-a) = [(a, a)] = [(0, 0)] = 0$.*

Definition 3.12 (Field). *A field is a set $\mathbb{F}$ with two binary operations called addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ such that*

- *$\mathbb{F}$ is an abelian group under addition with an identity we call 0.*
- *$\mathbb{F} \setminus \{0\}$ is an abelian group under multiplication with an identity we call 1.*
- *Multiplication and addition are compatible: multiplication distributes over addition,*

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

A totally ordered field is a field with a relation $\leq$ such that

- *$\leq$ is a total order,*
- *for any $a, b, c \in \mathbb{F}$, if $a \leq b$ then $a + c \leq b + c$,*
- *for any $a, b, c \in \mathbb{F}$, if $a \leq b$ and $0 \leq c$ then $a \cdot c \leq b \cdot c$,*

Remark 3.13. Note that a totally ordered field is not just a field with a total order. The order structure and addition and multiplication have to be compatible. For example, take arithmetic modulo 3. The elements of your set are the equivalence classes,

$$[0] = \{0, 3, 6, ...\}, \qquad [1] = \{1, 4, 7, ...\}, \qquad [2] = \{2, 5, 8, ...\}$$

with the total order

$$[0] \leq [0], \qquad [1] \leq [1], \qquad [2] \leq [2], \qquad [0] \leq [1], \qquad [0] \leq [2], \qquad [1] \leq [2],$$

and addition defined by $[a] + [b] = [a + b]$. Now,

$$[0] \leq [1], \quad \text{but} \quad [0] + [2] = [0 + 2] = [2] \not\leq [0] = [3] = [1 + 2] = [1] + [2].$$

Remark 3.14. We organise the rationals into positive, zero and negative rationals and then we can define a total order on them by stating that $x < y$ if $y - x$ is a positive rational and $x > y$ if $y - x$ is a negative rational. We say $x \leq y$ if either $x < y$ or $x = y$, similarly for $x \geq y$.

Above we reasoned that the naturals weren't enough to deal with subtraction, the integers weren't enough to deal with division but what's the issue with the rationals? Well, its an issue of 'completeness': there are numbers missing.

Proposition 3.15. *There does not exist a rational $q \in \mathbb{Q}$ with $q^2 \doteq q \cdot q = 2$.*

Proof. We approach this by contradiction. So, suppose otherwise, i.e. there is a $q \in \mathbb{Q}$ such that $q^2 = 2$. So, $q = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ such that $b \neq 0$. Pick a and b so that they have no common factors other than 1 (in this case we say a and b are coprime).

Now

$$2 = \left(\frac{a}{b}\right)^2 = \frac{a \cdot a}{b \cdot b} = \frac{a^2}{b^2}.$$

Therefore, $[(2, 1)] = [(a^2, b^2)]$ and this means $2b^2 = a^2$ by definition. This means a^2 is even, and, therefore, a is even. Indeed, if $a = 2k + 1$ for some $k \in \mathbb{Z}$, then

$$a^2 = (2k+1)^2 = 2\left(2k^2 + k\right) + 1,$$

which is an odd number, which is a contradiction.

Now if a is even we have $a = 2a'$ and, therefore, $b^2 = 2(a')^2$. Therefore, b is even, i.e. $b = 2b'$.

However, then a and b have a common factor, in this case 2, which is a contradiction. $\square$

This issue of completeness can be expressed in terms of supremum/infimum. So the problem with $\mathbb{Q}$ can be realised by looking for the supremum of the set

$$S = \{q \in \mathbb{Q} : q^2 < 2\}.$$

There is no supremum **in** $\mathbb{Q}$. The real numbers are the set which fixes this issue.

3.2 A Sketch of a Constructive Definition

We will not give the full details on the construction of the real numbers (there are two constructive definitions; we may return to this when we come to metric spaces). We will sketch the construction via Dedekind cuts.

Definition 3.16 (Dedekind Cut)**.** *A Dedekind cut is a partition of the rationals $\mathbb{Q}$ into two subsets L and R such that*

- *L is nonempty.*
- *$L \neq \mathbb{Q}$.*
- *If $x, y \in \mathbb{Q}$, $x < y$ and $y \in L$ then $x \in L$ (i.e. L contrains all rationals to the left of y, one says it's closed downwards.)*
- *If $x \in L$, then there exists a $y \in L$ such that $y > x$. (L has no greatest element.)*

Here, $R = \mathbb{Q} \setminus L$.

The idea is the following. We take sets such as

$$L = \{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\}, \qquad R = \mathbb{Q} \setminus L.$$

You should convince yourself this is a Dedekind cut. Since L in this definition completely determines R, we can simply use L as the representation of (L, R). The claim is that one can construct the real numbers from such sets. Effectively, we can represent any real number r in our system via a set L. For example, the 'missing number' the L above represents is $\sqrt{2}$. We can represent any rational $q \in \mathbb{Q}$ by the sets

$$L = \{x \in \mathbb{Q} : x < q\}.$$

Collecting all such sets defines the real numbers $\mathbb{R}$. One can define the familiar notions of addition, subtraction, division, multiplication, the 'usual' ordering on such sets and, moreover, there exist suprema and infima in $\mathbb{R}$. We omit the details here.

Remark 3.17. There is another constructive definition of the reals based on Cauchy sequences, limits and absolute value. We may comment on this more later.

3.3 Axiomatic Definition

We now have enough definitions to state quite simply what the real numbers are in axioms.

Definition 3.18 (The Real Numbers)**.** *The real numbers, denoted* $\mathbb{R}$*, is a totally ordered field that satisfies the supremum/least upper bound axiom: every non-empty subset* $A \subseteq \mathbb{R}$ *that has an upper bound has a supremum/least upper bound in* $\mathbb{R}$*.*

"Unpacking" this definition leads to the following list of properties:

Definition. *The real numbers are a totally ordered field, that is a set* $\mathbb{R}$ *with two binary operations, addition* $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *given by* $(x, y) \mapsto x + y$ *and multiplication* $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *given by* $(x, y) \mapsto x \cdot y$ *and a relation* $\leq$ *such that*

A) $(\mathbb{R}, +)$ *is a commutative group, that is,*

1) *for every* $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$ *(associativity),*
2) *there exists a unique element in* $\mathbb{R}$*, called* zero *and denoted* 0*, such that* $0 + a = a + 0 = a$ *for every* $a \in \mathbb{R}$ *(a neutral element),*
3) *for every* $a \in \mathbb{R}$ *there exists a unique element in* $\mathbb{R}$*, called the additive inverse of* a *and denoted* $-a$*, such that* $(-a) + a = a + (-a) = 0$ *(inverse elements),*
4) *for every* $a, b \in \mathbb{R}$, $a + b = b + a$ *(commutativity),*

M) *The operation* $\cdot$ *is associative, commutative and has a neutral element. Moreover,* $(\mathbb{R} \setminus \{0\}, \cdot)$ *is a commutative group, that is,*

1) *for every* $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
2) *there exists a unique element in* $\mathbb{R}$*, called* one *and denoted* 1*, such that* $1 \neq 0$ *and* $1 \cdot a = a \cdot 1 = a$ *for every* $a \in \mathbb{R}$,
3) *for every* $a \in \mathbb{R}$ *with* $a \neq 0$ *there exists a unique element in* $\mathbb{R}$*, called the multiplicative* inverse *of* a *and denoted* a^{-1}*, such that* $a^{-1} \cdot a = a \cdot a^{-1} = 1$,

4) for every $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$,

AM) The distributive property holds, that is for every $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

O) $\leq$ is a total order relation*, that is,*

1) for every $a, b \in \mathbb{R}$ either $a \leq b$ or $b \leq a$,
2) for every $a, b, c \in \mathbb{R}$ if $a \leq b$ and $b \leq c$, then $a \leq c$,
3) for every $a, b \in \mathbb{R}$ if $a \leq b$ and $b \leq a$, then $a = b$,
4) for every $a \in \mathbb{R}$ we have $a \leq a$,

AO) for every $a, b, c \in \mathbb{R}$ if $a \leq b$, $a + c \leq b + c$,

MO) for every $a, b, c \in \mathbb{R}$ if $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$,

S) the supremum property holds: every non-empty set $A \subset \mathbb{R}$ that has an upper bound has a supremum/least upper bound in $\mathbb{R}$.

Remark 3.19. We claim that the construction via Dedekind cuts satisfies the axioms of this definition; such an object exists! However, a word on uniqueness is in order.

This definition completely characterises the real numbers in the sense that if $(\mathbb{R}', \oplus, \odot, \preccurlyeq)$ satisfies the same properties, then there exists a bijection $T : \mathbb{R} \to \mathbb{R}'$ such that T is an isomorphism between the two fields, that is,

$$T(a + b) = T(a) \oplus T(b), \qquad T(a \cdot b) = T(a) \odot T(b)$$

for all $a, b \in \mathbb{R}$, and $a \leq b$ if and only if $T(a) \preccurlyeq T(b)$. Hence, for all practical purposes, we cannot distinguish $\mathbb{R}$ from $\mathbb{R}'$. It is in this sense that we can talk of 'the' real numbers and the real numbers can be used and manipulated, without referring to the method of construction.

Remark 3.20. For $a, b \in \mathbb{R}$, we write $a < b$ if $a \leq b$ and $a \neq b$.

Remark 3.21 (Natural/Integers/Rationals from the Reals). How do the naturals, integers and rationals fit into this axiomatic picture.

- By definition we have a multiplicative identity $1 \in \mathbb{R}$. One can then *define* the natural numbers by adding 1 itself n-times.

  $$n = 1 + 1 + ... + 1 \in \mathbb{R} \implies \mathbb{N} = \{0, 1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, ...\}.$$

  Let's adopt the convention that

  $$\mathbb{N}_k = \mathbb{N} \setminus \{n \in \mathbb{N} : n < k\}.$$

  So, the usual choice for the naturals in this notation are $\mathbb{N}_0 = \mathbb{N}$ and $\mathbb{N}_1 = \{1, 2, 3, 4, 5, ...\}$.

- By the definition of the reals we have an additive inverse of every element, therefore, $-n \in \mathbb{R}$ for any natural n. So we define the integers by taking the union of $\mathbb{N}$ with $-\mathbb{N}$.

  $$\mathbb{Z} = \{\pm n : n \in \mathbb{N}\}.$$

- By the definition of the reals for any $b \in \mathbb{Z} \setminus \{0\}$ there is a multiplicative inverse, $\frac{1}{b} \in \mathbb{R}$, so we can define the rationals via

  $$\mathbb{Q} = \Big\{\frac{a}{b} \doteq \frac{1}{b} \cdot a : a, b \in \mathbb{Z}, b \neq 0\Big\}.$$

  The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational numbers*.

In the above addition, multiplication and order on these sets are induced by $(+, \cdot, \leq)$ on $\mathbb{R}$.

Finally, let's define some privileged subsets of $\mathbb{R}$.

Definition 3.22 (Closed, Open Intervals)**.** *Let $a, b \in \mathbb{R}$ with $a \leq b$.*

- *A closed interval, denoted $[a, b]$, is the set $\{x \in \mathbb{R} : a \leq x \leq b\}$.*
- *A open interval, denoted (a, b), is the set $\{x \in \mathbb{R} : a < x < b\}$.*
- *The half-open intervals, denoted $[a, b)$ or $(a, b]$, are the sets $\{x \in \mathbb{R} : a \leq x < b\}$ and $\{x \in \mathbb{R} : a < x \leq b\}$, respectively.*

3.4 Exploring the Properties of the Real Numbers

First, we show that one could swap the supremum property for the infimum property:

Proposition 3.23. *Every non-empty set of real numbers that is bounded below has an infimum.*

Proof. Let S be a non-empty set that is bounded below. Denote the set of lower bounds for S as L. Since we assume that S is bounded below $L \neq \emptyset$. By the definition of a lower bound, the set L is bounded above by any member of S. Therefore, L is a non-empty set which is bounded above. By the supremum property, L has a least upper bound $\sup L$.

Now every $x \in S$ is an upper bound for L. Therefore, by definition $\sup L \leq x$ for any $x \in S$. So, $\sup L$ is a lower bound for S and, by definition, every lower bound of S is less than or equal to $\sup L$, which is the definition of an infimum. Hence, $\inf S = \sup L$. □

Next, we show that there are arbitrarily large natural numbers.

Theorem 3.24 (Archimedean Property)**.** *If $a \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $n > a$.*

Proof. Let's prove this by contradiction.

Assume that a is an upper bound for $\mathbb{N}$. We know that $\mathbb{N} \neq \emptyset$ since $1 \in \mathbb{N}$. So, therefore, $\mathbb{N}$ is a non-empty subset of $\mathbb{R}$ with an upper bound. By the supremum property, $s \doteq \sup \mathbb{N}$ exists.

By the definition of the supremum, $s - 1 \doteq s + (-1) < s$ is not a supremum (it's an exercise on the example sheets to show that $s - 1 < s$, which is equivalent to showing $1 > 0$, from the field axioms). So, we can find $n \in \mathbb{N}$ such that $s - 1 < n$, which, if we add 1 to both sides, gives $s < n + 1$. $n + 1$ is by definition a natural number and, therefore, shows that s is not an upper bound. This is a contradiction and, therefore, $\mathbb{N}$ is not bounded above. □

Proposition 3.25 (Characterisations of sup and inf)**.** *Let $S \subset \mathbb{R}$ and $s \in \mathbb{R}$. The real number $s = \sup S$ if and only if*

1. *$x \leq s$ for all $x \in S$,*
2. *for all $\varepsilon > 0$ there exists $x \in S$ such that $s - \varepsilon < x$.*

The real number $s = \inf S$ if and only if

1. *$s \leq x$ for all $x \in S$,*
2. *for all $\varepsilon > 0$ there exists $x \in S$ such that $x < s + \varepsilon$.*

Proof. Let's prove this 'if and only if' statement carefully.

($\Rightarrow$) Assume $s = \sup S$. Then 1 follows immediately by the definition of the supremum.

Let $\varepsilon > 0$. Then, $s - \varepsilon < s$. Hence, $s - \varepsilon$ cannot be an upper bound for S, i.e. there exists $x \in S$ such that $x > s - \varepsilon$.

($\Leftarrow$) Assume that 1 and 2 are true. 1 implies directly that s is an upper bound of S. Now, let $s' < s$ and set $\varepsilon = s - s' > 0$. By 2 there exists $x \in S$ such that $x \geq s - \frac{\varepsilon}{2} > s'$. Consequently, s' is not an upper bound for S. This means that s is the least upper bound for S, i.e. $s = \sup S$. $\square$

Proposition 3.26 (Bernoulli's Inequality)**.** *Let $x \geq -1$ and $n \in \mathbb{N} \setminus \{0\}$. Then,*

$$(1+x)^n \geq 1 + nx. \tag{3}$$

Proof. Let prove this by induction. So for the base case, let $n = 1$, we have $(1+x)^1 \geq 1 + 1x$, which is true.

Assume that for some $n \in \mathbb{N}$, the inequality (3) holds. We now want to prove that $(1+x)^{n+1} \geq 1 + (n+1)x$. Observe that

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n \geq (1+x)(1+nx) \\ &= 1 + (n+1)x + nx^2 \geq 1 + (n+1)x + 0 = 1 + (n+1)x,\end{aligned}$$

where in the first inequality we have used the fact that $1 + x \geq 0$. Thus if $(1+x)^n \geq 1 + nx$, then also $(1+x)^{n+1} \geq 1 + (n+1)x$ (for $x \geq -1$). Hence, by the principle of mathematical induction, the inequality (3) holds for every $n \in \mathbb{N}$. $\square$

Remark 3.27. For $x < -1$ the inequality (3) is false in general, take $x = -4$ and $n = 3 \in \mathbb{N}$. Then

$$(1+x)^n = (1-4)^3 = (-3)^3 = -27 \qquad 1 + nx = 1 + (3)(-4) = -11.$$

Let's do some examples using the supremum property.

Example 3.28. *$A = \{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\}$. We will show that $\sup A = \max A = 1$ and $\inf A = 0$.*

Clearly, $0 \leq \frac{1}{n}$ for all $n \in \mathbb{N} \setminus \{0\}$. So, 0 is a lower bound for A. Now, let $\varepsilon > 0$. By the Archimedean Principle, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $n > \frac{1}{\varepsilon}$. Hence, $\frac{1}{n} < \varepsilon$. This shows that $\varepsilon > 0$ is not a lower bound for A. Consequently, $0 = \inf A$. Moreover, $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N} \setminus \{0\}$ and $1 = \frac{1}{1} \in A$. So 1 is an upper bound for A in A so it is a maximum.

Example 3.29. *$A = \{\frac{n^2+2n}{n^2+2} : n \in \mathbb{N}\}$. We will show that $\inf A = \min A = 0$ and $\sup A = \frac{15}{11}$.*

First notice that $\frac{n^2+2n}{n^2+2} \geq 0$ for all $n \in \mathbb{N}$. So, 0 is a lower bound and $0 \in A$ (for $n = 0$). Hence, $\min A = \inf A = 1$.

Now for the sup*, let's write out the first few the elements of A as enumerated by the naturals:*

$$A = \left\{0, 1, \frac{4}{3}, \frac{15}{11}, \frac{4}{3}, \frac{35}{27}, \dots\right\}.$$

It looks like the maximum of A is achieved when $n = 3$ and decreases thereafter. Note that we can write

$$\frac{n^2 + 2n}{n^2 + 2} = 1 + \frac{2(n-1)}{n^2 + 2}.$$

Call $a_n = \frac{n-1}{n^2+2}$. So, if we can show that $a_{n+1} \le a_n$ for $n \ge 3$ then we know the maximum had to be achieved before or at $n = 3$ (this is a property known as monotonicity, which we will dicuss more later). If we compute

$$a_{n+1} - a_n = \frac{(n+1) - 1}{(n+1)^2 + 2} - \frac{n-1}{n^2+2} = -\frac{(n-3)(n+1) + 2n}{((n+1)^2 + 2)(n^2 + 2)} \le 0$$

for $n \ge 3$. So for $n \ge 3$, a_n decreases and, therefore, $a_3 \ge a_n$ for all $n \ge 3$. Hence, for all $n \ge 3$ it holds

$$\frac{n^2 + 2n}{n^2 + 2} = 1 + \frac{2(n-1)}{n^2 + 2} \le 1 + \frac{2(3-1)}{3^2 + 2} = \frac{15}{11}.$$

Therefore $\max A = \sup A = \max\{1, \frac{8}{6}, \frac{15}{11}\} = \frac{15}{11}$.

Theorem 3.30. *The rational numbers do not satisfy the supremum property.*

Proof. We need a nonempty set $A \subseteq \mathbb{Q}$ bounded from above but for which there exists no supremum in $\mathbb{Q}$. Define

$$A \doteq \left\{x \in \mathbb{Q} :\ 0 < x \text{ and } x^2 < 2\right\}.$$

Then A is nonempty, since $1 \in A$. Moreover, A is bounded from above, since 2 is an upper bound.

Assume for a contradiction that there exists $s \in \mathbb{Q}$ such that $s = \sup A$. It cannot be $s \le 0$, since $1 \in A$ and $1 > 0$. Hence, $s > 0$. We will split into two cases and show that $s^2 \not< 2$ and $s^2 \not> 2$ and, therefore, $s^2 = 2$, which is a contradiction to proposition 3.15.

So, let's prove that it cannot be that $s^2 < 2$. This will contradict that s is an upper bound as follows. Take the rational number $s + \frac{1}{n} \in \mathbb{Q}$ with $n \in \mathbb{N}_1$. Then

$$\left(s + \frac{1}{n}\right)^2 = s^2 + \frac{1}{n^2} + \frac{2s}{n} \le s^2 + \frac{1}{n} + \frac{2s}{n} = s^2 + \frac{2s+1}{n},$$

since $n \le n^2$. For the contradiction, we want the RHS to be less than 2, since then $s + \frac{1}{n} > s$ and is a member of A, so s is not an upper bound. So, we require an $n \in \mathbb{N}$ such that

$$s^2 + \frac{2s+1}{n} < 2 \Longleftrightarrow n > \frac{2s+1}{2 - s^2}. \tag{4}$$

Using the Archimedean property we can find $n \in \mathbb{N}_1$ such that the inequality in (4) is satisfied. Hence, $s + \frac{1}{n}$ belongs to A, which contradicts the fact that s is an upper bound of A.

Now, let's prove that it cannot be that $s^2 > 2$. We will show that there is a smaller upper bound than s, which is a contradiction. Consider $s - \frac{1}{n}$, for $n \in \mathbb{N}_1$. Let's prove that $s - \frac{1}{n}$ is an upper bound of A. We note

$$\left(s - \frac{1}{n}\right)^2 = s^2 - \frac{2s}{n} + \frac{1}{n^2} > s^2 - \frac{2s}{n} > 2$$

by picking n sufficiently large via the Archimedean property, i.e.

$$s^2 - \frac{2s}{n} > 2 \implies 2n < s^2 n - 2s \implies n > \frac{2s}{s^2 - 2}.$$

Similarly, $s - \frac{1}{n} > 0$. So, if $x \in A$, then $x > 0$ and $x^2 < 2 < \left(s - \frac{1}{n}\right)^2$ by the above argument. Hence,

$$0 < \left(s - \frac{1}{n}\right)^2 - x^2 = \left[\left(s - \frac{1}{n}\right) + x\right] \cdot \left[\left(s - \frac{1}{n}\right) - x\right]. \tag{5}$$

Since $s - \frac{1}{n} > 0$ and $x > 0$, we have that $\left(s - \frac{1}{n}\right) + x \ > 0$. It follows from the inequality (5) that $0 < \left(s - \frac{1}{n}\right) - x$, that is $x < \left(s - \frac{1}{n}\right)$. Hence, $s - \frac{1}{n}$ is an upper bound of A, which contradicts the fact that s is the least upper bound of A. Hence, $s^2 > 2$ is also excluded, Thus, $s^2 = 2$, which is a contradiction by proposition 3.15. ☐

Theorem 3.31. *The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is nonempty.*

Proof. Take

$$A \doteq \left\{x \in \mathbb{R} : 0 < x \text{ and } x^2 < 2\right\}.$$

Exactly as in the previous proof, we have that A is nonempty and bounded from above. Hence, by the supremum property, there exists $s \in \mathbb{R}$ such that $s = \sup A$. It follows as in the previous proof that $s^2 = 2$, and so s belongs to $\mathbb{R} \setminus \mathbb{Q}$. ☐

The number s is denoted by $\sqrt{2}$ and called **square root** of 2.

Remark 3.32. Similarly, for every $n \in \mathbb{N}$ with n even and every $y \in \mathbb{R}$ with $y \geq 0$, we can show that there exists a unique $x \in \mathbb{R}$ with $x \geq 0$ such that $x^n = y$. On the other hand, for every $n \in \mathbb{N}$ with n odd and every $y \in \mathbb{R}$, we can show that there exists a unique $x \in \mathbb{R}$ such that $x^n = y$. The number x is denoted $\sqrt[n]{y}$ and called **n-th root** of y.

Theorem 3.33 (Density of the rationals)**.** *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We want to find a $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$ such that

$$a < \frac{p}{q} < b \qquad \Leftrightarrow \qquad qa < p < qb.$$

Intuitively, to fit an integer in (qa, qb) we would like $qb - qa = q(b - a) > 1$. So, choose $q \in \mathbb{N}$ such that $q \geq \frac{2}{b-a} > \frac{1}{b-a}$, which can be done by the Archimedean property. Now, let $m \in \mathbb{Z}$ be the largest integer less or equal than $q \cdot a$ (See Problem Sheet). Set $p = m + 1 > qa$. Also, $p \leq 1 + qa$ since $m \leq qa$. From this we deduce that $p \leq 1 + qa < qb$ since we have chosen q such that $q(b - a) > 1$. ☐

Corollary 3.34 (Density of the irrationals)**.** *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.*

Proof. Since $a < b$, we have that $\sqrt{2}a < \sqrt{2}b$. By the density of the rationals, there exists $r \in \mathbb{Q}$ such that $\sqrt{2}a < r < \sqrt{2}b$. Without loss of generality, we may assume that $r \neq 0$ since if $r = 0$ then by the density of the rationals there is another rational $\tilde{r}$ such that $0 < \tilde{r} < \sqrt{2}b$. Hence, $a < \frac{r}{\sqrt{2}} < b$. Since $\frac{r}{\sqrt{2}}$ is irrational, the result is proved. $\square$

4 Sequences of Real Numbers

We want to talk about sequences of real numbers; intuitively infinite ordered lists of reals $1, \sqrt{2}, \frac{1}{2}, 10, \ldots$. We want to discuss when such lists tend asymptotically to some value, this is known as convergence. If we have an idea of what the list is tending toward then we might like to characterise convergence in terms of distance away from the asymptotic value. This leads us to a notion of distance between real numbers which can be defined the absolute value.

Definition 4.1 (Absolute Value/Distance)**.** *For $x \in \mathbb{R}$ we define*

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

We define the distance $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ between two real numbers $a, b \in \mathbb{R}$ as

$$d(a, b) = |b - a|.$$

Remark 4.2. This distance map is an example of a metric and $(\mathbb{R}, |\cdot|)$ is an example of a metric space. We will return to these notions shortly.

We briefly recall the well-known properties of the absolute value.

Theorem 4.3. *Let $x, y, z \in \mathbb{R}$. Then:*

1. *$|x| \geq 0$ for all $x \in \mathbb{R}$ and $|x| = 0$ if and only if $x = 0$,*
2. *$|-x| = |x|$ for all $x \in \mathbb{R}$,*
3. *for $y \geq 0$ and $x \in \mathbb{R}$ it holds $|x| \leq y$ if and only if $-y \leq x \leq y$,*
4. *$-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$,*
5. *$|xy| = |x||y|$ for all $x, y \in \mathbb{R}$,*
6. *$|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.*

Proof. Let's prove property 6 since it is arguably the most important inequality in analysis. (The rest is an exercise).

Suppose $b \leq a$ ($a \leq b$ follows from symmetry). We proceed in steps:

1. Suppose $a \geq b \geq 0$ then $|a + b| = a + b = |a| + |b|$ by the definition of the absolute value.
2. Suppose $0 \geq a \geq b$ then $|a + b| = -(a + b) = -|a| - |b|$ again by the definition of the absolute value.
3. Suppose $a \geq 0 \geq b$, then $|a + b| \leq \max(\{|a|, |b|\})$. Now clearly, $\max(\{|a|, |b|\}) \leq |a| + |b|$. So, $|a + b| \leq |a| + |b|$.

The slick way to do this is the following. We have

$$0 \geq |a + b|^2 = (a + b)^2 = a^2 + b^2 + 2ab = |a|^2 + |b|^2 + 2ab \leq |a|^2 + |b|^2 + 2|a||b| = (|a| + |b|)^2.$$

So,

$$|a + b|^2 \leq (|a| + |b|)^2 \implies |a + b| \leq |a| + |b|,$$

since $|a + b| \geq 0$ and $|a| + |b| \geq 0$. $\square$

4.1 Definition and Convergence

Definition 4.4. *A* ***sequence*** *of real numbers is a function*

$$a : \mathbb{N} \to \mathbb{R} \qquad n \mapsto a(n)$$

from the natural numbers to the real numbers.

To ease notation we write $a(n) = a_n \in \mathbb{R}$ for $n = 1, 2, \ldots$, usually we denote the sequence a by the symbol (a_n) or $(x_n)_{n\in\mathbb{N}}$ or $x_1, x_2, \ldots$.

Definition 4.5. *Let $(x_n)_n$ be a sequence of real numbers and let $l \in \mathbb{R}$. We say that $(x_n)_n$* ***converges*** *to $l \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ such that for all $n \geq N$*

$$|x_n - l| < \varepsilon.$$

In this case, we say that l is the ***limit*** *of (x_n) and we write*

$$\lim_{n\to\infty} x_n = l \text{ or } x_n \to l.$$

Example 4.6. *Define $(x_n)_n$ by $x_n = \frac{1}{n}$. We claim that $x_n \to 0$. Let $\varepsilon > 0$ be arbitrary. By the Archimedean Property $\mathbb{N}$ has no upper bound. Therefore, there exists an $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then, for all $n \geq N$, we have*

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Example 4.7. *Let $x > 0$ and $x_n = \sqrt[n]{x}$. We claim that $x_n \to 1$. For the proof, we distinguish two cases: $x > 1$ and $x \leq 1$.*

Suppose $x > 1$. Then, we claim $x_n = x^{\frac{1}{n}} > 1$ for all $n \in \mathbb{N}$. Suppose otherwise, then for some $n \in \mathbb{N}$, $0 \leq x_n \leq 1$. So,

$$x_n \leq 1 \implies x_n^2 \leq x_n \leq 1 \implies \cdots \implies x_n^n \leq \cdots \leq x_n \leq 1,$$

since if $c \geq 0$ and $a \leq b$ then $ac \leq bc$, which is a contradiction. Now write $x_n = 1 + a_n$ for some $a_n > 0$. By Bernoulli's inequality $x = x_n^n = (1 + a_n)^n \geq 1 + na_n$. Hence, $0 < a_n \leq \frac{x-1}{n}$. Now, let $\varepsilon > 0$. By the Archimedean Principle there exists $N \in \mathbb{N}$ such that $N > \frac{x-1}{\varepsilon}$. Then for all $n \geq N$ it holds

$$|x_n - 1| = |1 + a_n - 1| = a_n \leq \frac{x-1}{n} \leq \frac{x-1}{N} < \varepsilon.$$

Suppose $0 < x < 1$. It then holds that $0 < x_n < 1$ for all n (again, one can argue by contradiction). We write $x_n = \frac{1}{1+a_n}$ where $a_n > 0$. Bernoulli's inequality implies that

$$x = \frac{1}{(1 + a_n)^n} \leq \frac{1}{1 + na_n}.$$

Hence, $a_n \leq \frac{1-x}{xn}$. Now, let $\varepsilon > 0$. By the Archimedean Principle let $N \in \mathbb{N}$ such that $N > \frac{1-x}{x\varepsilon}$. Then it holds for all $n \geq N$ that

$$|x_n - 1| = \left|\frac{a_n}{1 + a_n}\right| = \frac{a_n}{1 + a_n} \leq a_n \leq \frac{1-x}{xn} \leq \frac{1-x}{xN} < \varepsilon.$$

Example 4.8. $x_n = (-1)^n$. *This sequence does not converge. Assume otherwise, i.e. that* $x_n \to l$. *It suffices to find an* $\varepsilon > 0$, *such that for every* $N \in \mathbb{N}$, *there exists* $n \geq N$ *such that* $|x_n - l| \geq \varepsilon$. *It holds for all odd* n,

$$|x_n - l| = |1 + l|,$$

and all even n,

$$|x_n - l| = |1 - l|.$$

Let $\varepsilon = \frac{1}{2}\max\{|1-l|, |1+l|\} > 0$. *Then if* $\max\{|1-l|, |1+l|\} = |1-l|$, *we have* $|1-l| \geq 2\varepsilon$. *So for even* n

$$|x_n - l| \geq 2\varepsilon > \varepsilon.$$

If $\max\{|1-l|, |1+l|\} = |1+l|$, *we have* $|1+l| \geq 2\varepsilon$. *So for odd* n

$$|x_n - l| \geq 2\epsilon > \varepsilon.$$

Definition 4.9. *We say that a sequence* (x_n) *of real numbers* ***diverges to plus infinity*** *if for any real number* $M \in \mathbb{R}$ *there exists* $N = N(M) \in \mathbb{N}$ *such that for all* $n \geq N$ *we have*

$$x_n > M. \tag{6}$$

In this case we write

$$\lim_{n\to\infty} x_n = \infty \quad or \quad x_n \to \infty.$$

Similarly we say that a sequence (x_n) *of real numbers* ***diverges to minus infinity*** *if for any real number* $M \in \mathbb{R}$ *there exists* $N = N(M) \in \mathbb{N}$ *such that for all* $n \geq N$ *we have*

$$x_n < -M. \tag{7}$$

In this case, we write

$$\lim_{n\to\infty} x_n = -\infty \quad or \quad x_n \to -\infty.$$

If (x_n) *does not converge and does not diverge to plus infinity or to minus infinity, we say that it* ***oscillates****.*

Remark 4.10. Some authors call a sequence divergent if it does not converge and speak of sequences which are converging to $\pm\infty$. This is a matter of convention.

4.2 Classification and Properties

Definition 4.11. *We say that a sequence* (x_n) *of real numbers is*

(i) ***bounded from above*** *if there exists* $M \in \mathbb{R}$ *such that* $x_n \leq M$ *for all* $n \in \mathbb{N}$;

(ii) ***bounded from below*** *if there exists* $M \in \mathbb{R}$ *such that* $x_n \geq M$ *for all* $n \in \mathbb{N}$;

(iii) ***bounded*** *if it is bounded from above and from below.*

(iv) ***eventually bounded*** *is* $\exists M \in \mathbb{R}$ *and* $\exists N \in \mathbb{N}$ *such that for all* $n \geq N$ *we have* $|x_n| \leq M$.

Remark 4.12. We can touch base with boundedness for sets. A sequence $(x_n)_{n\in\mathbb{N}}$ is bounded if and only if the set of its elements

$$S = \{x_n : n \in \mathbb{N}\}$$

is a bounded set.

Lemma 4.13. *Every eventually bounded sequence is bounded.*

Proof. Let $(a_n)_n$ be an eventually bounded sequence. Therefore, there exists a $C \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|a_n| \leq C$ for all $n \geq N$. If we now define $M = \max\{|a_1|, ..., |a_{N-1}|, C\}$, then $|a_n| \leq M$ for all n. □

Theorem 4.14 (Properties of Sequences)**.** *Let $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ be a sequences of real numbers. The following statements are true.*

(i) If the limit of (a_n) exists, it is unique.

(ii) If the limit of $(a_n)_n$ exists then $(a_n)_n$ is bounded.

(iii) Suppose $(a_n)_n$ is bounded and $b_n \to 0$. Then $a_n b_n \to 0$.

(iv) If $(a_n), (b_n)$ converge to a and b, respectively. Then the limit of $(a_n + b_n)_n$ exists and is equal to $a + b$.

(v) If $(a_n), (b_n)$ converge to a and b, respectively. Then the limit of $a_n b_n$ exists and is equal to ab.

(vi) If $(a_n), (b_n)$ converge to a and $b \neq 0$, respectively, and if $b_n \neq 0$ for all $n \in \mathbb{N}$. Then the limit of $({}^{a_n}/_{b_n})_n$ exists and is equal to ${}^{a}/_{b}$.

(vii) if $a_n \to a$ and $\lambda \in \mathbb{R}$. Then, $\lambda a_n \to \lambda a$.

(viii) Suppose $(a_n), (b_n)$ converge to a and b, respectively. If $a_n \geq b_n$ for all $n \geq N$ for some $N \in \mathbb{N}$ then $a \geq b$.

(ix) Suppose $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = a \in \mathbb{R}$, then $\lim\limits_{n\to\infty} b_n = a$.

Proof. (i) Let $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = l$. Let $\varepsilon > 0$ be arbitrary. By definition we can choose N such that

$$|a_N - a| < \frac{\varepsilon}{2}, \qquad |a_N - l| < \frac{\varepsilon}{2}.$$

Therefore, by the triangle inequality,

$$d(a, l) = |a - l| = |a - l + a_N - a_N| \leq |a_N - l| + |a_N - a| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the distance between a and l can be made arbitrarily small. Therefore, $a = l$.

(ii) Suppose $(a_n)_n$ is convergent with limit a. Take $\varepsilon = 1$. Then by the definition of convergence there exists $N = N(1)$ such that for all $n \geq N$, we have $|a_n - a| \leq 1$. By the triangle inequality,

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a|$$

for all $n \geq N$. So $(a_n)_n$ is eventually bounded. However, lemma 4.13, every eventually bounded sequence is bounded.

(iii) Since $(a_n)_n$ is bounded, we know that we can find $C > 0$ such that $|a_n| \leq C$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then, since $b_n \to 0$, we can find $N = N(\varepsilon) \in \mathbb{N}$ such that, for all $n \geq N$, $|b_n| \leq \frac{\varepsilon}{C}$. So, for all $n \geq N$

$$|a_n b_n - 0| = |a_n b_n| \leq C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

(iv) Let $\varepsilon > 0$. By the definition $\exists N_1(\varepsilon), N_2(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N_1$ $|a_n - a| < \frac{\varepsilon}{2}$ and for all $n \geq N_2$ $|b_n - b| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, we have (by the triangle inequality),

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| \leq \varepsilon.$$

(v) By adding and subtracting ab_n, we can write $a_n b_n - ab = (a_n - a)b_n + (b_n - b)a$. From (iv) we know that $a_n - a \to a - a = 0$ and $b_n - b \to 0$. Now from (ii) $(a_n)_n$ and $(b_n)_n$ are bounded. Combining with (iii), we have that $(a_n - a)b_n \to 0$ and $(b_n - b)a \to 0$. Therefore, using (iv) again, we have

$$a_n b_n - ab = (a_n - a)b_n + (b_n - b)a \to 0 \implies a_n b_n = (a_n b_n - ab) + ab \to 0 + ab = ab.$$

(vi) We can write $1/b_n - \frac{1}{b} = \frac{b - b_n}{b b_n}$. Since $b_n \to b$, for $\varepsilon = |b|/2 > 0$ there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|b - b_n| \leq \frac{|b|}{2}$. So, by the triangle inequality,

$$|b| = |b_n - b| + |b_n| \leq \frac{|b|}{2} + |b_n| \implies |b_n| \geq \frac{|b|}{2} \implies \frac{2}{|b|^2} \geq \frac{1}{|b_n|}\frac{1}{|b|} = \frac{1}{|bb_n|}.$$

Therefore, $(1/bb_n)_n$ is eventually bounded. Hence, by lemma 4.13, $(1/bb_n)_n$ is bounded. So, by property (iii) $1/b_n - 1/b = b - b_n / bb_n \to 0$ and, moreover, $1/b_n \to 1/b$. Using (v), then gives $a_n/b_n \to a/b$.

(vii) Follows from (v) with $b_n = \lambda$ for all n.

(viii) Suppose otherwise, i.e. that $b > a$. Take $\varepsilon = \frac{b-a}{2}$. Then by the definition fo the limit $\exists N \in \mathbb{N}$ such that, for all $n \geq N$,

$$a - \varepsilon < a_n < a + \varepsilon = \frac{a+b}{2}, \qquad b + \varepsilon > b_n > b - \varepsilon = \frac{a+b}{2}.$$

Therefore,

$$a_n < \frac{a+b}{2} < b_n$$

for all $n \geq N$, which is a contradiction.

(ix) Let $\varepsilon > 0$. Then by the definition of convergence we can find $N(\varepsilon)$ such that, for all $n \geq N$, we have $|a_n - a| < \varepsilon$ and $|c_n - a| < \varepsilon$ (this is done as above). So, for all $n \geq N$,

$$a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon \implies |b_n - a| < \varepsilon.$$

$\square$

Remark 4.15. The sequence $x_n = (-1)^n$ shows that the reverse of (ii) is not true. We will see later that the reverse is true for a subsequence.

Remark 4.16. You can prove (iii) from (ix).

Example 4.17. *Let $x_n = \frac{4n+3}{2n}$. We use a property from above to show*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{4+\frac{3}{n}}{2} = \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{3}{2}\frac{1}{n} = 2 + \frac{3}{2}\lim_{n\to\infty}\frac{1}{n} = 2 + 0 = 2.$$

Example 4.18. *Let $x_n = \frac{n^2+n}{n^2+2n-1}$. We write*

$$x_n = \frac{1+\frac{1}{n}}{1+\frac{2}{n}-\frac{1}{n^2}}.$$

Using the theorem above we find that

$$\lim_{n\to\infty} x_n = \frac{\lim_{n\to\infty} 1+\frac{1}{n}}{\lim_{n\to\infty} 1+\frac{2}{n}-\frac{1}{n^2}} = \frac{1+\lim_{n\to\infty}\frac{1}{n}}{1+\lim_{n\to\infty}\frac{2}{n}+\lim_{n\to\infty}\frac{1}{n^2}} = \frac{1}{1} = 1.$$

4.2.1 Monotone Sequences

Definition 4.19. *We say that a sequence (x_n) of real numbers is*

(i) ***increasing*** *if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$;*

(ii) ***decreasing*** *if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$;*

(iii) ***strictly increasing*** *if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$;*

(iv) ***strictly decreasing*** *if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.*

A sequence (x_n) of real numbers is a ***(strictly) monotone*** *sequence if it is (strictly) increasing or decreasing.*

The important property of monotone sequences is that they converge if and only if they are bounded. Essentially, they have nowhere to escape to if they are bounded.

Theorem 4.20. *Let $(a_n)_n$ be a bounded, monotone sequence of real numbers and let*

$$A = \{a_n : n \in \mathbb{N}\}.$$

(i) If $(a_n)_n$ is increasing, then the limit of $(a_n)_n$ exists and is equal to $\sup A$.

(ii) If $(a_n)_n$ is decreasing, then the limit of $(a_n)_n$ exists and is equal to $\inf A$.

Proof. We prove only (i). Let $s \doteq \sup A$. Note that since $(a_n)_n$ is bounded by assumption, the set A is bounded and hence, by the supremum property, its supremum exists. Fix $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound of A, there exists $x \in A$ such that $s - \varepsilon < x \leq s$. This means there must be a $N \in \mathbb{N}$ such that $s - \varepsilon < a_N \leq s$. Since $(a_n)_n$ is increasing, for all $n \geq N$, we have that

$$s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon.$$

Consequently, for all $n \geq N$ it holds $|a_n - s| < \varepsilon$. □

Example 4.21. *Let $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ the Fibonacci sequence:*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

Moreover, let $b_n = \frac{a_{2n+1}}{a_{2n}}$:

$$1, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, ...$$

We show that (b_n) converges by showing boundedness and monotonicity of (b_n).

Boundedness can be shown in the following way. We start by noting that $b_n \geq 0$ for all n. For $n \geq 2$,

$$b_n = \frac{a_{2n+1}}{a_{2n}} = \frac{a_{2n} + a_{2n-1}}{a_{2n}} = 1 + \frac{a_{2n-1}}{a_{2n}} = 1 + \frac{a_{2n-1}}{a_{2n-1} + a_{2n-2}} = 1 + \frac{1}{1 + \frac{1}{b_{n-1}}} \leq 2,$$

since $b_n > 0$. If $n = 0, 1$ then $b_n \leq 2$ also. Therefore, $0 \leq b_n \leq 2$ for all $n \in \mathbb{N}$.

For monotonicity, we note the following equalities

$$b_{n+1} - b_n = 1 + \frac{1}{1 + (b_n)^{-1}} - 1 - \frac{1}{1 + (b_{n-1})^{-1}} = \frac{b_n}{1 + b_n} - \frac{b_{n-1}}{1 + b_{n-1}} = \frac{b_n - b_{n-1}}{(1 + b_n)(1 + b_{n-1})}.$$

Hence, the sign of $b_{n+1} - b_n$ is the same as the one of $b_n - b_{n-1}$. We notice that

$$b_2 - b_1 = \frac{a_5}{a_4} - \frac{a_3}{a_2} = \frac{8}{5} - \frac{3}{2} = \frac{16}{10} - \frac{15}{10} = \frac{1}{10} > 0.$$

This shows inductively that (b_n) is monotonically increasing.

Thus, the previous theorem implies the convergence of (b_n). We call the limit L. Now, observe —using the rules for computing limits— that

$$b_n \to L, \qquad b_n = \frac{b_{n-1}}{1 + b_{n-1}} \to 1 + \frac{L}{1 + L}.$$

This implies that $L = 1 + \frac{L}{1+L}$ which can be rewritten as $L^2 - L - 1 = 0$. The two solutions of this equation are $\frac{1 \pm \sqrt{5}}{2}$. As L is clearly nonnegative, it follows that $L = \frac{1+\sqrt{5}}{2}$.

4.3 Subsequences and Bolzano-Weierstrass

Definition 4.22. *Given a sequence $(a_n)_n$ of real numbers and a strictly increasing sequence of natural numbers $(n_k)_k$ (that is $n_1 < n_2 < \ldots$) the sequence $(a_{n_k}) = (a_{n_k})_k$ is called a **subsequence** of (a_n).*

Theorem 4.23. *If a sequence of real numbers (x_n) is not bounded from above then there exists a subsequence (x_{n_k}) such that $x_{n_k} \to \infty$.*

Proof. Let $M_1 = 1$. As (x_n) is not bounded from above there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} > M_1$.
Let $M_2 = \max\{2, x_1, \ldots, x_{n_1}\}$. Again, there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} > M_2 \geq 2$. Note

that by construction $n_2 > n_1$.
Now, assume that $n_1 < \cdots < n_k$ are already constructed. Define $M_{k+1} = \max\{k + 1, x_1, \ldots, x_{n_k}\}$. As (x_n) is not bounded from above, there exists $n_{k+1} \in \mathbb{N}$ such that $x_{n_{k+1}} > M_{k+1}$. Again, note that by construction $n_{k+1} > n_k$. Hence (x_{n_k}) is a subsequence of (x_n) such that $x_{n_k} \geq k$. This implies that $x_{n_k} \to \infty$. □

Remark 4.24. If a sequence is not bounded from below then there exists a subsequence which diverges to $-\infty$.

Example 4.25. *Note that we can find a subsequence that diverges to ∞. It is not necessarily the sequence that does. For example,*

$$x_n = (-1)^n n$$

is unbounded but oscillates and, therefore, does not diverge to $\pm\infty$.

Next, we prove one of the most important theorems of real analysis which will have far-reaching consequences.

Theorem 4.26 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $(a_n)_n$ be a bounded sequence in $\mathbb{R}$. We define a **highpoint** of $(a_n)_n$ as a $N \in \mathbb{N}$ such that $a_N > a_m$ for all $m > N$.

Suppose now that the sequence $(a_n)_n$ has infinitely many highpoints $N_1 < N_2 < N_3 < ....$ Then, the subsequence $(a_{N_k})_k$ is monotonically decreasing. This is a bounded monotone sequence and, therefore, by theorem 4.20, converges.

Suppose that the sequence $(a_n)_n$ has finitely many highpoints $N_1 < ... < N_k$. Set $M = N_k + 1$ and consider the subsequence $(a_n)_{n=M}^{\infty}$. We now construct a monotonically increasing sequence as follows. We know that $n_1 = M$ is not a highpoint. So, there must exist a $n_2 > n_1$ such that $a_{n_1} \leq a_{n_2}$. However, n_2 cannot be a highpoint either, so there exists a $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. By induction, we construct a subsequence $(a_{n_k})_k$ which is monotonically increasing. Therefore, by theorem 4.20, it must converge. □

4.4 Cauchy Sequences, Limit Superior and Limit Inferior

Definition 4.27. *A sequence $(a_n)_n$ of real numbers is said to be a Cauchy sequence is $\forall \varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that for all $n, m \geq N$, we have $d(a_n, a_m) = |a_n - a_m| < \varepsilon$.*

What does this definition say intuitively? It says that terms in your sequence get closer and closer together, arbitrarily so! You would expect that every convergent sequence is Cauchy. However, what about the other way around, i.e. do Cauchy sequences converge? This leads to a very important property in analysis: completeness. We will visit this in further abstraction in due course but we will first study it in $\mathbb{R}$.

Lemma 4.28. *Every convergent sequence in $\mathbb{R}$ is Cauchy.*

Proof. Let $(a_n)_n$ be a convergent sequence in $\mathbb{R}$ with limit $a \in \mathbb{R}$. Let $\varepsilon > 0$. Pick N such that for all $n \geq N$

$$|a_n - a| < \frac{\varepsilon}{2}.$$

So, by the triangle inequality, we have that for all $n, m \geq N$,

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Lemma 4.29. *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Let (a_n) be a Cauchy sequence. It follows from the definition that there exists N such that for all $n, m \geq N$ we have that $|a_n - a_m| < 1$. In particular, for $n = N$, the reverse triangle inequality,

$$|x - y| \geq ||x| - |y||,$$

implies that for all $m \geq N$,

$$1 > |a_N - a_m| \geq ||a_N| - |a_m|| \implies |a_m| < 1 + |a_N|.$$

Thus, the sequence is eventually bounded and, therefore, by lemma 4.13, it is bounded. □

Theorem 4.30 ($\mathbb{R}$ is complete)**.** *Every Cauchy sequence in $\mathbb{R}$ converges.*

Proof. Let $(a_n)_n$ be the Cauchy sequence in $\mathbb{R}$. By Lemma 4.29 we get that $(a_n)_n$ is bounded, and therefore, using Bolzano-Weierstrass theorem, we conclude that $(a_n)_n$ has a convergent subsequence $(a_{n_k})_k \to l$, say. We will now show that also $\lim_{n\to\infty} a_n = l$.

To this end, let $\varepsilon > 0$. Then from the definitions of convergence and Cauchy sequence follows that

$$\begin{aligned} &\exists N_1 \quad \text{such that} \quad \forall n_k \geq N_1 \quad\implies |a_{n_k} - l| < \tfrac{\varepsilon}{2} \\ &\exists N_2 \quad \text{such that} \quad \forall m, n \geq N_2 \implies |a_n - a_m| < \tfrac{\varepsilon}{2}. \end{aligned}$$

Define $N = \max\{N_1, N_2\}$. Then, for $n \geq N$ we have that

$$|a_n - l| \leq |a_n - a_{n_k}| + |a_{n_k} - l| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

where $n_k \geq N$, finishes the proof. □

Note that within the proof of Theorem 4.30 we have shown the following fact.

Proposition 4.31. *Let $(a_n)_n$ be a Cauchy sequence which has a convergent subsequence $(a_{n_k})_k$ with limit $\bar{a}$. Then $(a_n)_n$ converges to $\bar{a}$.*

This demonstrates to check whether or not a sequence converges, it suffices to show that it is Cauchy. In practice this is easier to do than showing a sequence converges directly, as we do not need to guess a limit to begin with - we can just take the difference of two general terms in the sequence and see if we can make it small. Once we know a sequence is Cauchy and so converges, how do we find out what the limit is? This turns out to not be easy. However, there are two ‘natural’ quantities we can look at to find the limit.

Definition 4.32 (Limit Superior/Inferior)**.** *Let $(a_n)_n$ be a bounded sequence. We define the limit superior and limit inferior as*

$$\limsup_{n\to\infty} a_n \doteq \lim_{n\to\infty}\Big(\sup_{m\geq n} a_m\Big),$$
$$\liminf_{n\to\infty} a_n \doteq \lim_{n\to\infty}\Big(\inf_{m\geq n} a_m\Big),$$

respectively.

So these quantities look rather scary; you have a limit and a supremum/infimum appearing. Let's try to make them less obscure. Suppose you have a sequence $(a_n)_n$. To consider the lim sup you construct a new sequence $(b_n)_n$ with

$$b_1 \doteq \sup_{m\geq 1} a_m, \qquad b_2 \doteq \sup_{m\geq 2} a_m, \qquad b_3 \doteq \sup_{m\geq 3} a_m \qquad , \qquad \dots \qquad , \qquad b_n \doteq \sup_{m\geq n} a_m, \dots.$$

So each term in the sequence in $(b_n)_n$ is the largest the sequence can be at step/time n. Since at each step, we lose terms, the sequence is monotonically decreasing and bounded. Therefore, by theorem 4.20, $(b_n)_n$ must converge to a limit, which we call the lim sup. Similarly, for the lim inf. Therefore, both the lim inf and lim sup always exist for a bounded sequence.

Remark 4.33. If a sequence $(a_n)_n$ is unbounded above $\limsup_{n\to\infty} a_n = \infty$, if it is unbounded below $\limsup_{n\to\infty} a_n = -\infty$.

Proposition 4.34. *Let $(a_n)_n$ be a bounded sequence of real numbers. Then,*

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

Proof. We note that for all n

$$\inf_{m\geq n} a_m \leq \sup_{m\geq n} a_m.$$

The result now follows from property (viii) in theorem 4.14. □

Now, above we said that they are quantities that help us find the limit, if it exists. It can also help us show the limit doesn't exist. How does this work? Well, let's do three examples.

Example 4.35. *Let $x_n = (-1)^n$. Then,*

$$\liminf_{n\to\infty} (-1)^n = -1,$$
$$\limsup_{n\to\infty} (-1)^n = 1.$$

Example 4.36. *Let,*

$$a_1 \doteq -6, \quad a_2 \doteq \frac{7}{2}, \quad a_3 \doteq -\frac{8}{3}, \quad a_4 \doteq \frac{9}{4} \quad ,..., \quad a_n \doteq (-1)^n\Big(1+\frac{5}{n}\Big).$$

So, if $b_n \doteq \sup_{m\geq n} a_m$ then

$$b_1 = \frac{7}{2}, \quad b_2 = \frac{7}{2}, \quad b_3 = \frac{9}{4}, ...$$

So,

$$b_n \doteq \begin{cases} a_{n+1} = \left(1 + \frac{5}{n+1}\right) & n \ \ odd \\ a_n = \left(1 + \frac{5}{n}\right) & n \ \ even. \end{cases}$$

Therefore,

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1.$$

Similarly, if $c_n \doteq \inf_{m\geq n} a_m$ *then*

$$c_1 = -6, \quad c_2 = -\frac{8}{3}, \quad c_3 = -\frac{8}{3}, \ldots.$$

So,

$$c_n \doteq \begin{cases} a_n = -\left(1 + \frac{5}{n}\right) & n \ \ odd \\ a_{n+1} = -\left(1 + \frac{5}{n+1}\right) & n \ \ even. \end{cases}$$

Therefore,

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n = -1.$$

Example 4.37. *Let,*

$$a_1 \doteq -4, \quad a_2 \doteq \frac{7}{2}, \quad a_3 \doteq -\frac{2}{3}, \quad a_4 \doteq \frac{9}{4} \quad, \ldots, \quad a_n \doteq 1 + (-1)^n \frac{5}{n}.$$

So, if $b_n \doteq \sup_{m\geq n} a_m$ *then*

$$b_1 = \frac{7}{2}, \quad b_2 = \frac{7}{2}, \quad b_3 = \frac{9}{4}, \ldots.$$

So,

$$b_n \doteq \begin{cases} a_{n+1} = \left(1 + \frac{5}{n+1}\right) & n \ \ odd \\ a_n = \left(1 + \frac{5}{n}\right) & n \ \ even. \end{cases}$$

Therefore,

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1.$$

Similarly, if $c_n \doteq \inf_{m\geq n} a_m$ *then*

$$c_1 = -4, \quad c_2 = -\frac{2}{3}, \quad c_3 = -\frac{2}{3}, \ldots.$$

So,

$$c_n \doteq \begin{cases} a_n = \left(1 - \frac{5}{n}\right) & n \ \ odd \\ a_{n+1} = \left(1 - \frac{5}{n+1}\right) & n \ \ even. \end{cases}$$

Therefore,

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 1.$$

These examples illustrate that the lim sup and lim inf envelope the sequence; they restrict the sequence to lie in an interval which gets smaller and smaller, as shown in the following diagram:

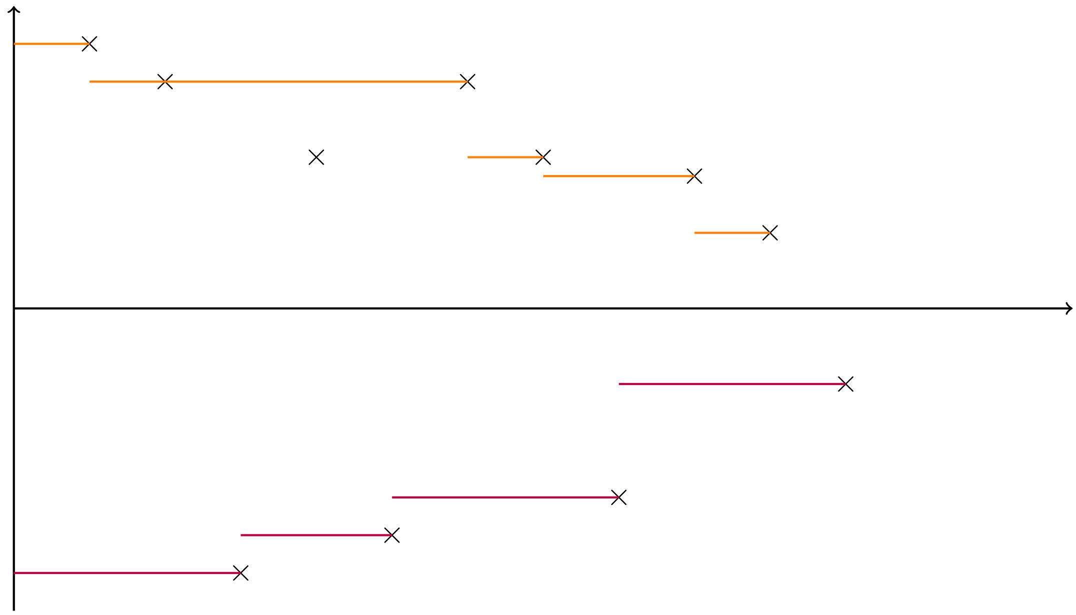


Figure 1: Illustration of limit superior and limit inferior.

Therefore, we would expect that if they agree then they are equal to the limit. This is what we now prove.

Theorem 4.38. *Let $(a_n)_n$ be a sequence of real numbers. Then,*

$$a_n \to a \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$$

Proof. ($\Rightarrow$) Suppose that $a_n \to a$ and let $\varepsilon > 0$. Then there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$

$$|a_n - a| < \varepsilon \iff a - \varepsilon < a_n < a + \varepsilon.$$

This means that $a - \varepsilon$ and $a + \varepsilon$ are lower and upper bounds for $\{a_n : n \geq N\}$, respectively. Therefore,

$$a - \varepsilon \leq \inf_{n\geq N} a_n < a + \varepsilon, \qquad a - \varepsilon < \sup_{n\geq N} a_n \leq a + \varepsilon.$$

Since ε was arbitrary, $\lim_{N\to\infty} \inf_{n\geq N} a_n = a$ and $\lim_{N\to\infty} \sup_{n\geq N} a_n = a$, which is the definition of lim inf and lim sup, respectively.

($\Leftarrow$) Suppose $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$ and let $\varepsilon > 0$. By the definition of convergence for the sequences $(\sup_{m\geq n} a_m)_n$ and $(\inf_{m\geq n} a_m)_n$, one can pick N such that for all $n \geq N$

$$|\sup_{m\geq n} a_m - a| < \varepsilon \iff a - \varepsilon < \sup_{m\geq n} a_m < a + \varepsilon,$$
$$|\inf_{m\geq n} a_m - a| < \varepsilon \iff a - \varepsilon < \inf_{m\geq n} a_m < a + \varepsilon.$$

So, for all $n \geq N$,

$$a - \varepsilon < \inf_{m \geq N} a_m \leq a_n \leq \sup_{m \geq N} a_m < a + \varepsilon \iff |a_n - a| < \varepsilon.$$

So $a_m \to a$. □

Example 4.39. *Let (a_n) and (b_n) two convergent sequences with limits a and b, respectively. Moreover, we define*

$$x_n = \begin{cases} a_k & \text{if } k \text{ is odd,} \\ b_k & \text{if } k \text{ is even.} \end{cases}$$

Then $\limsup x_n = \max\{a, b\}$ *and* $\liminf x_n = \min\{a, b\}$.

Example 4.40. *Consider the sequence $x_n = (-1)^n \frac{2n}{n+1}$. We claim that*

$$\limsup_{n \to \infty} (-1)^n \frac{2n}{n+1} = 2.$$

Fix $\varepsilon > 0$. We want to prove that there exists M such that for all $m \geq M$

$$2 - \varepsilon \leq \sup_{n \geq m} (-1)^n \frac{2n}{n+1} \leq 2 + \varepsilon$$

First note that

$$\sup_{n \geq m} (-1)^n \frac{2n}{n+1} = \sup_{\substack{n \geq m \\ n \text{ even}}} (-1)^n \frac{2n}{n+1} = \sup_{\substack{n \geq m \\ n \text{ even}}} \Big(2 - \frac{2}{n+1}\Big).$$

Clearly, $-\frac{2}{n+1} < \varepsilon$. So,

$$\sup_{n \geq m} (-1)^n \frac{2n}{n+1} \leq 2 + \varepsilon$$

for all $n \geq 1$. Additionally, we want $\frac{2}{m+1} > \varepsilon$ or equivalently $m \geq \frac{2-\varepsilon}{\varepsilon}$. So take $M = \frac{2-\varepsilon}{\epsilon}$ to give the desired inequality.

Example 4.41. *Let*

$$x_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases} \qquad y_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Then $\liminf(x_n + y_n) = \liminf 1 = 1$ *but* $\liminf x_n + \liminf y_n = 0 + 0 = 0$. *Hence, the usual rules for computations of limits do not apply in the same way for* $\limsup$ *and* $\liminf$.

One has an $\epsilon - N$ characterisation of $\limsup$ and $\liminf$.

Proposition 4.42. *Let $(a_n)_n$ be a bounded sequence and $a \in \mathbb{R}$. The number a is the* $\limsup_{n \to \infty} a_n$ *if and only if*

1. *for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_n < a + \varepsilon$ for all $n \geq N$.*
2. *for all $\varepsilon > 0$ and $\forall n_0 \in \mathbb{N}$, there exists $p \geq n_0$ such that $a_p > a - \varepsilon$.*

The number a is the $\liminf_{n\to\infty} a_n$ *if and only if*

1. *for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_n > a - \varepsilon$ for all $n \geq N$.*
2. *for all $\varepsilon > 0$ and $\forall n_0 \in \mathbb{N}$, there exists $p \geq n_0$ such that $a_p < a + \varepsilon$.*

Proof. ($\Rightarrow$) Suppose $a = \liminf_{n\to\infty} a_n$ then by the definition of the limit, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$a - \varepsilon < \inf_{m \geq n} a_m < a + \varepsilon \quad \forall n \geq N$$

We have that $a_n \geq \inf_{m\geq n} a_m$. So,

$$a - \varepsilon < \inf_{m \geq n} a_m \leq a_n \quad \forall n \geq N.$$

Now let $A_n = \{a_m : m \geq n\}$ $\inf_{m\geq n} a_m = \inf A_n$. So by proposition 3.25, we know that for all $\varepsilon > 0$, there exists $x \in A_n$ such that

$$\inf A_n \leq x < \inf A_n + \varepsilon.$$

Since A_n is constructed sequentially, we must have $x = a_p$ for some $p \geq n$. Therefore, for each n_0, there exists $p \geq n_0$ such that

$$\inf A_n \leq a_p < \inf A_n + \varepsilon < a + 2\varepsilon.$$

We can redefine $\varepsilon \mapsto \varepsilon/2$ to get the desired inequality.

($\Leftarrow$) Exercise for the reader. $\square$

The next two theorems are the analogues to the rules for computing limits for convergent sequences.

Theorem 4.43. *Consider two bounded sequences $(x_n)_n$ and $(y_n)_n$ of real numbers. Then,*

$$\begin{aligned}\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \leq \liminf_{n\to\infty} (x_n + y_n) &\leq \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \\ &\leq \limsup_{n\to\infty} (x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,\end{aligned}$$

Finally, if one of the two sequences has a limit, then

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = \liminf_{n\to\infty} (x_n + y_n)$$

and

$$\limsup_{n\to\infty} (x_n + y_n) = \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

Proof. We will prove that

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \leq \liminf_{n\to\infty} (x_n + y_n)\,. \tag{8}$$

To ease notation, let

$$\liminf_{n\to\infty} x_n = \ell_i(x), \quad \liminf_{n\to\infty} y_n = \ell_i(y), \quad \liminf_{n\to\infty} (x_n + y_n) = \ell_i(x + y).$$

By the characterisation of lim inf in proposition 4.42, there exist two positive integer N_1, $N_2 \in \mathbb{N}$ such that

$$\ell_i(x) - \varepsilon < x_n \quad \forall n \geq N_1, \quad \ell_i(y) - \varepsilon < y_n \quad \forall n \geq N_2.$$

Summing the two inequalities we obtain that

$$\ell_i(x) + \ell_i(y) - 2\varepsilon < x_n + y_n \quad \forall n \geq \max\{N_1, N_2\}.$$

By proposition 4.42 on the characterisation of lim inf, for each n, $\exists x_p + y_p \in A_n \doteq \{x_m + y_m : m \geq n\}$ such that

$$x_p + y_p < \ell_i(x+y) + \varepsilon$$

So, as we let n increase, we find infinitely many p such that

$$x_p + y_p < \ell_i(x+y) + \varepsilon.$$

By combining our bound from below and above, it follows that for infinitely many $n \geq \max\{N_1, N_2\}$,

$$\ell_i(x) + \ell_i(y) - 2\varepsilon < x_n + y_n < \ell_i(x+y) + \varepsilon.$$

Hence,

$$\ell_i(x) + \ell_i(y) < \ell_i(x+y) + 3\varepsilon$$

but this is true for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we conclude that $\ell_i(x) + \ell_i(y) \leq \ell_i(x+y)$. Hence (8) is true.

Now we prove that

$$\liminf_{n\to\infty} (x_n + y_n) \leq \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n. \tag{9}$$

Let

$$\limsup_{n\to\infty} x_n = l_s(x).$$

From proposition 4.42, for every $\varepsilon > 0$ there exists a positive integer $N_3 \in \mathbb{N}$ such that

$$x_n \leq l_s(x) + \varepsilon \quad \text{for all } n \geq N_3$$

and there exist infinitely many $n \in \mathbb{N}$ for which

$$y_n \leq \ell_i(y) + \varepsilon.$$

Summing the two inequalities we obtain that

$$x_n + y_n < l_s(x) + l_i(y) + 2\varepsilon \quad \text{for infinitely many } n \geq N_3.$$

On the other hand, there exists a positive integer $N_4 \in \mathbb{N}$ such that

$$\ell_i(x+y) - \varepsilon < x_n + y_n \quad \text{for all } n \geq N_4.$$

By combining the last two inequalities, it follows that for infinitely many $n \geq \max\{N_3, N_4\}$,

$$l_i(x+y) - \varepsilon < x_n + y_n \leq l_s(x) + l_i(y) + 2\varepsilon.$$

and so

$$l_i(x+y) - \varepsilon < l_s(x) + l_i(y) + 2\varepsilon.$$

but this is true for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain that (9) holds. The other inequalities are very similar and so we omit them. $\square$

Remark 4.44. A similar result is true for products, see homework.

4.5 The Exponential Function

In this subsection, we will show that the sequence $a_n = (1 + \frac{x}{n})^n$ converges for every $x \in \mathbb{R}$. We will need the following lemma:

Lemma 4.45. *Let $(a_n) \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} n \cdot a_n = 0$. Then,*

$$\lim_{n\to\infty} (1 + a_n)^n = 1.$$

Proof. By assumption there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $|n \cdot a_n| < \frac{1}{2}$. This implies that $|a_n| < \frac{1}{2n} \leq \frac{1}{2}$. From this one can show that $\forall n \geq N$,

$$\frac{2}{3} < \frac{1}{1+a_n} < 2 \implies -1 < \frac{a_n}{1+a_n} = 1 - \frac{1}{a_n+1} < \frac{1}{3}.$$

This means, that we can apply Bernoulli's inequality (twice) to obtain

$$1 + n \cdot a_n \leq (1+a_n)^n = \frac{1}{(1 - \frac{a_n}{1+a_n})^n} \leq \frac{1}{1 - \frac{na_n}{1+a_n}}.$$

Note that $n \cdot a_n > -\frac{1}{2} > -1$ and $\frac{a_n}{1+a_n} > -\frac{1}{3} > -1$ so use of Bernoulli's inequality is valid.

Now note that $\lim_{n\to\infty}(1 + n \cdot a_n) = 1$ and that $\lim_{n\to\infty} \frac{na_n}{1+a_n} = 0$, which implies that $\lim_{n\to\infty} \frac{1}{1 - \frac{na_n}{1+a_n}} = 1$. The claim now follows from the theorem about three sequences, Theorem 4.14 *(ix)*. $\square$

Theorem 4.46. *Let $x \in \mathbb{R}$ and define $a_n = (1 + \frac{x}{n})^n$. The limit $\lim_{n\to\infty} a_n$ exists for all $x \in \mathbb{R}$.*

Proof. We begin by showing that for n large enough, the sequence a_n is increasing. In particular, the claim is that if $n > -x \neq 0$, then $a_{n+1} \geq a_n$. First, note that if $n > -x$ then $1 + \frac{x}{n} > 0$. To show monotonicity we will study the ratio a_{n+1}/a_n and show that it is more than 1. Note, we can equivalently show,

$$\frac{a_{n+1}}{a_n} = \frac{(1 + \frac{x}{n+1})^{n+1}}{(1 + \frac{x}{n})^n} > 1 \iff \left(\frac{1 + \frac{x}{n+1}}{1 + \frac{x}{n}}\right)^{n+1} > \frac{n}{n+x}.$$

Now the RHS is equal to

$$\left(1-\frac{x}{(n+x)(n+1)}\right)^{n+1} \geq 1-(n+1)\frac{x}{(n+x)(n+1)} = 1-\frac{x}{n+x} = \frac{n}{n+x},$$

where one uses Bernoulli's inequality (note $\frac{-x}{(n+x)(n+1)} > -1$, since $n+x > x \neq 0$). Hence we have shown monotonicity for large enough n.

We have shown monotonicity for n large, we need boundedness to use theorem 4.20. We split into three cases:

1. If $x < 0$ and $n > -x$ then $1+\frac{x}{n} < 1$. Therefore, also $(1+\frac{x}{n})^n < 1$, which means that the sequence a_n is eventually bounded above and, therefore, bounded above. By theorem 4.20 it follows that a_n has a limit. Moreover, the limit is positive as $a_n > 0$ and it is an eventually increasing sequence.
2. If $x = 0$ then $a_n = 1$ for all n and so the limit exists and is equal to 1.
3. If $x > 0$ then we have

$$(1+\frac{x}{n})^n = \frac{(1-\frac{x^2}{n^2})^n}{(1+\frac{-x}{n})^n}.$$

From 1, the denominator has a positive limit since now $-x < 0$. The enumerator converges to 1, which follows from Lemma 4.45 since we have

$$\lim_{n\to\infty} n\cdot\frac{-x^2}{n^2} = 0.$$

Therefore, using Theorem 4.14, it follows that the sequence a_n converges to a positive limit. This finishes the proof.

□

Definition 4.47. *We define the function* $\exp : \mathbb{R} \to (0,\infty)$ *to be*

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Proposition 4.48. *For any* $x, y \in \mathbb{R}$ *we have that* $\exp(x+y) = \exp(x)\cdot\exp(y)$.

Proof of this proposition will be left to the reader.

Definition 4.49. *We define the function* $\ln : (0,\infty) \to \mathbb{R}$ *to be the inverse function of* $\exp$.

5 Series

Definition 5.1 (Series)**.** *Let $(a_n)_n$ be a real sequence. Define a new sequence $(S_n)_n$, the sequence of N^{th} partial sums, by*

$$S_N \doteq \sum_{n=1}^{N} a_n.$$

If $(S_N)_N$ converges to a limit $S \in \mathbb{R}$ we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S.

Proposition 5.2. *If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. We have that $S_N \to S$ and $S_{N-1} \to S$. So by the addition property of limits (theorem 4.14) $S_N - S_{N-1} \to 0$. Note that then $a_N = S_N - S_{N-1} \to 0$. $\square$

Remark 5.3. The statement above implies in particular that if $a_n \not\to 0$ then $\sum_{k=1}^{\infty} a_k$ does not converge.

It's important to note that if $a_n \to 0$ then this does not necessarily imply that the series converges.

Example 5.4 (Harmonic Series)**.** *Let $a_n = 1/n$. Then $a_n \to 0$ but the series does not converge. In order to see the divergence we compute for $n \in \mathbb{N}$*

$$S_{2^n} - S_{2^{n-1}} = \sum_{k=1}^{2^n} \frac{1}{k} - \sum_{k=1}^{2^{n-1}} \frac{1}{k} = \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k} \geq \frac{1}{2^n} \sum_{k=2^{n-1}+1}^{2^n} 1 = (2^n - (2^{n-1}+1)+1)\frac{1}{2^n} = \frac{1}{2},$$

since $\frac{1}{k} \geq \frac{1}{2}^n$. Therefore,

$$\begin{aligned} S_{2^n} &= S_{2^n} - (S_{2^{n-1}} - S_{2^{n-1}}) - (S_{2^{n-2}} - S_{2^{n-2}}) + ... - (S_{2^1} - S_{2^1}) - (S_{2^0} - S_{2^0}) \\ &= (S_{2^n} - S_{2^{n-1}}) + (S_{2^{n-1}} - S_{2^{n-2}}) + \cdots + (S_{2^1} - S_{2^0}) + S_1 \\ &\geq S_1 + \frac{n}{2} = 1 + \frac{n}{2}. \end{aligned}$$

Hence, the partial sums are unbounded, and so do not converge!

Example 5.5 (Geometric Series)**.** *Let $\rho \in \mathbb{R}$ such that $|\rho| < 1$. We have*

$$S_N = \sum_{n=0}^{N} \rho^n = 1 + \rho + \rho^2 + ... + \rho^N,$$
$$\rho S_N \rho + \rho^2 + ... + \rho^{N+1} = S_N - 1 + \rho^{N+1}.$$

Therefore,

$$S_N(1-\rho) = 1 - \rho^{N+1} \implies S_N = \frac{1-\rho^{N+1}}{1-\rho}.$$

As $N \to \infty$, $\rho^{N+1} \to 0$, so

$$S_\infty = \sum_{n=1}^{\infty} \rho^n = \frac{1}{1-\rho}.$$

Exercise: $\lim_{n\to\infty} x^n = 0$ *for x with $|x| < 1$.*

5.1 Absolute and Unconditional Convergence

So far we have discussed convergence of a series. There are two stronger notions of convergence one can define: absolute and unconditional convergence. These turn out to be equivalent.

Definition 5.6. *Let (x_n) be a sequence of real numbers. We say that $\sum_{k=1}^{\infty} x_k$* converges absolutely *if the series $\sum_{k=1}^{\infty} |x_k|$ converges.*

As the notion suggests, absolute convergence implies convergence.

Theorem 5.7. *If a series converges absolutely, it converges.*

Proof. Let (a_n) be a sequence of real numbers. Define $A_n = \sum_{k=1}^{n} a_k$ and $\tilde{A}_n = \sum_{k=1}^{n} |a_k|$. Since $\tilde{A}_n$ converges by assumption, it also forms a Cauchy sequence.

Now let $\varepsilon > 0$. For $m \geq n$ we have

$$\begin{aligned} |A_n - A_m| &= \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| = \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| = \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k| \\ &= \tilde{A}_n - \tilde{A}_m = |\tilde{A}_n - \tilde{A}_m|. \end{aligned}$$

The inequality follows by the triangle inequality. Since $\tilde{A}_n$ is convergent, it is Cauchy, this estimate then shows that also the partial sums A_n are Cauchy. As $\mathbb{R}$ is complete, this proves the convergence of the series $\sum_{k=1}^{\infty} a_k$. □

You may wonder about the formal manipulations we did in the introduction, i.e. rearranging the order of summation. When is this valid? For this we study unconditional convergence.

Definition 5.8 (Unconditional Convergence)**.** *A series $\sum_{n=1}^{\infty} a_n$ converges unconditionally if any rearrangement of the series converges, i.e. if $\sum_{n=1}^{\infty} a_{\Pi(n)}$ converges for any $\Pi : \mathbb{N} \to \mathbb{N}$.*

Theorem 5.9. *A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if it converges unconditionally. Moreover, if the series converges absolutely, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\Pi(n)}$ for any rearrangement $\Pi : \mathbb{N} \to \mathbb{N}$.*

Proof. ($\Rightarrow$) If S_p is the p^{th} partial sum of $a_{\Pi(n)}$ for some rearrangement Π. Then, if $p > q$,

$$|S_p - S_q| = \Big| \sum_{k=q+1}^{p} a_{\Pi(k)} \Big| \leq \sum_{k=q+1}^{p} |a_{\Pi(k)}|, \tag{10}$$

by the triangle inequality. The idea is to show that $(S_p)_p$ is Cauchy and, therefore, by completeness of $\mathbb{R}$, converges. So given $\varepsilon > 0$, we would like the RHS of (10) to be bounded by ε. We have control on $\sum_{n=1}^{\infty} |a_n|$. In particular, since $\sum_{k=M+1}^{\infty} |a_k|$ converges, we can pick M such that

$$\sum_{k=M+1}^{\infty} |a_n| < \varepsilon.$$

Since Π is a rearrangement, it must eventually hit $1, ..., M$. So we can pick $N \geq M$ such that $\{n\}_{n=1}^{M} \subset \{\Pi(k)\}_{k=1}^{N}$. So, if $p \geq n \geq q > N$, then $\Pi(n) > M$. So, from (10), we find

$$|S_p - S_q| \leq \sum_{k=q+1}^{p} |a_{\Pi(n)}| \leq \sum_{n=M+1}^{\infty} |a_n| < \varepsilon.$$

($\Leftarrow$) Here we prove the contrapositive: if $\sum_{n=1}^{\infty} a_n$ does not converge absolutely then it does not converge unconditionally. So we suppose that $\sum_n |a_n| = \infty$. Let us divide up the sequence $(a_n)_n$ into a positive subsequence $(b_n)_n$ and a negative subsequence $(c_n)_n$. If both were to converge then for all $M \in \mathbb{N}$

$$\sum_{n=1}^{M} |a_n| \leq \sum_{n=1}^{M} |b_n| + \sum_{n=1}^{M} |c_n| = \sum_{n=1}^{M} b_n - \sum_{n=1}^{M} c_n.$$

Therefore, $\sum_{n=1}^{\infty} |a_n| < \infty$, which is a contradiction. So, assume $\sum_n b_n = \infty$ (you can run the same argument for if $\sum_n c_n = \infty$). We now reorder $(a_n)_n$ into a sequence $(d_n)_n$ to show divergence. Choose $d_1 = c_1$ and pick $N_1 > 0$ such that $c_1 + b_1 + \cdots + b_{N_1} \geq 1$ and $(d_2, \ldots d_{N_1+1}) = (b_1, \ldots, b_{N_1})$. Next, pick $d_{N_1+2} = c_2$ and $N_2 > N_1$ such that $c_2 + b_{N_1+1} + \cdots + b_{N_2} \geq 1$ and $(d_{N_1+2}, d_{N_1+3}, \ldots d_{N_2+2}) = (c_2, b_{N_1+1}, \ldots, b_{N_2})$. Carry on iteratively and call the rearrangement Π. We have the partial sums

$$S_N = \sum_{n=1}^{N} a_{\Pi(n)} \geq \sum_{n=1}^{N} 1 = N \to \infty.$$

So, we have constructed a rearrangement that does not converge. We leave the equality of limits as an exercise. $\square$

Remark 5.10. The contrapositive: suppose you have 'if statement A then statement B is true', i.e. $A \implies B$. The contrapositive is 'if B is not true then A is not true', i.e. $\neg B \implies \neg A$, where $\neg$ denotes logical negation. We can show the equivalence as follows. Suppose that we have: if A is true then B is true, and B is not true. Then assume A is true, then B is true. So we have a contradiction, i.e. we have shown $(A \implies B) \implies (\neg B \implies \neg A)$. To prove the other direction, you assume $\neg B \implies \neg A$ and A is true. You then use contradiction, as above, to show that $(A \implies B) \impliedby (\neg B \implies \neg A)$.

5.2 Convergence Tests

We start with an easy property of series with non-negative summands. You can check the convergence of arbitrary real series by checking absolute convergence via the tests below.

Theorem 5.11. *Let $(a_n)_n$ be a sequence of non-negative real numbers. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums is bounded.*

Proof. We notice that the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ is increasing. Hence, it converges if and only if it is bounded by theorem 4.20. $\square$

Theorem 5.12 (Comparison Test)**.** *Let (a_n) and (b_n) be two sequences of non-negative real numbers. Suppose that $\exists C \geq 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \leq Cb_n$. Then if $\sum_{n=1}^{\infty} b_n$ converges so does $\sum_{n=1}^{\infty} a_n$.*

Proof. Let $M > N$, then

$$S_M - S_N = \sum_{n=N+1}^{M} a_n \leq C \sum_{n=N+1}^{M} b_n \leq C \sum_{n=1}^{\infty} b_n.$$

So,

$$S_M \leq S_N + C \sum_{n=1}^{\infty} b_n.$$

Therefore, $(S_M)_M$ is eventually bounded and, therefore, bounded. The $a_n \geq 0$, which means S_M is increasing and bounded. Therefore, by theorem 4.20, $(S_N)_N$ convergences. □

Example 5.13. *Consider the series given by* $\sum_{n=0}^{\infty} \frac{2^n \sin^2(5n)}{4^n+\cos^2(n)}$. *Let* $a_n = \frac{2^n \sin^2(5n)}{4^n+\cos^2(n)}$. *Then, we have that* $a_n \geq 0$ *for all* $n = 0, 1, 2, \ldots$ *and* $a_n \leq \frac{2^n \cdot 1}{4^n+0} = \frac{1}{2^n} =: b_n$. *Note that* $\sum_{n=0}^{\infty} b_n$ *is a geometric series which converges since* $\frac{1}{2} < 1$. *It follows by the comparison test that* $\sum_{n=0}^{\infty} \frac{2^n \sin^2(5n)}{4^n+\cos^2(n)} < \infty$ *also converges.*

Example 5.14. *Consider the series* $\sum_{n=1}^{\infty} \frac{n}{2^n}$. *We will show that this series converges. Indeed, for* $n \geq 4$ *we have that* $n \leq 2^{n/2}$. *This follows by induction since* $4 = 2^{4/2}$ *and for* $n \geq 4$ *we have* $1 \leq 4(\sqrt{2}-1) \cdot 2^{n/2-2}$ *(since* $4(\sqrt{2}-1) > 1$ *and* $2^{n/2-2} \geq 1$.

Hence, by the comparison test,

$$\sum_{n=4}^{\infty} \frac{n}{2^n} \leq \sum_{n=4}^{\infty} \frac{2^{n/2}}{2^n} = \sum_{n=4}^{\infty} 2^{-n/2},$$

which converges since it is geometric, and so does our series.

Remark 5.15. You can use the contrapositive of this to show divergence: Assume that $\sum_{n=1}^{\infty} a_n = \infty$ and that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $b_n \geq a_n$. Then $\sum_{n=1}^{\infty} b_n$ also diverges.

Example 5.16. *Consider the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. *We have shown in the Example 5.4 that the series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges. Moreover, note that* $0 < \sqrt{n} \leq n$ *for all* $n = 1, 2, \ldots$. *It follows that the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges.*

Proposition 5.17 (Condensation Test)**.** *Let* $(a_n)_n$ *be a decreasing sequence of non-negative reals. Then*

$$\sum_{n=1}^{\infty} a_n < \infty \Longleftrightarrow \sum_{n=1}^{\infty} 2^n a_{2^n} < \infty.$$

Proof. (⇒) Since the sequence (a_n) is decreasing, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= a_1 + a_2 + (a_3 + a_4) + (a_5 + \ldots + a_8) + (a_9 + \ldots a_{16}) + \ldots \\
&\geq a_1 + a_2 + 2a_4 + 4a_8 + 8a_{16} + \ldots \\
&= a_1 + \tfrac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n}.
\end{aligned}$$

Since $a_n \geq 0$ for all $n = 1, 2, \ldots$, the partial sums are an increasing sequence which is bounded by $\sum_{n=1}^{\infty} a_n < \infty$. Therefore, it has a finite limit and so $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

($\Leftarrow$) Similarly, since (a_n) is decreasing we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= a_1 + (a_2 + a_3) + (a_4 + \ldots + a_7) + (a_8 + \ldots + a_{15}) + \ldots \\ &\leq a_1 + 2a_2 + 4a_4 + \ldots \\ &= a_1 + \sum_{n=1}^{\infty} 2^n a_{2^n}.\end{aligned}$$

Again, it follows that if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges and so must $\sum_{n=1}^{\infty} a_n$ by the monotone sequence property.

□

Example 5.18. *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

Proof. If $p \leq 0$, then the series is clearly divergent (e.g. by Proposition 5.2). If $p > 0$, then we can use the condensation test since the sequence is decreasing (and nonnegative). We get that it is sufficient to analyse the convergence of the series

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=1}^{\infty} (2^{(1-p)})^k.$$

Since the series above is geometric, it converges if and only if $2^{(1-p)} < 1$, i.e. if $p > 1$. □

There are many different criteria for the convergence of series which are useful in different situations. We will present a few of the most important ones, but a number of others can be found (for example on wikipedia). Both criteria are relatively simple corollaries of the comparison criterion where we compare a series with the geometric series $\sum_{k=0}^{\infty} x^k$.

Theorem 5.19 (Ratio Test)**.** *Let (a_n) be a sequence of positive real numbers and define*

$$R = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}, \qquad r = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Then the series over a_n converges if $R < 1$ and does not converge if $r > 1$.

Proof. First, assume that $R < 1$. Choose $\varepsilon > 0$ such that $R+\varepsilon < 1$. Then, by proposition 4.42, there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \leq R + \varepsilon$$

for all $n \geq N$, i.e.

$$a_{n+1} \leq (R+\varepsilon)a_n \leq (R+\varepsilon)^2 a_{n-1} \leq \cdots \leq (R+\varepsilon)^{n+1-N} a_N.$$

We find

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1} \leq (R+\varepsilon)^{-N} a_N \sum_{n=0}^{\infty} (R+\varepsilon)^{n+1}.$$

So the series converges by comparison with the geometric series

$$\sum_{n=0}^{\infty}(R+\varepsilon)^{n+1},$$

where our constant of comparison is $C=(\varepsilon+R)^{-N}a_N$.

Now assume that $r>1$. Choose $\varepsilon>0$ such that $r-\varepsilon>1$. Then, by proposition 4.42, there exists $N\in\mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n}\geq r-\varepsilon$$

for all $n\geq N$ and $a_N\neq 0$, i.e.

$$a_{n+1}\geq(r-\varepsilon)\,a_n\geq\cdots\geq(r-\varepsilon)^{n+1-N}\,a_N.$$

Since $|a_N|\neq 0$, this means that $|a_n|\to\infty$, so the series cannot converge even conditionally. $\square$

The quotient test is easy to apply and should be the first test you check, but not as sharp as the next test.

Theorem 5.20 (Root Test)**.** *Let (a_n) be a nonnegative sequence and define*

$$\beta=\limsup_{n\to\infty}\sqrt[n]{a_n}.$$

Then the series, $\sum a_n$ converges if $\beta<1$ and does not converge if $\beta>1$.

Proof. First assume that $\beta<1$ and choose $\varepsilon>0$ such that $\beta+\varepsilon<1$. Then there exists $N\in\mathbb{N}$ such that

$$\sqrt[n]{a_n}\leq\beta+\varepsilon\quad\forall\ n\geq N\qquad\Rightarrow\qquad a_n\leq(\beta+\varepsilon)^n\quad\forall n\geq N.$$

The series converges by comparison with the geometric series.

Assume that $\beta>1$ and choose $\varepsilon>0$ such that $\beta-\varepsilon>1$. Then, by proposition 4.42, we have

$$\sqrt[n]{a_n}\geq 1\qquad\Rightarrow\qquad a_n\geq 1$$

for infinitely many $n\in\mathbb{N}$. Thus (a_n) does not converge to zero and the series cannot converge. $\square$

Remark 5.21. Sharpness: consider the series $\sum_{n=1}^{\infty}3^{-n-(-1)^n}$. Then

$$\frac{a_{n+1}}{a_n}=3^{-1+2(-1)^n}\implies\sup_{m\geq n}\frac{a_{n+1}}{a_n}=3\implies\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=3$$
$$\inf_{m\geq n}\frac{a_{n+1}}{a_n}=\frac{1}{27},\qquad\liminf n\to\infty\frac{a_{n+1}}{a_n}=\frac{1}{27}.$$

So the ratio test is inconclusive. However,

$$(a_n)^{\frac{1}{n}}=3^{-(1+\frac{(-1)^n}{n})},\implies\sup_{m\geq n}(a_n)^{\frac{1}{n}}=\frac{1}{3^{1-\frac{1}{n}}},\quad\limsup_{n\to\infty}(a_n)^{\frac{1}{n}}=\frac{1}{3}<1.$$

So the series converges.

Remark 5.22. In the borderline cases of the theorems above the tests are inconclusive. In fact, $\sqrt[n]{n^s} = (\sqrt[n]{n})^s \to 1$ but $\sum_{n=1}^{\infty} \frac{1}{n^s}$ does not converge for $s = 1$ and does converge for all $s > 1$.

Proof. (Proof of $\sqrt[n]{n^s} \to 1$) Let $x = \sqrt[n]{n}$ for $n > 1$. Then $x^n = n$. Assume that $0 < x \leq 1$, then $x^n \leq x^{n-1} \leq \cdots \leq x \leq 1 < n$. Contradiction. So $\sqrt[n]{n} > 1$. Therefore,

$$\sqrt[n]{\sqrt{n}} = n^{\frac{1}{2n}} = \sqrt[2n]{n} = 1 + x_n$$

with $x_n \geq 0$. Now,

$$\sqrt{n} = \left(\sqrt[n]{\sqrt{n}}\right)^n = (1 + x_n)^n \geq 1 + nx_n\,.$$

This shows that $0 \leq x_n \leq \frac{1}{\sqrt{n}} - \frac{1}{n} \to 0$. This shows that $(\sqrt[n]{n})^s = (1 + x_n)^{2s} \to 1$. ☐

Theorem 5.23 (Alternating Sequence Test)**.** *Let (a_n) be decreasing and nonnegative such that $a_n \to 0$. Then the series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.*

Proof. First, consider

$$A_{2n+1} = \sum_{k=1}^{2n+1} (-1)^k a_k = A_{2(n-1)+1} + \underbrace{(a_{2n} - a_{2n+1})}_{\geq 0} \geq A_{2(n-1)+1}.$$

Hence, the subsequence (A_{2n+1}) is increasing. On the other hand

$$A_{2n+1} = (-a_1 + a_2) + \cdots + (-a_{2n-1} + a_{2n}) - a_{2n+1} \leq -a_{2n+1} \leq 0.$$

Hence, the subsequence is also bounded from above. This implies convergence of (A_{2n+1}). We call its limit L.
Moreover, $A_{2n} = A_{2n+1} + a_{2n+1} \to L + 0 = L$. This shows that the whole sequence (A_n) converges to L. ☐

Example 5.24. *The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges, but the series $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge, so the first series does not converge absolutely. So convergence does not imply absolute convergence (so not unconditionally either). Note that you can rearrange this series as follows. Pick $N_1 \in \mathbb{N}$ and such that*

$$2 < \sum_{n=1}^{N_1} \frac{1}{2n} < 3 \implies 1 < \sum_{n=1}^{N_1} \frac{1}{2n} - 1 < 2.$$

Next, pick $N_2 \in \mathbb{N}$,

$$3 < \sum_{n=1}^{N_1} \frac{1}{2n} - 1 + \sum_{n=(N_1+1)}^{N_2} \frac{1}{2n} < 4 \implies 2 < \sum_{n=1}^{N_1} \frac{1}{2n} - 1 + \sum_{n=(N_1+1)}^{N_2} \frac{1}{2n} - \frac{1}{3} < 3.$$

Carry on inductively, pick $N_n \in N$ such that

$$n + 1 < \sum_{n=1}^{N_n} \frac{1}{2n} - \sum_{k=1}^{n-1} \frac{1}{2k-1} < n + 2 \implies n < \sum_{n=1}^{N_n} \frac{1}{2n} - \sum_{k=1}^{n} \frac{1}{2k-1} < n + 1.$$

Defining $\Pi : \mathbb{N} \to \mathbb{N}$ in this manner gives a sequence of unbounded partial sums.

5.3 Power Series

Many series depend on a parameter x. For example, consider the finite series

$$S_n(x) = \sum_{k=0}^{n} a_k\, x^k.$$

Series of this type are called power series. Many classical functions can be written as power series. Moreover, every well-behaved function (to be made precise later) can locally be written as a power series.

Definition 5.25. *The radius of convergence of a power series $A_n(x)$ is $R \in [0, \infty]$ given by*

$$R \doteq \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

The significance of this definition will be explained in the next theorem.

Theorem 5.26. *Let $S_n(x)$ be a power series and R its radius of convergence. If $R \in (0, \infty]$, then $S_n(x)$ converges absolutely for all x with $|x| < R$. It does not converge for $x \in \mathbb{R}$ with $|x| > R$.*

Proof. Fix $x \in \mathbb{R}$ and consider the series

$$S(x) = \sum_{n=0}^{\infty} a_n\, x^n.$$

By the root criterion, it converges absolutely if

$$|x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|\, |x|^n} < 1$$

and diverges if the same expression is > 1. □

Remark 5.27. The theorem does not make a statement about $x = \pm R$. The next example shows that no general statement can be made.

Example 5.28. *Consider the power series $\sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k}$. The radius of convergence can be computed to be $R = 1$. Hence, the power series converges for all $|x| < 1$. By the alternating sign test, it also converges for $x = 1$. On the other hand, the power series does not converge for $x = -1$ since it becomes the harmonic series.*

Here are some important functions which can be represented by power series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \qquad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

We will explore this further when we discuss Taylor's theorem.

Remark 5.29. Some comments:

1. The power series in the definition above converges absolutely for all $x \in \mathbb{R}$.
2. Note that it follows directly from the definition that $e^{ix} = \cos(x) + i \sin(x)$. From there one can prove the classical trigonometric identities.
3. All statements proved above remain valid if the series are of the form

$$\sum_{n=0}^{\infty} a_n\, (x - x_0)^n, \qquad x_0 \in \mathbb{R}.$$

6 Metric Spaces

So far, we have emphasised the order properties of the real numbers – concepts like suprema or upper and lower limits critically depend on the order, and monotone sequences are a useful tool. In this chapter, we will introduce *metric* properties. These cover many other important spaces ($\mathbb{R}^n$, curved spaces like spheres or tori (donut surfaces), infinite-dimensional function spaces etc.), although we will mostly focus on the case of the real line.

6.1 Definition and Examples

Definition 6.1. *A* ***metric space*** *is a pair* (X, d) *where* X *is a set and* d *is a function* $d : X \times X \to \mathbb{R}$*, called the metric on* X*, which satisfies the following properties:*

(i) $d(x, y) \geq 0$ *for all* $x, y \in X$*.*

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$ *(triangle inequality),*

(iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *(symmetry),*

(iv) $d(x, y) = 0$ *if and only if* $x = y$*.*

Remark 6.2. In general relativity, the metric on the set (called in this context a spacetime manifold, which comes with extra structure) captures the notion of distance. However, it has some rather unusual properties. Property (i) is relaxed to allow for negative metric distances, the triangle inequality no longer is satisfies (in fact for points with negative distances the inequality is reversed) and, finally, (iv) is replaced with a non-degeneracy condition $d(x, y) = 0$ for all $y \in X$ if and only if $x \equiv 0$. These properties can be shown to capture physically what we experience as causality.

Example 6.3. *Here are some examples of metric spaces.*

(i) In $\mathbb{R}$ *we have that* $d(x, y) \doteq |x - y|$ *is a metric.*

(ii) In $\mathbb{R}^n$, $n \geq 1$*, for* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *the function*

$$d(x, y) \doteq \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

is a metric.

(iii) In $\mathbb{R}^n$, $n \geq 1$ *for* $x, y \in \mathbb{R}^n$ *as above, the taxicab or Manhattan distance is defined by*

$$d(x, y) \doteq |x_1 - y_1| + \ldots + |x_n - y_n|$$

(iv) Suppose (X, d_X) *is a metric space and* $Y \subseteq X$*. Then,* (Y, d_Y) *is a metric space, where* $d_Y(a, b) \doteq d_X(a, b)$ *for all* $a, b \in Y$*. In this case, we would call* Y *a (metric) subspace. A concrete example of this case is* $Y = \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ *defined by*

$$\mathbb{S}^n \doteq \{x \in \mathbb{R}^{n+1} : ||x|| = d(x, 0) = 1\}.$$

(v) In $\mathbb{R}$ *we have that*

$$d_1(x, y) \doteq \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right| \tag{11}$$

is a metric.

(vi) In $\mathbb{R}$ the so-called discrete metric is given by $d_{discrete}(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$

(vii) Let $G = (V, E)$ be a connected graph with vertex set V, and edges E. Let $x, y \in V$ and define $d(x,y)$ to be the number of edges on the shortest path connecting x to y. Then d is a metric on G. It is often called the graph metric, geodesic distance or shortest-path distance.

(viii) In the field of information theory, one can assign a distance between two strings of characters of different lengths. The distance between strings of characters is given by the number of substitutions one would have to make to change one string to another. For example, on strings of characters of length 3

$$d(dog, cat) = 3, \qquad d(bat, cat) = 1, \qquad d(jag, dog) = 2.$$

6.2 Sequences in Metric Spaces

As before, a sequence in a metric space is a map $f : \mathbb{N} \to X$ and is usually denoted by $(x_n)_n$.

Definition 6.4 (Convergence)**.** *A sequence $(x_n)_n$ in a metric space (X, d) converges to $x \in X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $d(x_n, x) < \varepsilon$.*

Definition 6.5 (Boundedness)**.** *A sequence $(x_n)_n$ in a metric space (X, d) is bounded if there exist $x \in X$ and $C > 0$ such that $d(x_n, x) \leq C$ for all $n \in \mathbb{N}$.*

Definition 6.6 (Cauchy Sequence)**.** *A sequence $(x_n)_n$ in a metric space (X, d) is a Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$.*

Remark 6.7. If (X, d) is $(\mathbb{R}, d)$ with the metric $d(x,y) = |x - y|$, this concept of convergence agrees with the previous definition.

Proposition 6.8. *Let (X, d) be a metric space and $(x_n)_n$ a sequence in X. Then the following hold true.*

1. *If $(x_n)_n$ converges to $x \in X$, then $(x_n)_n$ is a Cauchy sequence.*
2. *If $(x_n)_n$ is a Cauchy sequence, it is bounded.*
3. *The limit of a convergent sequence is unique.*
4. *If $(x_n)_n$ is a Cauchy sequence with a convergence subsequence $(x_{n_k})_k$ with limit $\tilde{x}$. Then $x_n \to \tilde{x}$.*

Proof. These proofs are essentially the same as in $(\mathbb{R}, |\cdot - \cdot|)$.

1. Since (x_n) converges to $x \in X$, given $\varepsilon > 0$, consider $\frac{\varepsilon}{2}$ in the definition of convergence. Then there exists $N_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\varepsilon}{2}$$

for all $n \geq N_\varepsilon$. Hence, by the triangle inequality and symmetry of d, if $n, m \geq N_\varepsilon$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

2. Assume that (x_n) is Cauchy. Then there exists $N > 0$ such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. In particular $d(x_n, x_N) \leq 1$ for all $n \geq N$. This implies that

$$d(x_n, x_N) \leq \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\} < \infty,$$

so (x_n) is bounded.

3. Suppose that $x_n \to x$ and $x_n \to y$. Let $\varepsilon > 0$. We can find N_1, N_2 such that, for all $n \geq \max\{N_1, N_2\}$, $d(x_n, x) < \varepsilon/2$ and $d(x_n, y) < \varepsilon/2$. So from non-negativity, symmetry and the triangle inequality:

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) = d(x_n, x) + d(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\epsilon > 0$ was arbitrary, $x = y$.

4. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that for all $n, m \geq N$ it holds $d(x_n, x_m) < \frac{\varepsilon}{2}$.Next, let $K \in \mathbb{N}$ such that for all $k \geq K$ it holds $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Let $N_1 = \max\{N, K\}$ and $k \geq N_1$. Then $n_k \geq k \geq N_1$. It follows that

$$d(x_k, x) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $x_n \to x$. □

The opposite is not true, that is, there are Cauchy sequences that do not have a limit in certain 'bad' metric spaces.

Example 6.9. *Let $X = (0, 1)$ with the metric $d(x, y) = |x - y|$. The sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ converges to 0 in $\mathbb{R}$, and so it is a Cauchy sequence in $\mathbb{R}$. In particular, it is a Cauchy sequence in X. However, it does not converge to an element of X, since $0 \notin X$.*

Definition 6.10. *A metric space (X, d) is said to be* ***complete*** *if every Cauchy sequence converges.*

Intuitively, a space which is not complete is 'missing points' in between somewhere. We saw above that the Bolzano-Weierstrass theorem implies that $(\mathbb{R}, d)$ is a complete metric space. See Theorem 4.30.

6.3 Inducing Metrics from Norms and Inner Products

Metric spaces are the most general setting for studying many of the concepts of mathematical analysis and geometry. Note that while the metric gives a way to measure the distance between two objects it does not assign 'size' to any given object. This leads to the notion of a *normed space*. We first recall what a vector space is.

Definition 6.11. *A vector space over a field $\mathbb{F}$ is a non-empty set V together with two binary operations: addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{F} \times V \to V$ that satisfy the eight axioms listed below. In this context, the elements of V are commonly called vectors, the elements of $\mathbb{F}$ are called scalars, and we say that V is a vector space over $\mathbb{F}$.*

Let $v, w \in V$ and $a, b \in \mathbb{F}$

i) Associativity of vector addition: $u + (v + w) = (u + v) + w$

ii) Commutativity of vector addition: $u + v = v + u$

iii) Identity element of vector addition: There exists an element $0 \in V$*, called the zero vector, such that* $v + 0 = v$ *for all* $v \in V$*.*

iv) Inverse elements of vector addition: For every $v \in V$*, there exists an element* $-v \in V$*, called the additive inverse of* v*, such that* $v + (-v) = 0$*.*

v) Compatibility of scalar multiplication with field multiplication: $a(bv) = (ab)v$

vi) Identity element of scalar multiplication: $1v = v$*, where* 1 *denotes the multiplicative identity in* $\mathbb{F}$*.*

vii) Distributivity of scalar multiplication with respect to vector addition: $a(u+v) = au+av$

viii) Distributivity of scalar multiplication with respect to field addition: $(a + b)v = av + bv$

When the scalar field is the real numbers the vector space is called a real vector space. When the scalar field is the complex numbers, the vector space is called a complex vector space. These two cases are the most common ones, but vector spaces with scalars in an arbitrary field $\mathbb{F}$ are also commonly considered. Such a vector space is called an $\mathbb{F}$-vector space or a vector space over $\mathbb{F}$.

Definition 6.12. *A* norm *defined on a vector space* X *is a function* $\|\cdot\| : X \to \mathbb{R}$ *that satisfies*

i) nonnegativity: $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$*.*

ii) homogeneity: $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{R}$ *and for all* $x \in X$

iii) triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$

A normed space *is a vector space equipped with a norm.*

The additional simple structure of the normed space leads to many fascinating phenomena and, considered together with linear functionals defined on these spaces, gave rise to a field of mathematics called *functional analysis.*

Note that the norm induces a metric. In other words, by knowing the size (norm) of any object we get a way of measuring the distances via the size of their difference. In particular, we use the fact that we are working with a vector space to take the differences of two elements.

Proposition 6.13. *Let* $(X, \|\cdot\|)$ *be a normed space. Then* $d(x, y) \doteq \|x - y\|$ *is a metric.*

Proof. Note that the nonnegativity of d is inherited from the nonnegativity of the norm. Moreover, since $\|v\| = 0$ if and only if $v = 0$, we get that $d(x, y) = 0 \iff x = y$. The symmetry of the metric follows from the homogeneity of the norm. Indeed, we have

$$d(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = d(y, x).$$

Finally, the triangle inequality can be simply checked by

$$d(x, z) + d(z, y) = \|x - z\| + \|z - y\| \geq \|x - z + z - y\| = d(x, y).$$

It follows that such defined $d : X \times X \to \mathbb{R}$ is a metric.

□

Example 6.14. *A classical example is that of the vector space $\mathbb{R}^n$ (over $\mathbb{R}$) with the norm given by*

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

The space $(\mathbb{R}^n, \|\cdot\|_2)$ is often called the Euclidean space or ℓ_2-space. The Euclidean distance between two points x and y is given by $\|x-y\|_2$ and was intended to 'represent physical space' (in the case $n = 1$, $n = 2$ or $n = 3$). This should remind you of Example 6.3 (ii).

Example 6.15. *The generalization of the ℓ_2 norm is called ℓ_p norm, and is given by*

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p},$$

for a real number $p \geq 1$. For $p = \infty$ we define the ℓ_∞ norm by $\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$. For any $p \geq 1$, the pair $(\mathbb{R}^n, \|\cdot\|_p)$ is called the ℓ_p-space.

6.3.1 Inner product spaces

We define a notion of an inner product, which gives us further information about the relative position of two vectors.

Definition 6.16. *Let X be a real vector space. The function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called the* inner product *if the following conditions are satisfied*

i) $\langle x, x \rangle \geq 0$ *for all* $x \in X$, *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$,

ii) $\langle x, y \rangle = \langle y, x \rangle$ *for all* $x, y \in X$,

iii) $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y, z \in X$.

We call the pair $(X, \langle \cdot, \cdot \rangle)$ the inner product space.

Note that the inner product can be equivalently described as a symmetric positive definite bilinear form.

Example 6.17. *If $X = \mathbb{R}^n$ then $\langle x, y \rangle \doteq \sum_{i=1}^n x_i y_i$ is an inner product (or dot/scalar product).*

Theorem 6.18 (Cauchy-Schwarz inequality)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then,*

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

Proof. By the properties of the inner product, we have that for all $\lambda \in \mathbb{R}$

$$0 \leq \langle x + \lambda y, x + \lambda y \rangle = \langle x, x \rangle + 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle.$$

This means that the right-hand side is a quadratic polynomial in λ that is non-negative. Consequently, this polynomial has at most one real root. If a polynomial $ax^2 + bx + c$ has one real root then $b^2 - 4ac = 0$ and if it has has no real roots if $b^2 - 4ac < 0$. Applying this formula to our case gives

$$4\langle x, y \rangle^2 - 4\langle x, x \rangle \langle y, y \rangle \leq 0,$$

which, divided by 4 and rearranged, yields the result. □

Proposition 6.19. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then,*

$$\|x\| \doteq \sqrt{\langle x, x \rangle}$$

is a norm. We call this norm the Euclidean norm.

Proof. Nonnegativity and the fact that the norm is zero if and only if the vector is the zero vector follow clearly from the first property of the inner product. The homogeneity can be seen from the *iii)* property as follows

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{\lambda^2 \langle x, x \rangle} = |\lambda| \sqrt{\langle x, x \rangle} = |\lambda| \|x\|.$$

Finally, the triangle inequality follows from the Cauchy-Schwarz inequality

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \quad \square$$

6.4 Topology of Metric Spaces

Next, we will discuss certain topological notions. The goal is to understand the notion of compactness which here can be understood from the point of view of sequences.

6.4.1 Open and Closed Sets

Definition 6.20 (Open/Closed Balls)**.** *Let (X, d) be a metric space. For $x \in X$ and $r \geq 0$ we define the open (metric) ball of radius r centred at x to be*

$$B_r(x) \doteq \{y \in X : d(y, x) < r\}.$$

We define the closed ball of radius r centred at x to be

$$\overline{B}_r(x) \doteq \{y \in X : d(y, x) \leq r\}.$$

Remark 6.21. Often, open balls are also denoted as $B_r(x) = B(x, r)$.

Example 6.22. *These objects are natural generalisations of spherical balls in $\mathbb{R}^n$; although we call $B(x, r)$ a 'ball', the actual shape depends heavily on the metric. Recall the ℓ_p norms defined on $\mathbb{R}^n$ in Example 6.15. For any $p \geq 1$ the open ball of radius r centered at x is the set*
$\{y \in \mathbb{R}^n : \|x - y\|_p \leq r\}$. Here are some pictures in $\mathbb{R}^2$:

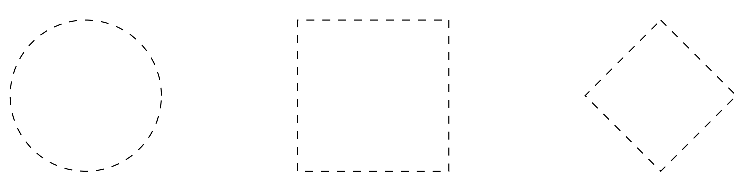


Figure 2: These are examples of open balls $B((x_1, x_2), r)$ in $\mathbb{R}^2$ with respect to the metrics $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, and $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$, respectively.

Remark 6.23. Any normed space $(\mathbb{R}^n, \|\cdot\|)$ is characterised by its "unit ball", i.e. the set $\bar{B}(0,1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. One can check that the unit ball is an *origin symmetric* ('Def'. If $x \in \bar{B}(0,1)$ then also $-x \in \bar{B}(0,1)$) and *convex* ('Def'. If $x, y \in \bar{B}(0,1)$ then for any $t \in [0,1]$ we have $tx + (1-t)y \in \bar{B}(0,1)$). In fact, every non-empty, compact (to be defined), origin symmetric and convex set in $\mathbb{R}^n$ is the closed unit ball of a norm on $\mathbb{R}^n$. This allows for a geometric approach to the analysis of normed spaces on $\mathbb{R}^n$.

Definition 6.24 (Open/Closed Sets)**.** *Let (X,d) be a metric space. A subset $U \subseteq X$ is open if for every $x \in U$ there exists $\delta = \delta(x) > 0$ such that $B_\delta(x) \subseteq U$. A subset $U \subseteq X$ is closed if $U^c = X \setminus U$ is open.*

In metric spaces, a set is open if you can find an open ball around every point. You should think that the set has no boundary; we will draw open sets with a dashed boundary.

Proposition 6.25. *Open balls are open. Closed balls are closed.*

Proof. Let $B_r(x_0)$ be an open ball of radius $r > 0$ around $x_0 \in X$. Let $x \in B_r(x_0)$. Take $\delta = r - d(x_0, x) > 0$ and let $y \in B(x, \delta)$. So, $d(x,y) < \delta$. Now,

$$d(x_0, y) \leq d(x_0, x) + d(x, y) < d(x_0, x) + \delta = r$$

which means that $y \in B(x_0, r)$, so in total $B(x, \delta) \subseteq B(x_0, r)$ which means that $B(x_0, r)$ is open. □

Remark 6.26. Open and closed sets are not a dichotomy; there exist sets which are neither or both. The subsets of $\mathbb{R}$ with the usual metric are neither open or closed:

1. $A_1 = (0,1]$ is not open since $B_\delta(1) \not\subseteq A_1$ for any $\delta > 0$. A_1 is not closed since $\mathbb{R} \setminus A_1 = (-\infty, 0] \cup (1, \infty)$ is not open as it does not contain any ball around 0.
2. $A_2 = \mathbb{Q}$ is not open since any ball around a rational contains an irrational (irrationals are dense). A_2 is not closed since $\mathbb{R} \setminus A_2$ is the irrationals and any ball around an irrational contains a rational (rationals are dense).
3. $A_3 = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Example 6.27. *Some simple examples of sets that are open and of some that are not.*

1. *In $(\mathbb{R}, |\cdot|)$, the set $(a, \infty) = \{x \in \mathbb{R} : x > a\}$ is open. Indeed, if $x > a$, take $r \doteq x - a > 0$. Then $B(x, r) \subset (a, \infty)$. Similarly, the set $(-\infty, a)$ is open.*
2. *In $(\mathbb{R}, |\cdot|)$, the set $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ is open. Indeed, given $a < x < b$, take $r \doteq \min\{b - x, x - a\} > 0$. Then $B(x,r) \subseteq (a,b)$.*
3. *In $(\mathbb{R}, |\cdot|)$, the set $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ is not open, since b belongs to the set but there is no ball $B(b, r)$ contained in $(a, b]$.*
4. *In $([0,1], |\cdot|)$, $[0, 1/2)$ is open since $B_{\frac{1}{2}}(0) = [0, 1/2)$ is an open ball and open balls are open.*
5. *The set $\{1\}$ is open in $\mathbb{N}$ with the distance function $d(n,m) = |n - m|$, but it is not open in $\mathbb{R}$ with the distance function $d(x,y) = |x - y|$.*
6. *If $a \in \mathbb{R}$, then $\{a\}$ is closed since $\{a\}^c = (-\infty, a) \cup (a, \infty)$ is open.*

7. *If $a < b$, then $[a, b]$ is closed.*

8. *If $a \in \mathbb{R}$, then both $[a, \infty)$ and $(-\infty, a]$ are closed.*

9. *$\mathbb{N}$ is closed in $\mathbb{R}$.*

Remark 6.28. Note that openness/closedness depends heavily on the subset, the ambient set of our metric space and the metric!

6.4.2 Topological Properties of Open and Closed Sets

Proposition 6.29. *Let (X, d) be a metric space. The following properties hold:*

(i) $\emptyset$ and X are open.

(ii) If $U_i \subseteq X$, $i = 1, \ldots, n$, is a finite family of open sets of X, then $U_1 \cap \cdots \cap U_n$ is open.

(iii) If $\{U_i\}_{i\in I}$ is an arbitrary collection of open sets of X, then $\bigcup_{i\in I} U_i$ is open.

(iv) $\emptyset$ and X are closed.

(v) If $C_i \subseteq X$, $i = 1, \ldots, n$, is a finite family of closed sets of X, then $C_1 \cup \cdots \cup C_n$ is closed.

(vi) If $\{C_i\}_{i\in I}$ is an arbitrary collection of closed sets of X, then $\bigcap_{i\in I} C_i$ is closed.

Proof. Property (i) is by definition: it is vacuously true for the empty set and if $x \in X$ then, by definition, an open ball around x of radius r is the set of points y *in* X such that $|y-x| < r$, so $B_r(x) \subseteq X$.

To prove (ii), let $x \in U_1 \cap \cdots \cap U_n$. Then $x \in U_i$ for every $i = 1, \ldots, n$, and since U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$. Take $r \doteq \min\{r_1, \ldots, r_n\} > 0$. Then

$$B(x, r) \subseteq U_1 \cap \cdots \cap U_n,$$

which shows that $U_1 \cap \cdots \cap U_n$ is open.

To prove (iii), let $x \in U \doteq \bigcup_{i\in I} U_i$. Then there is $i \in I$ such that $x \in U_i$ and since U_i is open, there exists $r > 0$ such that $B(x, r) \subseteq U_i \subseteq U$. This shows that U is open.

(iv)-(vi) Homework! □

Remark 6.30. Properties (i)-(iii) are used to define topological spaces: a topological space is a set X with a collection of subsets $\tau = \{U_\alpha\}_\alpha$, called a topology, which satisfy (i)-(iii). The U_α are called open subsets by definition.

Remark 6.31. The intersection of infinitely many open sets is not open in general. Take $U_n \doteq \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

but $\{0\}$ is not open. Indeed, for every $r > 0$, the ball $(-r, r)$ is not contained in $\{0\}$.

6.4.3 Characterisations of Openness and Closedness: Neighbourhoods and Adherent Points

Definition 6.32 (Neighbourhood)**.** *Let X be a metric space. If $x \in X$, we say that $A \subseteq X$ is a neighbourhood of x if there exists an open subset $U \subseteq A$ with $x \in U$. In particular, an open neighbourhood of x is an open set $U \subseteq X$ with $x \in U$.*

We can study convergence of sequences via open neighbourhoods:

Lemma 6.33. *Let $(x_n)_n$ be a sequence in a metric space X. Suppose U is an open neighbourhood of x and $x_n \to x$. Then, $\exists N$ such that, for all $n \geq N$, $x_n \in U$.*

Proof. For an open neighbourhood U, we must have $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. From the definition of convergence $(x_n)_n$ to x, there exists an $N(\epsilon) \in \mathbb{N}$, such that for all $n \geq N$, $d(x_n, x) < \epsilon$. This means that for all $n \geq N$, $x_n \in B_\epsilon(x) \subseteq U$. □

Definition 6.34. *Let X be a metric space and $A \subseteq X$.*

i) The point $x \in X$ is an adherent point of A if for all $\epsilon > 0$, the ball $B_\epsilon(x)$ has non-empty intersection with A.

ii) The point $x \in X$ is called a limit point (or accumulation point) if it is adherent to $A \backslash \{x\}$.

iii) If $x \in A$ and there exists $\delta > 0$ such that $d(x, y) > \delta$ for all $y \in A \setminus \{x\}$ then we say x is an isolated point.

Proposition 6.35. *Let X be a metric space and $A \subseteq X$. If the point $x \in X$ is adherent then x is either a limit point or an isolated point. Conversely, if x is either a limit point or an isolated point then it is adherent.*

Proof. Note that the second statement follows by definition (limit and isolated points are adherent by definition).

Suppose $x \in X$ is an adherent point of A but not a limit point. We will show that it must be an isolated point. Indeed, we have that there exists an $\epsilon > 0$, $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$, since it is not a limit point, but $B_\epsilon(x) \cap A \neq \emptyset$ since it is adherent. Therefore, $x \in A$ since it is in the intersection of $B_\epsilon(x)$ and A. Additionally, since $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$, we have that for all $y \in A \setminus \{x\}$ we have

$$d(x, y) \geq \varepsilon.$$

Therefore, letting $\delta = \frac{\varepsilon}{2}$ it follows that x is an isolated point. □

Lemma 6.36. *Let X be a metric space and $A \subseteq X$. The point $x \in X$ is a adherent point of A if and only if there exists a sequence $(x_n)_n$ such that $x_n \in A$ for all n and $x_n \to x$.*

To prove this lemma we will use *axiom of choice*. Informally put, the axiom of choice says that given any collection of sets, each containing at least one element, it is possible to construct a new set by arbitrarily choosing one element from each set, even if the collection is infinite! More formally,

Definition 6.37 (Axiom of Choice)**.** *Given a family of nonempty sets A_α indexed by $\alpha \in I$, there exists a choice function $f : I \to \bigcup_\alpha A_\alpha$ such that $f(\alpha) \in A_\alpha$ for all α.*

Example 6.38. *Here we discuss a few examples of choice functions:*

1. *Let* $\{\{1,2\},\{3,4\},\{5,6\}\}$*, then we have* 3 *sets* $A_i = \{i, i+1\}$ *for* $i \in I = \{1,3,5\}$*. we can define a choice function* $f : I \to \bigcup_{i\in I} A_i = \{1,2,3,4,5,6\}$ *by*
$$f(1) = 2, \qquad f(3) = 3, \qquad f(5) = 6.$$
2. *Let* $A_i = \{i, i+1\}$ *for* $i \in I = \{2k+1 : k \in \mathbb{N}_1\}$*. We can define a choice function* $f : I \to \bigcup_{i\in I} A_i = \mathbb{N}$ *by*
$$f(i) = \frac{i-1}{2}$$
3. *Suppose* $A_\alpha = (\alpha, \alpha+1)$ *for any* $\alpha \in R$*. we can define a choice function* $f(\alpha) = \alpha + \frac{1}{2}$*.*
4. *Let* $\{A_\alpha\}_\alpha$ *be the collection of all non-empty subsets of the* $\mathbb{R}$*. How does one* ***construct*** *a choice function? However, the axiom of choice tells that there exists such a function.*

Remark 6.39. Many mathematicians issue with the axiom of choice is that it allows one to proofs in non-constructive manners. It also can be proven to be logically equivalent to non-intuitive results.

In this course we only need a weaker form of the axiom of choice:

Definition 6.40 (Axiom of Countable Choice)**.** *Given a family of nonempty sets* A_i *indexed by* $i \in \mathbb{N}$*, there exists a choice function* $f : \mathbb{N} \to \bigcup_{i\in\mathbb{N}} A_i$ *such that* $f(i) \in A_i$ *for all* $i \in \mathbb{N}$*, i.e. there exists a sequence* $(x_n \doteq f(n))_{n\in\mathbb{N}}$ *in* $\bigcup_{i\in\mathbb{N}} A_i$ *such that* $x_n \in A_n$ *for each* $n \in \mathbb{N}$*.*

Proof. ($\Rightarrow$) Suppose x is adherent. Then for all $n > 0$, $B_{1/n}(x) \cap A \neq \emptyset$. In each $B_{1/n}(x)$, we can pick $x_n \in B_{1/n}(x)$ (by the axiom of countable choice). Now let $\varepsilon > 0$ and pick $N \geq \frac{1}{\varepsilon}$. Then, for any $n \geq N$
$$x_n \in B_{\frac{1}{N}}(x) \implies d(x_n, x) < \frac{1}{N} \leq \varepsilon.$$

($\Leftarrow$) Suppose $x_n \to x \in X$ with $x_n \in A$ for all n. Then for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that, for $n \geq N$, $d(x_n, x) < \varepsilon$. Therefore, for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that, for $n \geq N$, $x_n \in B_\epsilon(x) \cap A$. Hence, for all $\epsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$, which means x is adherent. $\square$

Proposition 6.41. *Let* (X, d) *be a metric space and* $C \subseteq X$*. Then* C *is a closed subset if and only if every adherent point of* C *is an element of* C*.*

Proof. For the forward direction we proceed by contradiction and for the reverse we prove the contrapositive: C not closed $\implies$ there exists a adherent point of C not in C.

($\Rightarrow$) Suppose C is closed and let x be a adherent point of C but $x \notin C$. Then $X \setminus C$ is open by definition and $x \in X \setminus C$. By Lemma 6.36, we can find a sequence $(x_n)_n$ such that $x_n \in C$ with $x_n \to x$. Since $X \setminus C$ is an open neighbourhood of x, Lemma 6.33 gives us that $x_n \in X \setminus C$ for all $n \geq N$ for some N. However, we have reached a contradiction, since $x_n \in C$ for all n. So, in fact, $x \in C$.

($\Leftarrow$) Suppose that C is not closed. Therefore, $X \setminus C$ is not open (this is the contrapositive of 'C is closed $\implies$ $X \setminus C$ is open'). Assuming (without loss of generality, since X is closed) $C \neq X$ then we can find $x \in X \setminus C$ such that $B_\delta(x) \not\subset X \setminus C$ for any $\delta > 0$ (since $X \setminus C$ is not open). This means that $B_{1/n}(x) \cap C \neq \emptyset$ for all $n \in \mathbb{N}$. So choose $x_n \in B_{1/n}(x) \cap C$ for each n. This means that we have a sequence $(x_n)_n$ in C with $d(x_n, x) < \frac{1}{n} \to 0$. In otherwords, $x_n \to x$. So, by Lemma 6.36, x is a adherent point of C, which is not in C. $\square$

6.4.4 Closure, Interior and Boundary

Definition 6.42. *Let (X, d) be a metric space, $A \subseteq X$ be a set and denote:*

$$\mathscr{C}_A \doteq \{C \subseteq X : A \subseteq C \text{ and } C \text{is closed in } X\}.$$

The closure of A, denoted $\overline{A}$ or $\mathrm{cl}(A)$, is the intersection of all closed sets that contain A:

$$\overline{A} = \bigcap_{C \in \mathscr{C}_A} C.$$

Remark 6.43. The closure of A is the smallest closed set that contains A. It follows by proposition 6.29 that $\overline{A}$ is closed.

Proposition 6.44. *Let $A \subseteq X$. The closure of A is equal to the set of all adherent points of A.*

Proof. Let us denote the set of adherent points of a set S as $\mathrm{ad}(S)$. We want to show $\mathrm{cl}(A) = \mathrm{ad}(A)$.

"$\supseteq$" Suppose $A \subseteq C$ for a closed subset C of X. Then, it should be clear by definition, $\mathrm{ad}(A) \subseteq \mathrm{ad}(C)$. By proposition 6.41, since C is closed $\mathrm{ad}(C) = C$. Therefore, $\mathrm{ad}(A) \subseteq C$. Thus we have shown that if $C \in \mathscr{C}_A$ then $\mathrm{ad}(A) \subset C$, i.e. $\mathrm{ad}(A) \subset C$ for all $C \in \mathscr{C}_A$. Hence,

$$\mathrm{ad}(A) \subseteq \bigcap_{C \in \mathscr{C}_A} C = \mathrm{cl}(A)$$

where the last equality is by definition.

"$\subseteq$" Note that if $x \in A$ then $x \in \mathrm{ad}(A)$. So it suffices to show that $\mathrm{ad}(A)$ is closed then $\mathrm{ad}(A) \in \mathscr{C}_A$ because then $\mathrm{ad}(A) \in \mathscr{C}_A$. So, suppose $x \in X$ is a adherent point of $\mathrm{ad}(A)$. Then for any $\epsilon > 0$ there exists $y \in \mathrm{ad}(A)$ such that $d(x, y) < \varepsilon/2$. Now y is an adherent point of A, so there exists $z \in A$ such that $d(y, z) < \epsilon/2$. So, by the triangle inequality,

$$d(x, z) \leq d(x, y) + d(y, z) < \varepsilon.$$

Therefore, in fact, $x \in \mathrm{ad}(A)$. Therefore, any adherent point of $\mathrm{ad}(A)$ is an element of $\mathrm{ad}(A)$. So, by Proposition 6.41, $\mathrm{ad}(A)$ is closed. □

Definition 6.45. *Let (X, d) be a metric space, $A \subseteq X$ be a set and denote:*

$$\mathscr{O}_A \doteq \{O \subseteq X : O \subseteq A \text{ and } O \text{ is open in } X\}.$$

The interior of A, denoted $\mathrm{int}(A)$ or A^o, is the union of all open sets that contain A:

$$\mathrm{int}(A) = \bigcup_{O \in \mathscr{O}_A} O.$$

Remark 6.46. The interior of A is the largest open set that is contained in A. It follows by proposition 6.29 that A^o is open.

Proposition 6.47. *Let $A \subseteq X$. Then*

(i) $\mathrm{int}(A)$ *is an open subset of A,*

(ii) a point $x \in A$ is an interior point of A if there exists $r > 0$ such that $B_r(x) \subseteq A$.

(iii) *A is open if and only if* $A = \mathrm{int}(A)$,

(iv) $\mathrm{int}\,(\mathrm{int}(A)) = \mathrm{int}(A)$.

Proof. Homework. $\square$

Definition 6.48. *Given a set* $A \subseteq X$, *a point* $x \in X$ *is a* ***boundary point*** *of* A *if for every* $r > 0$ *the ball* $B(x, r)$ *contains at least one point of* A *and one point of* $X \setminus A$. *The set of boundary points of* A *is denoted* ∂A.

Theorem 6.49. *Let* $A \subseteq X$. *Then*

(i) $\overline{A} = A \cup \partial A$,

(ii) *A is closed if and only if it contains all its boundary points,*

(iii) $\partial A = \partial (X \setminus A)$,

(iv) $\partial A = \overline{(X \setminus A)} \cap \overline{A}$.

Proof. Exercise $\square$

Now, we prove a version of the Bolzano-Weierstrass theorem.

Theorem 6.50 (Bolzano-Weierstrass II)**.** *Let* $A \subseteq \mathbb{R}$ *be bounded with infinitely many elements. Then* A *has at least one limit point.*

Proof. We define a sequence as follows. Let $x \in A$. Inductively, assume that $x_1, \dots, x_k \in A$ are pairwise different. Then let $x_{k+1} \in A \setminus \{x_1, \dots, x_k\}$, which is not empty since A is infinite.

By construction $x_k \neq x_j$ for all $k \neq j$ and $x_n \in A$ for all $n \in \mathbb{N}$. Moreover, as A is bounded, the constructed sequence (x_n) is also bounded. By the Bolzano-Weierstrass theorem it contains a convergent subsequence (x_{n_k}) with limit $x \in \mathbb{R}$. We claim that x is a limit point of A.

Let $r > 0$. Then there exists $N \in \mathbb{N}$ such that for all $k \geq N$ it holds that $|x_{n_k} - x| < r$, i.e. $x_{n_k} \in B(x, r)$. As the sequence elements of (x_n) are pairwise different, it follows that

$$B(x, r) \cap \{x_{n_k} : k \in \mathbb{N}\} \neq \emptyset.$$

$\square$

6.4.5 Compactness

Compactness is a very useful property. You can think of it is the next best thing to finiteness. The definition is the following:

Definition 6.51 (Compact Set)**.** *Let* (X, d) *be a metric space. A set* $K \subseteq X$ *is compact if for every open cover of* K *(for every collection* $\{U_i\}$ *of open sets such that* $\bigcup_i U_i \supset K$*) there exists a finite subcover (a finite subcollection of* $\{U_i\}$ *whose union still contains* K*).*

Why is this useful? Well suppose you know that a property $\mathcal{P}$ of some mathematical object is true locally on K, i.e. you can find a (possibly infinite) collection of open sets which cover K such $\mathcal{P}$ holds on each of them. The question you may be faced with is: does $\mathcal{P}$ hold globally on K? If K is compact you can find a finite subcollection of your open sets, which again cover K, on which $\mathcal{P}$ holds. You're now dealing with a finite problem, which typically means its much easier to show that the property is true for all of K.

Example 6.52. *The family of sets* $\{(\frac{1}{n}, 1) : n = 1, 2, \ldots\}$ *is a collection of open sets which covers* $(0, 1)$*. Does it have a finite subcover? No. Since this collection of intervals is nested upward, any finite subcollection has a largest interval* $(\frac{1}{m}, 1)$ *for some* m*. Then the union of the sets in the subcollection is just* $(\frac{1}{m}, 1)$ *and this does not contain* $(0, 1)$*. This means that the set* $(0, 1)$ *is not compact.*

Example 6.53. *The family of sets* $\{(\frac{1}{n}, 2) : n = 1, 2, \ldots\}$ *together with a set* $(-1, t)$ *for some fixed* $t > 0$ *is a open cover of* $[0, 1] \subseteq \mathbb{R}$*. We can choose a finite subcover letting* N *be such that* $\frac{1}{N} < t$ *and by taking* $(-1, t)$ *together with* $\{(\frac{1}{n}, 2) : n = 1, 2, \ldots, N\}$*. Note that this does not prove that* $[0, 1]$ *is compact. Nevertheless,* $[0, 1]$ *is compact and we will show it due time.*

There is a very nice notion of compactness arising from a sequential definition that is often easier to work with and as we will state below often equivalent to Definition 6.51.

Definition 6.54. *A set* $K \subseteq X$ *is sequentially compact if for every sequence* $(x_n)_n \subseteq K$*, there exist a subsequence* (x_{n_k}) *of* (x_n) *and* $x \in K$ *such that* $x_{n_k} \to x$ *as* $k \to \infty$*.*

Example 6.55. *A finite set* $K \subseteq \mathbb{R}$ *is sequentially compact. The set* $E = [0, 1)$ *is not sequentially compact, since the sequence* $x_n = 1 - \frac{1}{n}$ *converges to* 1*, which does not belong to* E*. The problem here is that* E *is not closed.*

The set $F = [0, \infty)$ *is closed but not sequentially compact, since the sequence* $\{n\}$ *converges to* ∞*, which does not belong to* F*. The problem here is that* F *is not bounded.*

Theorem 6.56. *Let* (X, d) *be a metric space. Then* $K \subseteq X$ *is compact if and only if it is sequentially compact.*

Proof. Omitted. □

Remark 6.57. In metric spaces, both concepts agree, while in more general *topological spaces*, they do not necessarily.

The following theorems are the main results of this section. The second theorem is the *Heine-Borel theorem.*

Theorem 6.58. *Let* $K \subseteq X$ *be sequentially compact. Then* K *is closed and bounded.*

Proof. To see that K is closed, we consider the adherent points of K, since if K contains its adherent points it is closed. Let $(x_n)_n$ be a sequence in K such that $x_n \to x$ for some $x \in X$. Since K is sequentially compact, there exist a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and $y \in K$ such that $x_{n_k} \to y$. By the uniqueness of limits, it follows that $x = y \in K$, which shows that $K = \overline{K}$, i.e. K is closed.

To prove that K is bounded, we proceed by contradiction. Fix $x_0 \in X$. If K is not bounded, then for every $n \in \mathbb{N}$ we would find $x_n \in K$ such that $d(x_0, x_n) \geq n$, but then $d(x_0, x_n) \to \infty$. Take a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which converges to $x \in K$ and observe that

$$d(x_0, x_{n_k}) \leq d(x_0, x) + d(x, x_{n_k}) \leq d(x_0, x) + 1$$

for all sufficiently large $k \in \mathbb{N}$. The right-hand side is finite and, therefore, we have a contradiction to the fact that $d(x_0, x_{n_k}) \to \infty$, so K must be bounded. □

Theorem 6.59 (Heine-Borel)**.** *Let $K \subset \mathbb{R}^n$ with the usual metric. Then K is compact if and only if K is closed and bounded.*

Proof. Note that it is enough to prove K is sequentially compact if and only if K is closed and bounded since, by theorem 6.56, sequential compactness and compactness are equivalent.

($\Rightarrow$) Proved in Theorem 6.58.

($\Leftarrow$) (We will only prove this for $\mathbb{R}$). Assume that K is a closed and bounded subset of $\mathbb{R}$ and let $(x_n)_n$ be a sequence in K. Since K is bounded, so is $(x_n)_n$. Hence, by the Bolzano-Weierstrass theorem there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \to x \in \mathbb{R}$. Therefore, x is an adherent point. Since K is closed every adherent point is an element of K. Therefore, $x \in K$. This shows that K is sequentially compact. □

Remark 6.60. Closed and bounded $\implies$ compact holds on any **finite dimensional** normed space due to the following observation we won't prove but you can find in courses on linear algebra and functional analysis: any finite dimensional normed space is isomorphic to $\mathbb{R}^n$ with the usual norm.

However, this property fails in general metric spaces. In particular, it fails for infinite dimensional spaces incomplete metric spaces and can fail even for $\mathbb{R}$ without its usual metric.

Example 6.61. *The set $E = (-\infty, 1]$ is not compact. For this consider the open cover $U_i = (-i, 2)$ for $i \in \mathbb{N}$. The problem here is that E is unbounded.*

Example 6.62. *The set $E = (-1, 1]$ is not compact. Consider the family of open sets $U_n = \left(-1 + \frac{1}{n}, 2\right)$. Then*

$$\bigcup_{n=1}^{\infty} \left(-1 + \frac{1}{n}, 2\right) = (-1, 2) \supseteq E,$$

but if we consider a finite number of the sets U_n, say, $U_{n_1}, \ldots, U_{n_\ell}$, letting $N = \max\{n_1, \ldots, n_\ell\}$, we have that

$$\bigcup_{k=1}^{\ell} \left(-1 + \frac{1}{n_k}, 2\right) = \left(-1 + \frac{1}{N}, 2\right),$$

which does not cover E. The problem is that E is not closed.

Corollary 6.63. *Suppose $A \subseteq \mathbb{R}$ is compact. Then A has a maximum.*

Proof. Since A is compact, the Heine-Borel theorem 6.59 tells us that A is bounded. So by the supremum property of $\mathbb{R}$, there exists $a \in \mathbb{R}$ such that $a = \sup A$. It now suffices to show that $a \in A$ since by definition $a \geq x$ for all $x \in A$.

Suppose that $a \notin A$. Then $a \in \mathbb{R} \setminus A$, which is a open set since A is closed. This means that $\exists \varepsilon > 0$ such that $B_\varepsilon(a) \subseteq \mathbb{R} \setminus A$. Therefore, $|x - a| \geq \varepsilon$ for all $x \in A$. Noting that, since a is an upper bound, $x - a < 0$ gives $x \leq a - \varepsilon$ for all $x \in A$. This contradicts the statement that a is the supremum. Hence, $a \in A$. □

7 Continuous Functions on Metric Spaces

Our discussion on metric spaces allows us to discuss continuous functions and establish many familiar properties from functions on the real line. We begin with extending the definition of continuity to a function between metric spaces. There are three notions of continuity we will discuss. In metric spaces, these three notions are all equivalent. In more general topological spaces, they are not necessarily.

7.1 Definitions and Characterisations

Definition 7.1 (Continuity of a Function). *Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f : X \to Y$ is continuous at $x \in X$ if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon, x) > 0$, such that for all $y \in B(x, \delta)$, we have $d(f(y), f(x)) < \varepsilon$. The function f is called continuous if it is continuous at all $x \in X$.*

Definition 7.2 (Sequential Continuity of a Function). *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called sequentially continuous at a point $x \in X$ if $f(x_n) \to f(x)$ in d_Y whenever $x_n \to x$ in d_X. The function f is called sequentially continuous if it is sequentially continuous at all $x \in X$.*

Theorem 7.3 (Characterisations of Continuity for Metric Spaces). *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function. The following are equivalent:*

(i) The function f is continuous on X.

(ii) The function f is sequentially continuous on X.

(iii) For any closed subset $C \subseteq Y$, the pre-image $f^{-1}(C) = \{x \in X : f(x) \in C\}$ is closed in X.

(iv) For any open subset $U \subseteq Y$, the pre-image $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X.

Remark 7.4. The characterisation (iv) of continuity in terms of open sets is the usual definition of continuity for topological spaces. Note that this is a statement about the preimage of open sets; images of open sets are usually not open.

Proof. (i)⇒(ii) Assume that f is continuous at $x \in X$ and that $x_n \to x$. Let $\varepsilon > 0$. By the definition of continuity, we can choose $\delta > 0$ such that if $y \in X$ with $d_X(y, x) < \delta$ then this implies that $d_Y(f(y), f(x)) < \varepsilon$. Additionally, by the definition of convergence, there exists $N \in \mathbb{N}$ such that $d_X(x_n, x) < \delta$ for all $n \geq N$. Therefore, combining these two definitions (with $y = x_n$) gives

$$d_Y(f(x_n), f(x)) < \varepsilon \qquad \forall\, n \geq N \implies f(x_n) \to f(x).$$

(ii)⇒(iii) Suppose $C \subseteq Y$ is closed and $x_n \to x$ with $x_n \in f^{-1}(C)$ for all n. Then x is an adherent point of $f^{-1}(C)$. Since f is sequentially continuous, $y_n \doteq f(x_n) \to f(x)$ and $y_n \in C$ for all n. Therefore, $f(x)$ is an adherent point of C. By proposition 6.41, since C is closed, $f(x) \in C$. Therefore, $x \in f^{-1}(C)$. So, $f^{-1}(C)$ contains all adherent points which, by proposition 6.41, implies $f^{-1}(C)$ is closed.

(iii)⇒(iv) Suppose $O \subseteq Y$ is open. Then $Y \setminus O$ is closed. So by (iii), the preimage of $Y \setminus O \subseteq Y$ is closed in X. This means $f^{-1}(Y \setminus O) = X \setminus f^{-1}(O)$ is closed, which implies $f^{-1}(O)$ is open in X.

(iv)⇒(i) Assume that the pre-images of open sets are open. Let $x \in X$ and $\varepsilon > 0$. Since $B_\varepsilon(f(x))$ is open, also its pre-image $f^{-1}(B_\varepsilon(f(x)))$ is open. Clearly, $x \in f^{-1}(B_\varepsilon(f(x)))$. Additionally, since $f^{-1}(B_\varepsilon(f(x)))$ is an open set, it means we can find an open ball in contained in this set, i.e. there exists $\delta > 0$ such that

$$B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0))).$$

This says that if $y \in B_\delta(x_0)$ then $f(y) \in B_\varepsilon(f(x_0))$, i.e. if $y \in X$ such that $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$. Hence, f is continuous at x. Since x was generic, f is continuous at x for all $x \in X$ and thus continuous. □

Example 7.5. *For $f : \mathbb{R} \to \mathbb{R}$ continuous and $t \in \mathbb{R}$ the set $\{x \in \mathbb{R} : f(x) < t\} = f^{-1}((-\infty, t))$ is open.*

Example 7.6. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x^2 - 3x + 2$ *is continuous,*

Example 7.7. $f : (0, \infty) \to (0, \infty), \quad f(x) = \frac{1}{x}$ *is continuous,*

Example 7.8. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ *is not continuous.*

More examples are given by all power series within their radius of convergence as the next theorem shows.

Theorem 7.9. *Let $\sum_{k=0}^{\infty} a_k x^k$ be a power series and $R > 0$ its radius of convergence. Then the function $f : (-R, R) \to \mathbb{R}$ defined by $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is continuous.*

Proof. Homework. □

Remark 7.10. The theorem above shows that the exponential function, cos, and sin are continuous.

Theorem 7.11. *Let X, Y, Z be metric spaces and $f : X \to Y$ be continuous in $x_0 \in X$ and $g : Y \to Z$ continuous in $f(x_0)$. Then their composition, $g \circ f : X \to Z$, defined by $(g \circ f)(x) \doteq g(f(x))$, is continuous in x_0.*

Proof. Let (x_n) a sequence in X such that $x_n \to x_0$. By the continuity of f it follows that $f(x_n) \to f(x_0)$. The continuity of g implies then that $g \circ f(x_n) = g(f(x_n)) \to g(f(x_0)) = g \circ f(x_0)$. □

Finally, we end this section on characterisations of continuity by giving a familar definition from calculus in one variable. Namely, a function is continuous at a point if the value of the function at that point agrees with the limit of the function at that point. First we need to define what we mean by a limiting value of a function.

Definition 7.12. *Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $x_0 \in X$ an adherent point of E and $f : E \to Y$ a function. We say that $f(x)$ converges to L in Y as x converges to x_0 in E, and write*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = L,$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_X(x, x_0) < \delta$ for $x \in E$ then $d_Y(f(x), L) < \varepsilon$.

Remark 7.13. We will often write $\lim_{x \to x_0; x \in E} f(x)$ as $\lim_{x \to x_0} f(x)$ when it's clear what space x lies in.

Proposition 7.14. *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a function. The function f is continuous at $x_0 \in X$ if and only if*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Thus, f is continuous on X if and only if $\lim_{x \to x_0} f(x) = f(x_0)$ for all $x_0 \in X$.

Proof. Compare the definitions. □

Theorem 7.15. *Let (X, d) be a metric space and $A, B \subseteq X$. Moreover, let $f : A \cup B \to \mathbb{R}$. Let $x \in A \cup B$ be an limit point of A and B. Then f is continuous at x_0 if*

$$\lim_{\substack{x \to x_0 \\ x \in A}} f(x) = \lim_{\substack{x \to x_0 \\ x \in B}} f(x) = f(x_0).$$

Example 7.16. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ x^3 & \text{if } x > 0. \end{cases}$$

Clearly f is continuous in all $x \neq 0$. Moreover, we see easily that

$$\lim_{\substack{x \to 0 \\ x > 0}} f(x) = \lim_{\substack{x \to 0 \\ x \leq 0}} f(x) = 0 = f(0).$$

Hence, by the previous theorem the function is continuous in 0.

Remark 7.17. We will write $\lim_{x \to 0^-}$ for $\lim_{x \to 0, x \leq 0}$ and $\lim_{x \to 0^+}$ for $\lim_{x \to 0, x \geq 0}$.

7.2 Compactness and Continuity

Continuity and compactness interact in very nice ways. For example, continuous functions map compact sets to compact sets. In particular, we have the following theorem.

Theorem 7.18. *Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$ a continuous function and $K \subseteq X$ compact. Then $f(K) \subseteq Y$ is compact.*

Proof. In view of theorem 6.56, it suffices to prove that $f(K)$ is sequentially compact. Let y_n be a sequence in $f(K) \subset Y$. Then by definition, for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that $y_n = f(x_n)$. Since K is (sequentially) compact, there exists a subsequence $(x_{n_k})_k$ and $x \in K$ such that $x_{n_k} \to x$. Since f is continuous (and, therefore, sequentially continuous by theorem 7.3),

$$y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$$

since $x \in K$. Therefore, $f(K)$ is sequentially compact. $\square$

Remark 7.19. Pre-images of compact sets are usually not compact.

As a corollary, we show here that continuous and real-valued functions on a compact domain attain their extrema.

Definition 7.20 (Bounded Function)**.** *Let (X, d_X) be a metric space and $f : X \to \mathbb{R}$ (where $\mathbb{R}$ is endowed with the usual metric). The function f is bounded on $S \subseteq X$ if there exist $C \in \mathbb{R}$ such that $|f(x)| \leq C$ for all $x \in S$.*

Theorem 7.21 (Extreme Value Theorem)**.** *Let (X, d) be a metric space and $K \subseteq X$ compact. Moreover, let $f : K \to \mathbb{R}$ be continuous. Then f is bounded and there exist $a, b \in K$ such that $f(a) = \sup_{x \in K} f(x)$ and $f(b) = \inf_{x \in K} f(x)$*

Proof. By theorem 7.18, if K is compact then the image of K, $f(K) \subseteq \mathbb{R}$, is compact in $\mathbb{R}$. Therefore, $f(K)$ is closed and bounded, i.e. f is bounded on K. By corollary 6.63, $f(K)$ has a maximum: $s = \sup(f(K)) \in f(K)$. Therefore, there exists $a \in K$ such that $f(a) = s$. $\square$

7.3 Continuous Functions on the Reals

Let's specialise to continuous functions that map from $A \subseteq \mathbb{R}$ to $\mathbb{R}$.

Proposition 7.22. *Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $x \in A$. Then,*

(i) $f + g$, defined as $(f + g)(x) = f(x) + g(x)$, is continuous at x.

(ii) $f \cdot g$, defined as $(f \cdot g)(x) = f(x)g(x)$, is continous at x.

(iii) if $g(x) \neq 0$, defined as $({}^{f}/_{g})(x) = {}^{f(x)}/_{g(x)}$, ${}^{f}/_{g}$ is continuous at x.

Proof. By theorem 7.3, we can prove sequential continuity to prove continuity since these are equivalent on metric spaces and here we are dealing with the metric space $(\mathbb{R}, |\cdot|)$. This result is now proved from the properties of sequences 4.14. For example, let $(x_n)_n$ be a sequence in A such that $x_n \to x$. By the continuity of f and g and the addition property of limits,

$$(f + g)(x_n) = f(x_n) + g(x_n) \to f(x) + g(x) = (f + g)(x).$$

The rest are analogous. $\square$

Remark 7.23. Once you convince yourself that constant functions and $f(x) = x$ are continuous, induction in conjunction with proposition 7.22 tells you that polynomials $p(x)$ are continuous. Additionally, it tells us that all rational functions $\frac{p(x)}{q(x)}$ are continuous except where $q(x) = 0$.

We have already prove the extreme value theorem which states that if a continuous function on a closed and bounded interval is bounded then it attains its bounds. Another nice theorem about continuous functions is the intermediate value theorem, which states that, if a continuous function is negative and positive on an interval, it must be 0 somewhere on that interval. More precisely:

Theorem 7.24 (Intermediate Value Theorem)**.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) < 0 < f(b)$. Then, there exists $x \in (a, b)$ such that $f(x) = 0$.*

Proof. Let $A \doteq \{x \in [a, b] : f(x) < 0\}$ and denote $s \doteq \sup(A)$. Clearly, $A \subseteq [a, b]$ and $s \in [a, b]$ since $f(a) < 0 < f(b)$. Our strategy is to show that $f(s)$ cannot be negative or positive and, therefore, $f(s) = 0$. The basic idea here is that if f is negative at s, $f(s) < 0$, continuity then implies that f is negative in a small neighbourhood around s. This means that there is another point $s + \Delta \in A$ with $\Delta > 0$ where $f(s + \Delta) < 0$ then s is not an upper bound for A which is a contradition. An analogous argument works for $f(s) > 0$.

Suppose $f(s) < 0$. Continuity of f at s means that for any $\varepsilon > 0$ we can find a $\delta = \delta(\varepsilon, s) > 0$ such that if $|y - s| < \delta$, then $|f(y) - f(s)| < \varepsilon$. So, in particular, for $\varepsilon = {}^{|f(s)|}/_{2}$, we can find a δ such that $|f(y) - f(s)| < {}^{|f(s)|}/_{2} = -\frac{f(s)}{2}$. This means that if $|y - s| < \delta$ then

$$\frac{f(s)}{2} < f(y) - f(s) < -\frac{f(s)}{2} \implies f(y) < \frac{f(s)}{2}.$$

So, in particular, $f(y = s + {}^{\delta}/_{2}) < 0$. Therefore, any $y \in [s, s + \delta)$ is also in A and, therefore, s is not an upper bound for A. This is a contradiction and, therefore, $f(s) \geq 0$.

Suppose $f(s) > 0$. Continuity of f at s means that for $\varepsilon = {}^{|f(s)|}/_{2}$ we can find a $\delta = \delta(\varepsilon, s) > 0$ such that if $|y - s| < \delta$, then $|f(y) - f(s)| < {}^{|f(s)|}/_{2} = \frac{f(s)}{2}$. Again, this means if $|y - s| < \delta$ then

$$-\frac{f(s)}{2} < f(y) - f(s) < \frac{f(s)}{2} \implies f(y) > \frac{f(s)}{2}.$$

Therefore, for any $y \in (s - \delta, s]$, $f(y) > 0$. Hence, $y \notin A$, which means $s - {}^{\delta}/_{2}$ is a smaller upper bound than s. This contradicts s being the supremum/least upper bound. So $f(s)$ cannot be strictly positive. $\square$

Corollary 7.25. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) < c < f(b)$. Then, there exists $x \in (a, b)$ such that $f(x) = c$.*

Proof. Define g as the function $g(x) = f(x) - c$. Then g is a continuous function on $[a, b]$ with $g(a) < 0 < g(b)$. Then we know from the intermediate value theorem that there exists $x \in (a, b)$ such that $g(x) = 0$. Hence $f(x) = c$. $\square$

Remark 7.26. Note that the converse of the intermediate value theorem is not true. Suppose $f : [a, b] \to \mathbb{R}$ and that f satisfies that for any $c \in \mathbb{R}$ such that $f(a) < c < f(b)$ there exists $x \in [a, b]$ such that $f(x) = c$. It is not true that f must be continuous. For example, $\sin({}^{1}/_{x})$ on $[-1, 1] \setminus \{0\}$ is not continuous at 0 no matter what you define the function to be there.

Remark 7.27. Here $[a, b]$ is compact by Heine-Borel. However, this is not the essential requirement here. It is the fact that $[a, b]$ has the topological property of connectedness. Consider the function:

$$f : [1, 2] \cup [3, 4] \to \mathbb{R} \qquad f(x) = \begin{cases} -1 & x \in [1, 2] \\ 1 & x \in [3, 4]. \end{cases}$$

This is clearly continuous and $[1, 2] \cup [3, 4]$ is a compact set. However, it is disconnected. Clearly, there is no $x \in [1, 2] \cup [3, 4]$ such that $f(x) = 0$.

The Intermediate Value Theorem implies that the image of an interval under a continuous function is also an interval.

Corollary 7.28. *Suppose $f : [a, b] \to \mathbb{R}$ continuous. Then the image of $[a, b]$ under f is an interval.*

Proof. Take $y_1, y_2 \in f([a, b])$. Then there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) = y_1, f(x_2) = y_2$. Up to relabelling, we may assume that $x_1 < x_2$. Because $[a, b]$ is an interval, we find that $[x_1, x_2] \subset [a, b]$. Now let y between y_1 and y_2 and observe that by the intermediate value theorem, there exists $x \in [x_1, x_2]$ such that $y = f(x)$. This means that also $y \in f([a, b])$, so every point between y_1 and y_2 also belongs to $f([a, b])$. This means that $f([a, b])$ is an interval. $\square$

Corollary 7.29. *Suppose $f : [a, b] \to [c, d]$ is continuous and strictly increasing with $f(a) = c$ and $f(b) = d$. Then the inverse of f, f^{-1}, exists and is also continuous.*

Proof. Since f is increasing it is an injection: if $x \neq y$ then up to a relabelling $x < y$, so f being strictly increasing implies that $f(x) < f(y)$ so $f(x) \neq f(y)$.

The intermediate value theorem gives that it is a surjection: for any $y \in (f(a) = c, f(b) = d)$, there exists $x \in (a, b)$ such that $f(x) = y$.

By the above f is a bijection and, hence, invertible. We now prove that f^{-1} has to be strictly increasing also. Let $y_1 < y_2$, then since f is bijective (and, therefore, so is f^{-1}) there exist $x_1, x_2 \in [a, b]$ such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Therefore, $f(x_1) = y_1$ and $f(x_2) = y_2$. Assume that $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$ since f is strictly increasing and hence $y_1 \geq y_2$, which is a contradiction.

Now we argue that f^{-1} is continuous at $y \in [c, d]$, i.e. for any $\varepsilon > 0$, there exists a $\delta(y, \varepsilon) > 0$ such that, if $|z - y| < \delta$, then $|f^{-1}(z) - f^{-1}(y)| < \varepsilon$. Let $x = f^{-1}(y)$ and $\varepsilon > 0$, so that $f(x) = y$. Since f is strictly increasing $f(x-\varepsilon) < f(x) = y < f(x+\varepsilon)$. So we have an open interval containing $y = f(x)$. Therefore, $\exists \delta > 0$ such that $(y - \delta, y + \delta) \subseteq (f(x - \varepsilon), f(x + \varepsilon))$. So, if $z \in (y - \delta, y + \delta)$, then $f^{-1}(z) \in (f^{-1}(y - \delta), f^{-1}(y + \delta)) \subseteq (x - \varepsilon, x + \varepsilon)$ since f^{-1} is strictly increasing. $\square$

Remark 7.30. This shows that $\sqrt{x}$ is continuous when x^2 is strictly increasing (when $x > 0$).

7.4 Stronger Notions of Continuity: Uniform, Hölder and Lipschitz Continuity

Next we introduce the notion of uniform continuity. Uniform continuity is stronger than continuity. For uniform continuity, δ may no longer depend on the point $x \in X$; there must be one $\delta > 0$ for all $x \in X$, i.e. that we may choose δ uniformly in x. The notion will see application when we study the Riemann integral.

Definition 7.31. *Let (X, d) be a metric space, $A \subseteq X$ and $f : A \to \mathbb{R}$ be a function. Then we say f is uniformly continuous on A if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $d(x, y) < \delta$ we have $|f(x) - f(y)| < \varepsilon$.*

Proposition 7.32. *Let (X, d) be a metric space, $A \subseteq X$ and $f : A \to \mathbb{R}$ be a function. If f is uniformly continuous on A then f is continuous on A.*

Proof. Write out the definitions. □

Remark 7.33. If you like to think in quantifiers you can see the distinction between uniform continuity and continuity in the placement of $\forall x \in A$:

1. Continuity on A:

$$\forall x \in A, \forall \epsilon > 0, \exists \delta > 0 : \forall y \in A : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$

2. Uniform continuity on A:

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, y \in A : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

So continuity is a local property of a function and uniform continuity is a global property.

Example 7.34. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. For $x, y \in \mathbb{R}$ we find*

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y|.$$

Now let $\delta > 0$ and $n \geq \delta^{-1}$. Set $x = n$, $y = n + \frac{\delta}{2}$. Then $|x - y| < \delta$ but $|f(x) - f(y)| \geq 1$. Hence, f is not uniformly continuous. The problem in this particular case seems to be that the domain of f is not bounded. Indeed, if we restrict f to some bounded interval $[a, b]$ then it becomes uniformly continuous (simply choose $\delta = \frac{\varepsilon}{2\max\{|a|,|b|\}}$).

Example 7.35. *Let $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. We claim that f is not uniformly continuous. Indeed, let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. By choosing n large enough, we can make the distance between x_n and y_n as small as we want. On the other hand it holds for all n that*

$$|f(x_n) - f(y_n)| = |n - (n + 1)| = 1.$$

The problem here seems to be the fact that f is not continuous in 0 and therefore values close to zero can have very different function-values. So in some sense the problem is the non-closedness of the domain.

Conversely to proposition 7.32, continuity implies uniform continuity on compact domains. Here we will see an application of compactness being the next best thing to finiteness and extraction of a local property to a global one.

Theorem 7.36. *Let (X,d) be a metric space, $K \subseteq X$ be compact and $f : K \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$. By continuity, we can find, for each $x \in K$, a $\delta(x) > 0$ such that for all $y \in B_{\delta(x)}(x)$ it holds that $f(y) \in B_{\varepsilon/2}(f(x))$. The balls $B_{\delta(x)/2}(x)$ form an open cover of K:

$$K \subseteq \bigcup_{x \in K} B_{\frac{\delta(x)}{2}}(x).$$

Thus, compactness allows us to find finitely many points $x_1, \ldots, x_N \in K$ such that $K \subseteq \bigcup_{i=1}^{N} B(x_i, \delta(x_i)/2)$.

We now show that two points sufficently close lie in the same i^{th} ball of the subcover and, therefore, continuity at x_i implies uniform continuity. Define $\delta = \frac{1}{2}\min\{\delta(x_1), ..., \delta(x_N)\} > 0$ and let $z, y \in K$ such that $d(z,y) < \delta$. First, there exists $i \in \{1, \ldots, N\}$ such that $z \in B_{\delta(x_i)/2}(x_i)$. As $\delta \leq \frac{1}{2}\delta(x_i)$, the triangle inequality implies

$$d(y, x_i) \leq d(x,y) + d(x, x_i) < \delta + \frac{\delta(x_i)}{2} \leq \delta(x_i) \implies d(y, x_i) < \delta(x_i).$$

So, we have shown that $y, z \in B_{\delta(x_i)}(x_i)$.

Our function is continuous at x_i so $d(f(z), f(x_i)) < \varepsilon/2$ and $d(f(y), f(x_i)) < \frac{\varepsilon}{2}$. Using the triangle inequality,

$$d(f(x), f(y)) \leq d(f(x), f(x_i)) + d(f(x_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Simple examples of uniformly continuous functions are α-Hölder and Lipschitz continuous functions, which see application in functional analysis and solving PDE.

Definition 7.37. *Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subseteq X$ and $f : A \to Y$ a function. Then we say that f is α-Hölder continuous on A if there exists $C > 0$ and $\alpha > 0$ such that*

$$d_Y(f(x), f(y))| \leq C d_X(x,y)^\alpha,$$

for all $x, y \in A$. If f is α-Hölder continuous with $\alpha = 1$, we say that f is Lipschitz continuous.

Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subseteq X$ we will denote:

- $C^0(A, Y)$ to be the space of continuous functions with domain A and codomain Y.
- $C^{0,\alpha}(A, Y)$ to be the space of α-Hölder continuous functions with domain A and codomain Y.
- $C^{0,1}(A, Y)$ to be the space of Lipschitz continuous functions with domain A and codomain Y.
- $C^{0,1}_{\text{loc}}(A, Y)$ to be the space of locally Lipschitz continuous functions with domain A and codomain Y. A function is locally Lipschitz on A if for each $x \in A$ there exists a neighbourhood U of x such that f restricted to U is Lipschitz. (You can also localise the definition of Hölder too).

If $Y = \mathbb{R}$ we will often drop the Y in the above notation.

Remark 7.38. Hölder spaces see use in the analysis of PDE. They interact nicely with compactness, i.e. bounded sequences have convergent subsequences. If one has a PDE with appropriately regular (differentiable) co-efficients then often one knows a priori (beforehand) that any solution will have 'good' regularity properties. Lipschitz continuity is often enough to give existence and uniqueness of solutions for ordinary differential equations.

We will see later that the Lipschitz condition can be linked to the boundedness of the derivative.

Proposition 7.39. *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be Lipschitz continuous. Then f is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. Since f is Lipschitz continuous, there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|,$$

for all $x, y \in E$. Now pick $\delta = \frac{\varepsilon}{2C}$. If $x, y \in E$ with $|x - y| \leq \delta$, then

$$|f(x) - f(y)| \leq C|x - y| \leq C\delta = C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2}.$$

□

On the other hand not every uniformly continuous function is Lipschitz continuous.

Example 7.40. *The function $f(x) = \sqrt{x}$, $x \in [0, 1]$, is uniformly continuous (as $[0, 1]$ is compact), but not Lipschitz. Indeed, let $x_n = 0$ and $y_n = \frac{1}{n}$. Then*

$$\left|\frac{f(x_n) - f(y_n)}{x_n - y_n}\right| = \left|\frac{0 - \sqrt{\frac{1}{n}}}{0 - \frac{1}{n}}\right| = \sqrt{n} \to \infty,$$

which shows that f is not Lipschitz continuous.

Proposition 7.41. *We have the following inclusions:*

$$C^{0,1}([a, b]) \subsetneq C^{0,\alpha}([a, b]) \subsetneq \{\textit{uniformly continuous functions on } [a, b]\} = C^0([a, b]),$$

where $\alpha \in (0, 1]$.

Proof. We have already proved {uniformly continuous functions on $[a, b]$} $= C^0([a, b])$.

We will now show that $C^{0,1}([a, b]) \subseteq C^{0,\alpha}([a, b])$. We have

$$|f(x) - f(y)| \leq C|x - y| = C|x - y|^\alpha \left(\frac{|x - y|}{|b - a|}\right)^{1-\alpha} |b - a|^{1-\alpha}$$

Now since $|x - y| \leq |b - a|$, we have $\frac{|x-y|}{|b-a|} \leq 1$. So,

$$|f(x) - f(y)| \leq K|x - y|^\alpha,$$

where $K = C|b-a|^{1-\alpha}$.

Finally, we will show $C^{0,\alpha}([a,b]) \subseteq \{\text{uniformly continuous functions on } [a,b]\}$. Fix $\varepsilon > 0$. Since f is Hölder continuous, there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C|x-y|^{\alpha},$$

for all $x, y \in [a,b]$. Now pick $\delta = (\frac{\varepsilon}{2C})^{\frac{1}{\alpha}}$. If $x, y \in [a,b]$ with $|x-y| \leq \delta$, then

$$|f(x) - f(y)| \leq C|x-y|^{\alpha} \leq C\delta^{\alpha} = C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2} < \varepsilon.$$

The strictness of these inclusions is an exercise: construct counterexamples. □

8 Differentiation of Functions on the Real Line

8.1 Definition and Basic Properties

Definition 8.1 (Derivative/Differentiability)**.** *Let $A \subseteq \mathbb{R}$, f be a function $f : A \to \mathbb{R}$ and $a \in A$ be a limit point of A. We say that f is differentiable at a with derivative $\lambda \in \mathbb{R}$ if*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lambda \quad \text{or equivalently} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lambda$$

and write $^{df}/_{dx}(a) = f'(a) = \lambda$. We say that f is differentiable *on A if, for every limit point $a \in A$, the function f is differentiable at a.*

If the limit does not exist, or $a \notin A$ or a is not a limit point of A then we say that $f'(a)$ is undefined and f is not differentiable at a.

Remark 8.2. You may ask why we insist on a being a limit point. By Proposition 6.35, a is either a limit or an isolated point. If a is an isolated point,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is automatically undefined.

Remark 8.3. Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are differentiable at $a \in A$ and $f = g$ on A then $f' = g'$ on A. However, suppose that $f \neq g$ on all of A except at $a \in A$, i.e. merely $f(a) = g(a)$.This does *not* imply that $f'(a) = g'(a)$. For example, $f(x) = x$ and $g(x) = 1$ on $[0, 2]$. Clearly, $f(1) = g(1)$ but $f'(1) = 1$ and $g'(1) = 0$. Differentiability is about the local behaviour of a function around a point, not its values at a point.

Example 8.4. *Lets consider the differentiability of $f(x) = |x|$ at 0 on $\mathbb{R}$ or $[0, \infty)$. Consider*

$$\lim_{x \to 0, x \in \mathbb{R}} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0, x \in \mathbb{R}} \frac{\operatorname{sign}(x) x}{x} = \lim_{x \to 0, x \in \mathbb{R}} \operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

However,

$$\lim_{x \to 0, x \in [0,\infty)} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0, x \in [0,\infty)} \frac{\operatorname{sign}(x) x}{x} = \lim_{x \to 0, x \in x \in [0,\infty)} \operatorname{sign}(x) = 1.$$

Proposition 8.5 (Newton's/Linear Approximation)**.** *Let $A \subset \mathbb{R}$, $a \in A$ be a limit point of A and $f : A \to \mathbb{R}$ be a function which is differentiable at a, i.e. $f'(a)$ exists. Then $\forall \varepsilon > 0$, there exists $\delta > 0$ such that, if $|x - a| < \delta$, then*

$$|f(x) - (f(a) + f'(a)(x - a))| < \varepsilon |x - a|.$$

Proof. We have that for all ϵ, $\exists \delta > 0$ such that if $|x - a| < \delta$ then

$$\Big| \frac{f(x) - f(a)}{x - a} - f'(a) \Big| < \varepsilon \iff \Big| \frac{f(x) - (f(a) + f'(a)(x - a))}{x - a} \Big| < \varepsilon.$$

Rearranging gives the result. $\square$

Remark 8.6. One informally writes $f(x) \approx f(a) + f'(a)(x-a)$. Note that the converse is also true: suppose that $\exists \lambda \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such if $|x-a| < \delta$ then

$$|f(x) - (f(a) + \lambda(x-a))| < \varepsilon |x-a|.$$

Then f is differentiable at a with derivative λ.

Theorem 8.7. *Let $A \subseteq \mathbb{R}$ and let $a \in A$ be an limit point of A. If a function $f : A \to \mathbb{R}$ is differentiable at a, then it is locally Lipschitz (and, therefore, continuous) at a.*

Proof. We have that as

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

So there exists a $\delta > 0$ such that if $|x - a| < \delta$, then

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < 1 \iff f'(a) - 1 < \frac{f(x) - f(a)}{x - a} < 1 + f'(a).$$

This implies,

$$\left| \frac{f(x) - f(a)}{x - a} \right| < 1 + |f'(a)|.$$

So, we have that f is Lipschitz on $B_\delta(a)$, which by proposition 7.39 implies continuity. $\square$

The converse of this theorem is not true.

Example 8.8. *The function $f(x) = |x|$ is Lipschitz but not differentiable everywhere.*

Remark 8.9. Continuity does not imply differentiability. In fact, there exist functions which are continuous but nowhere differentiable! However, one can show that Lipschitz functions are differentiable 'almost everywhere'. This means they are differentiable every where except a set of 'measure zero': a set which can be covered by a countable union of intervals of arbitrarily small length. For example, a finite number of points in $\mathbb{R}$ is a set of measure zero, the naturals are measure zero, the rationals are measure zero ($I_n = (q_n - \varepsilon/2^n, q_n + \varepsilon/2^n)$). So Lipschitz is the 'next best thing' to differentiability and Hölder is the next best thing to Lipschitz. In some sense, the Hölder condition gives you quantification of how 'fractionally differentiable' your function is, without having to make use or sense of the Fourier transform. There are many notions of fractional differentiation. Informally, suppose that the Fourier transform $\hat{f}$ of a function exists, you can now invert this to show

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \qquad \forall x \in \mathbb{R}$$

If $\alpha \in (0,1)$, you could define a fractional derivative of $f : \mathbb{R} \to \mathbb{R}$ by

$$\left(\frac{d}{dx}\right)^\alpha f(x) = \int_{\mathbb{R}} (2\pi i \xi)^\alpha \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

Now some functions do not interaction well with the Fourier transform so the Hölder condition is a different way of understanding this.

Theorem 8.10. *Let $A \subseteq \mathbb{R}$ and let $a \in A$ be a limit point of A. Given two functions f, $g : A \to \mathbb{R}$ assume that f and g are differentiable at a. Then*

(a) $f + g$ is differentiable at a and $(f+g)'(a) = f'(a) + g'(a)$,

(b) $f \cdot g$ is differentiable at a and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$;

(c) if $g(a) \neq 0$ then $\dfrac{f}{g}$ restricted to the set $B \doteq \{x \in A : g(x) \neq 0\}$ differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g^2(a)}.$$

Proof. The first claim is essentially obvious from the linearity of addition (try writing it out as an exercise).

For the second one, observe that

$$\begin{aligned}
\frac{(fg)(a+h) - (fg)(a)}{h} &= \frac{f(a+h)\big[g(a+h) - g(a)\big] + g(a)\big[f(a+h) - f(a)\big]}{h} \\
&= f(a+h)\,\frac{g(a+h) - g(a)}{h} + g(a)\,\frac{f(a+h) - f(a)}{h} \\
&\to f(a)\,g'(a) + g(a)\,f'(a)
\end{aligned}$$

as $h \to 0$, where we use the properties of limits of functions and the fact that since f and g are differentiable at a they are continuous at a.

For the third claim, we proceed similarly

$$\begin{aligned}
\frac{(f/g)(a+h) - (f/g)(a)}{h} &= \frac{1}{g(a)\,g(a+h)}\,\frac{f(a+h)g(a) - f(a)\,g(a+h)}{h} \\
&= \frac{1}{g(a)\,g(a+h)}\left(g(a)\frac{f(a+h) - f(a)}{h} - f(a)\,\frac{g(a+h) - g(a)}{h}\right) \\
&\to \frac{1}{g(a)^2}\big(g(a)\,f'(a) - f(a)\,g'(a)\big).
\end{aligned}$$

Again, we use the fact that f and g are continuous at a, which also implies that $g(a+h) \neq 0$ for all small enough h. $\square$

Theorem 8.11 (Chain rule)**.** *Let A, $B \subseteq \mathbb{R}$ and let $a \in A$ be a limit point of A. Given two functions $f : A \to B$ and $g : B \to \mathbb{R}$ assume that f is differentiable at a, that $f(a)$ is a limit point of B and that g is differentiable at $f(a)$. Then $g \circ f : A \to \mathbb{R}$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof. Consider the function

$$h(y) \doteq \begin{cases} \frac{g(y) - g(f(a))}{y - f(a)} & \text{if } y \neq f(a), \\ g'(f(a)) & \text{if } y = f(a). \end{cases}$$

Since g is differentiable at $f(a)$, we have that $\lim_{y \to f(a)} h(y) = h(f(a)) = g'(f(a))$. For $x \in A$, $x \neq a$, write

$$\frac{g(f(x)) - g(f(a))}{x - a} = h(f(x))\,\frac{f(x) - f(a)}{x - a}.$$

Note that if $f(x) = f(a)$ for $x \neq a$ then the two sides agree and if $f(x) \neq f(a)$ then we have that the denominator of h cancels with the numerator of the adjacent fraction, so again they agree. Since f is differentiable at a, it is continuous at a. But h is continuous at $f(a)$, thus the composition $h \circ f$ is continuous at a. It follows that $\lim_{x \to a} h(f(x)) = h(f(a)) = g'(f(a))$. Hence, by the product of limits

$$\frac{g(f(x)) - g(f(a))}{x - a} = h(f(x)) \frac{f(x) - f(a)}{x - a} \to g'(f(a)) \cdot f'(a)$$

as $x \to a$. $\square$

Example 8.12. *The function*

$$f(x) \doteq \begin{cases} x^2 \sin \dfrac{1}{x} & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0 \end{cases}$$

is continuous on $\mathbb{R}$ *since* $x^2 \to 0$ *and* $\sin(1/x)$ *is bounded, so*

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 = f(0).$$

For $x \neq 0$ *we have*

$$\frac{d}{dx}\left(x^2 \sin \frac{1}{x}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

For $x = 0$ *we have to use the definition (we cannot apply the previous theorems since* $\sin(1/x)$ *is not defined at that point). We have*

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x} \to 0$$

as $x \to 0$ *since* $\sin(1/x)$ *is bounded and so* $f'(0) = 0$. *Note that*

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0 \end{cases}$$

is not continuous at $x = 0$ *since the limit*

$$\lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)$$

does not exist (exercise).

Theorem 8.13 (Differentiability of the Inverse)**.** *Let* $A \subseteq \mathbb{R}$ *and let* $a \in A$ *be an limit point of* A. *Given a function* $f : A \to \mathbb{R}$ *assume that* f *is one-to-one and differentiable at* a. *Suppose also that the inverse function* $f^{-1} : f(A) \to \mathbb{R}$ *is continuous at* $f(a)$. *Then* $f(a)$ *is a limit point of the set* $f(A)$ *and* f^{-1} *is differentiable at* $f(a)$ *if and only if* $f'(a) \neq 0$ *and in this case*

$$\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}.$$

Proof. We first prove that $f(a)$ is a limit point of the set $f(A)$. Since a is a limit point of A, there exists a sequence $\{a_n\} \subseteq A \setminus \{a\}$ which converges to a. Define $b_n \doteq f(a_n)$. Since f is one-to-one and $\{a_n\} \subseteq A \setminus \{a\}$ we have that $\{b_n\} \subseteq f(A) \setminus \{f(a)\}$. By continuity of f at a we have that $b_n = f(a_n) \to f(a)$. Hence $f(a)$ is a limit point of the set $f(A)$.

Next we show that $f^{-1} : f(A) \to \mathbb{R}$ is differentiable at $f(a)$ if and only if $f'(a) \neq 0$. Consider the quotient

$$\frac{f^{-1}(x) - f^{-1}(f(a))}{x - f(a)}$$

for $x \in f(A) \setminus \{f(a)\}$. For every $x \in f(A)$ there exists a unique $y \in A$ such that $f(y) = x$ and so we may write

$$\frac{f^{-1}(x) - f^{-1}(f(a))}{x - f(a)} = \frac{y - a}{f(y) - f(a)}.$$

By assumption the function f^{-1} is continuous at $f(a)$ and so $y = f^{-1}(x) \to a = f^{-1}(f(a))$ as $x \to f(a)$. Since f is differentiable at a we have that

$$\frac{f^{-1}(x) - f^{-1}(f(a))}{x - f(a)} = \frac{1}{\dfrac{f(y) - f(a)}{y - a}} \to \frac{1}{f'(a)}$$

as $y \to f(a)$. □

Remark 8.14. Note that by writing $b = f(a)$, the previous theorem says that

$$\frac{df^{-1}}{dy}(b) = \frac{1}{f'(f^{-1}(b))}.$$

Example 8.15. *1. We defined the* log *as the inverse function of the exponential function whose derivative is simple. Using the theorem above we can also compute*

$$\log'(x) = \frac{1}{e^{\log(x)}} = \frac{1}{x}.$$

2. As an exercise, show that

$$\arctan'(x) = \frac{1}{1 + x^2} \text{ and } \arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}.$$

8.2 Maxima and Minima

Definition 8.16. *Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, and let $a \in A$. We say that*

i) f attains a local minimum *at a if there exists $r > 0$ such that $f(x) \geq f(a)$ for all $x \in A \cap B_r(a)$,*

ii) f attains a global minimum *at a if $f(x) \geq f(a)$ for all $x \in A$,*

iii) f attains a local maximum *at x_0 if there exists $r > 0$ such that $f(x) \leq f(a)$ for all $x \in A \cap B_r(a)$,*

iv) f attains a global maximum *at x_0 if $f(x) \leq f(a)$ for all $x \in E$.*

Theorem 8.17 (Fermat's Interior Extremum Theorem)**.** *Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$. If f attains a local minimum or maximum at some interior point $a \in A$ and if f is differentiable at a, then $f'(a) = 0$.*

Remark 8.18. Why insist on an interior point? Recall that $a \in A$ is an interior point if $\exists r > 0$ such that $B_r(a) \subseteq A$. In $\mathbb{R}$ with the usual metric an interior point is a limit point since if a is interior then

$$a \pm \frac{\min\{\varepsilon, r\}}{2} \in (B_r(a) \setminus \{a\}) \cap (B_\varepsilon(a) \setminus \{a\}) \subseteq B_\varepsilon(a) \cap (A \setminus \{a\}).$$

However, not every limit point $a \in A$ of A is an interior point. For example $0 \in [0, 1]$ is a limit point but no ball around 1 is contained in $[1, 2]$. The problem is that the theorem is trivially false if we relax to limit points in A. For example take $f(x) = x$ on $[1, 2]$. Clearly, $f(1) = 1$ is a global and local minimum and $f(2) = 2$ is a global and local maximum. However,

$$\frac{df}{dx}(1) = 1 = \frac{df}{dx}(2).$$

So, $f'(1) = f'(2) \neq 0$.

Remark 8.19. A small digression on interior vs. limit points. In general metric spaces, there is no correspondence. For example, take $\mathbb{R}$ with the discrete metric:

$$d_D(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Then, if $r \leq 1$, $B_r(a) = \{a\}$. Then $a \in A$ is interior since $B_r(a) = \{a\} \subset A$ but a is not a limit point since

$$B_r(a) \cap (A \setminus \{a\}) = \emptyset.$$

Proof. Assume that f attains a local minimum. Then there exists $r > 0$ such that $f(x) \geq f(a)$ for all $x \in A \cap B(a, r)$. Since $a \in A$ is an interior point, by taking r smaller, we can assume that $B(a, r) \subseteq A$ so that $f(x) \geq f(a)$ for all $x \in B(a, r)$. If $x > a$, then $f(x) - f(a) \geq 0$ and so

$$\frac{f(x) - f(a)}{x - a} \geq 0.$$

Letting $x \to a^+$ and using the fact that f is differentiable at a, we get that $f'(a) \geq 0$.

If $x < a$, then $f(x) - f(a) \geq 0$ and so

$$\frac{f(x) - f(a)}{x - a} \leq 0.$$

Letting $x \to a^-$ and using the fact that f is differentiable at a, we get that $f'(a) \leq 0$. This shows that $f'(a) = 0$.

The result for the maximum can be derived by the case above by considering a function $g = -f$. $\square$

Remark 8.20. In view of the previous theorem (and recall the Extreme Value Theorem 7.21), when looking for minima and maxima of a continuous function on a compact set, we have to search among the following:

- Interior points at which f is differentiable and $f'(x) = 0$, these are called *critical points*. Note that if $f'(a) = 0$, the function f may not attain a local minimum or maximum at a. Indeed, consider the function $f(x) = x^3$. Then $f'(0) = 0$, but f is strictly increasing, and so f does not attain a local minimum or maximum at 0.

- Interior points at which f is not differentiable. The function $f(x) = |x|$ attains a global minimum at $x = 0$, but f is not differentiable at $x = 0$.
- Boundary points.

Example 8.21. *Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = |x| + (x - 1)^2$. We want to find the minimum of f. The function f is not differentiable at 0. For $x \neq 0$, we find*

$$f'(x) = \frac{x}{|x|} + 2(x - 1).$$

It follows that $f'(x) = 0$ implies that $x = \frac{1}{2}$. It follows that the minimal value of f on $[-1, 1]$ is in the set $\{f(-1), f(1), f(0), f(\frac{1}{2})\} = \{5, 1, 1, \frac{3}{4}\}$. Hence, the minimal value of f in $[-1, 1]$ is $\frac{3}{4}$ and attained at $x = \frac{1}{2}$.

8.3 Mean Value Theorem

Rolle's theorem is a precursor to the mean value theorem. It says that, if a differentiable function f starts and ends at the same value, f must be flat somewhere.

Theorem 8.22 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ with $a < b$ and differentiable in (a, b). Suppose that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.*

Proof. Since $[a, b]$ is compact, f has a global maximum and a global minimum in $[a, b]$. If

$$\max_{[a,b]} f = \min_{[a,b]} f,$$

then f is constant, and so $f'(x) = 0$ for all $x \in (a, b)$. If $\max_{[a,b]} f > \min_{[a,b]} f$, then since $f(a) = f(b)$, it follows that f admits one of them at some interior point $c \in (a, b)$. Since f is differentiable in the interior of its domain, by Theorem 8.17 we have that $f'(c) = 0$. □

Remark 8.23. Note that this theorem is explicitly false for just continuous functions. Take $f : [-1, 1] \to \mathbb{R}$ as $x \mapsto |x|$.

Rolle's theorem assumes that the function is differentiable and that it assumes the same value at the start and end points. The mean value theorem generalises this situation to different values at endpoints. It says that if you connect two points on a graph of a differentiable function with a straight line then at some point in between of them the function must have a tangent parallel to the line.

Theorem 8.24 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. The proof follows from Rolle's Theorem 8.22 applied for the function

$$h(x) \doteq f(x) - l(x),$$

where $l(x) = \frac{f(b)-f(a)}{b-a}(x - a) + f(a)$ is the linear function passing though points $(a, f(a))$ and $(b, f(b))$. It follows that $h(a) = 0 = h(b)$ and, since f is differentiable over (a, b), so is g.

Furthermore, since f is continuous on $[a, b]$, h is also continuous on $[a, b]$. Hence, h satisfies assumptions of Rolle's theorem that implies that there exists $c \in (a, b)$ such that $h'(c) = 0$. Since

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

plugging in $x = c$ and using that $h'(c) = 0$ we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which concludes the proof. □

Theorem 8.25 (Generalised Mean Value Theorem). *Let $f, g : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in (a, b). Moreover, assume that $g(a) \neq g(b)$. Then there exists $c \in (a, b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Apply the Mean Value theorem to the function

$$h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x).$$

□

We now prove some corollaries of the MVT.

Corollary 8.26. *Let $f : (a, b) \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

Proof. Let $x, y \in (a, b)$ with $x < y$. Note that by differentiability of f on (a, b) is continuous on $[x, y]$. By the mean value theorem on $[x, y]$, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x) = 0$$

Hence, $f(y) - f(x) = 0$, which shows that f is constant. □

Corollary 8.27. *Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Then f is increasing if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.*

Proof. If f is increasing, then $f(x) \geq f(y)$ for $x > y$, and so

$$\frac{f(x) - f(y)}{x - y} \geq 0.$$

Letting $x \to y$, we get $f'(y) \geq 0$.

Conversely, suppose $f' \geq 0$ for all $x, y \in (a, b)$ with $x < y$. Note that f is continuous on $[x, y]$ by differentiability of f. So, by the mean value theorem, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x) \geq 0.$$

Hence, $f(y) \geq f(x)$. □

Remark 8.28. If $f' > 0$ in (a, b), then with the same proof we can show that f is strictly increasing, but the opposite is not true. Indeed, the function $f(x) = x^3$ is strictly increasing but $f'(x) = 3x^2$, which is zero for $x = 0$.

Corollary 8.29. *Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable on (a, b) with f' bounded. Then f is Lipschitz continuous.*

Proof. Let $L > 0$ be such that $|f'(x)| \leq L$ for all $x \in (a, b)$. By the mean value theorem, for all $x, y \in [a, b]$ with $x < y$, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x).$$

Hence,

$$|f(y) - f(x)| \leq L(y - x),$$

which shows that f is Lipschitz continuous. □

Theorem 8.30 (1D Inverse Function Theorem)**.** *Let f be a function with continuous derivative on (a, b). Let $x \in (a, b)$, and suppose $f'(x) \neq 0$. Then there exists an open interval (u, v) containing x, on which f is invertible. Moreover, if g is the inverse then g is differentiable with $g'(f(z)) = \frac{1}{f'(z)}$ for all $z \in (u, v)$.*

Proof. Without loss of generality, suppose $f'(x) > 0$. Then by continuity of f', we can find $\delta > 0$ such that $f'(z) > 0$ for every $z \in (x - \delta, x + \delta)$. Hence, by Corollary 8.27 of the mean value theorem, f is strictly increasing on $(x - \delta, x + \delta)$. One can now apply corollary 7.29 to show existence of the continuous inverse. Then one can apply Theorem 8.13 to get the statement about the derivative. □

The above theorem gives a criterion of when we can locally invert a function and can be generalised, for example, for functions of many variables.

8.4 L'Hôpital's Rule

Next, we prove l'Hôpital's rule which is sometimes handy to compute limits. We state three different versions of the theorem and only prove the first one.

Theorem 8.31 (L'Hôpital's Rule ("$\frac{0}{0}$"))**.** *Assume f and g are continuous functions defined on an open interval I containing a, and assume that f and g are differentiable on this interval, with the possible exception of the point a. If $f(a) = g(a) = 0$, and $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$, then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \in [-\infty, \infty] \quad \textit{implies that} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Consider a sequence $(a_n) \subseteq I \setminus \{a\}$ such that $a_n \to a$. Apply the Generalised Mean Value Theorem 8.25 in the interval of endpoints a_n and a to find b_n between a_n and a such that

$$\frac{f(a_n) - f(a)}{g(a_n) - g(a)} = \frac{f'(b_n)}{g'(b_n)}.$$

As $a_n \to a$, we have that $b_n \to a$, and so, by our assumption that $f(a) = 0 = f(b)$, we get

$$\frac{f(a_n)}{g(a_n)} = \frac{f'(b_n)}{g'(b_n)} \to L.$$

Since this is true for every sequence $(a_n)_n$ converging to a, it follows that there exists

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L. \qquad \square$$

Another version of L'Hôpital's theorem is the following:

Theorem 8.32 (L'Hôpital's Rule ("$\frac{\infty}{\infty}$")). *Let $I \subset \mathbb{R}$ be a closed interval containing a and let $f : I \setminus \{a\} \to \mathbb{R}$ and $g : I \setminus \{a\} \to \mathbb{R}$ be continuous in $I \setminus \{a\}$ and that f and g are differentiable in $I \setminus \{a\}$ with $g(x), g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. Further, assume that $\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = \infty$. Then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \in [-\infty, \infty], \qquad \textit{implies that} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Another version of the previous theorem is the following.

Theorem 8.33 (L'Hôpital's Rule ("limit at infinity")). *Let $f : [a, \infty) \to \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$ be continuous in $[a, \infty)$ and differentiable in (a, ∞). Assume that $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ and that $g(x), g'(x) \neq 0$ for all $x \in (a, \infty)$. Then*

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L \in [-\infty, \infty], \qquad \textit{implies that} \qquad \lim_{x \to \infty} \frac{f(x)}{g(x)} = L.$$

The proof is similar and we omit it.

Remark 8.34. A similar result holds if $\lim_{x \to \infty} |f(x)| = \lim_{x \to \infty} |g(x)| = \infty$.

Example 8.35. *Let's calculate*

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

Consider the functions $f(x) = \sin x - x$ and $g(x) = x^3$. Then $f(0) = g(0) = 0$ and both functions are differentiable. We have that $f'(x) = \cos x - 1$ and $g'(x) = 3x^2$. Let's calculate

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}. \tag{12}$$

We get $\frac{0}{0}$. Consider the functions $f_1(x) = \cos x - 1$ and $g_1(x) = 3x^2$. Then $f_1(0) = g_1(0) = 0$ and both functions are differentiable. We have that $f_1'(x) = -\sin x$ and $g_1'(x) = 6x$. Let's calculate

$$\lim_{x \to 0} \frac{f_1'(x)}{g_1'(x)} = \lim_{x \to 0} \frac{-\sin x}{6x} = -\frac{1}{6}. \tag{13}$$

By (13) and l'Hôpital's theorem, there exists

$$\lim_{x \to 0} \frac{f_1(x)}{g_1(x)} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = -\frac{1}{6}.$$

Finally, by (12) and l'Hôpital's theorem, there exists

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

Remark 8.36. Using l'Hôpital's theorem to compute the limit $\frac{\sin x}{x}$ as $x \to 0$ is contentuous. Since you need to know the derivative of $\sin x$ at 0, and that, in theory, involves the limit

$$\frac{\sin x - \sin(0)}{x - 0} = \frac{\sin x}{x},$$

in some sense this is circular logic. Whilst in hindsight this is ok, if you want to establish that the of $\frac{\sin x}{x}$ as $x \to 0$ is 1 you need to make a different argument. You proved in the problem sheet $\sin' = \cos$ via the power series, so this is slightly more satisfactory. However, you can argue geometrically from a unit circle that

$$1 > \frac{\sin x}{x} > \cos x.$$

Since $\cos x$ is continuous, you can conclude via the squeeze theorem.

Remark 8.37. The limit of $\frac{f'}{g'}$ existing is stronger than the limit of $\frac{f}{g}$ existing: the converse of L'Hôpital's theorem does not hold, that is, if there exists $\lim_{x\to a} \frac{f(x)}{g(x)}$, we cannot conclude that there exists the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$. To see this, take

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and $g(x) = x$. Then f and g are continuous and for $x \neq 0$, have that $f'(x) = 2x \sin \frac{1}{x} - x^2 \left(-\frac{1}{x^2}\right) \cos \frac{1}{x}$ and $g'(x) = 1$. Then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x\to 0} x \sin \frac{1}{x} = 0,$$

since $0 \leq \left| x \sin \frac{1}{x} \right| \leq |x| \to 0$, but the limit

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2x \sin \frac{1}{x} + \cos \frac{1}{x}}{1}$$

does not exist (we see an oscillatory behaviour).

8.5 Taylor's Theorem and Higher Order Derivatives

Definition 8.38. *We say that a function f is $(n+1)$-*times differentiable* if it is n-times differentiable, and its n-the derivative, denoted by $f^{(n)}$, is also differentiable.*

The class of n-times differentiable functions play an important role in many areas of mathematics, for example, in partial differential equations.

Definition 8.39. *Let $n \in \mathbb{N}$ and $[a, b] \subseteq \mathbb{R}$ an interval. We denote*

$$C^n([a, b]) = \{f : [a, b] \to \mathbb{R} : f \text{ is } n\text{-times differentiable and } f^{(k)} \text{ is continuous for } 0 \leq k \leq n\}.$$

We call this the space of n-times continuously differentiable functions. We define smooth functions as functions n-times differentiable functions,

$$C^\infty([a, b]) \doteq \bigcap_{n\in\mathbb{N}_0} C^n([a, b]).$$

Remark 8.40. For $n \in \mathbb{N}$, $C^n(I)$ is a complete metric space with the metric

$$d(f,g) = ||f-g||_{C^k([a,b])}$$

defined through the norm

$$||f||_{C^k([a,b])} = \sum_{k=0}^{n} \sup_{x\in[a,b]} |f^{(k)}(x)|.$$

We will not prove this result here.

You can extend this to Hölder spaces and define the space of k-times continuously differentiable functions with α-Hölder continuous k^{th} derivative:

$$C^{k,\alpha}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \in C^k([a,b]),\ f^{(k)} \text{ is } \alpha\text{-Hölder continuous } \}.$$

You can induce the metric from the norm

$$||f||_{C^{k,\alpha}([a,b])} = ||f||_{C^k([a,b])} + \sup_{\substack{x\in[a,b]\\ x\neq y}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^{\alpha}}.$$

This turns out to also be complete. These are examples of very important objects in mathematics known as Banach spaces: complete normed spaces.

Note that any polynomial (of an arbitrary degree) is a smooth function. The class of polynomials is "very well-behaved" and one may want to use them to approximate other, more complicated functions. This is the idea of the Taylor series.

Theorem 8.41 (Taylor's Formula)**.** *Let I be an open, bounded interval centered at a and let $f \in C^{(n)}(I)$. Then, for every $x \in I$, we have*

$$f(x) = T_n f(x,a) + R_n(x,a).$$

where $T_n f(x,a)$ is the Taylor polynomial of order n around a given by

$$T_n f(x,a) \doteq f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and $R_n(x,a)$ is called the remainder and satisfies

$$\lim_{x\to a} \frac{R_n(x,a)}{(x-a)^n} = 0.$$

Proof. Given a polynomial of degree n,

$$p(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n = \sum_{i=0}^{n} c_i (x-a)^i,$$

we can take k-derivatives to find,

$$p^{(k)}(x) = \sum_{i=k}^{n} i(i-1)\cdots(i-k+1)\, c_i (x-a)^{i-k}.$$

Evalutating at a gives

$$p^{(k)}(a) = k!c_k.$$

For the Taylor polynomial, $p(\cdot) = T_n f(\cdot, a)$, this implies that $p^{(k)}(a) = f^{(k)}(a)$ for all $0 \le k \le n$. Therefore, if we define

$$g(x) \doteq f(x) - T_n f(x, a),$$

we find $g^{(k)}(a) = 0$ for all $0 \le k \le n$.

Now we can apply L'Hôpital's rule to find the limit $\lim_{x \to a} \frac{g(x)}{(x-a)^n}$:

$$\lim_{x \to a} \frac{g(x)}{(x-a)^n} = \lim_{x \to a} \frac{g'(x)}{n(x-a)^{n-1}} = \cdots = \lim_{x \to a} \frac{g^{(n)}(x)}{n!} = 0.$$

We define $R_n(x, a) = g(x)$. □

Remark 8.42. At the end of the course we will discuss a result that looks similar at the surface level: the Weierstrass approximation theorem. This roughly states that you can approximate a continuous function on an interval $[a, b]$ with a polynomial arbitrarily well. This is different for two reasons:

1. Taylor's theorem is local about a given point rather than on an interval, i.e. you have to get arbitrarily closer and closer to the point a for the approximation to improve.
2. Taylor's theorem requires differentiability.

Example 8.43. *Let $f(x) = e^x$. Then $T_n f(x, 0) = \sum_{k=0}^{n} \frac{x^k}{k!}$.*

Example 8.44. *Let $g(x) = \log(1 + x)$ for $x > -1$. Then $T_n g(x, 0) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k}$. As $n \to \infty$ this series converges exactly for all $x \in (-1, 1]$.*

Via the Taylor expansion we can define one other class of function, real analytic functions.

Definition 8.45 (Real Analyticity)**.** *Let $I \subseteq \mathbb{R}$ be an open interval interval centred at $a \in I$. We denote $C^\omega(I)$ the space of functions which are smooth on I and have convergent Taylor series at each point $x \in I$, i.e. if*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

converges to $f(x)$ for all $x \in I$.

Remark 8.46. Analytic functions are very nice to deal with: they can be differentiated and integrated termwise. They also have unique complex analytic (holomorphic) extensions, which allows for access to techniques from complex analysis.

Example 8.47. *The functions e^x, $\sin(x)$ and $\cos x$ are analytic.*

The last result of this chapter shows how higher regularity of a function may be used to obtain further insights into the behaviour of the function.

Corollary 8.48. *Let I be an interval and let $f : I \to \mathbb{R}$ be a function of class $C^n(I)$, $n \geq 2$, and let $a \in I$ be such that*

$$f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0 \tag{14}$$

for some $2 \leq m \leq n$. Then the following hold:

(i) if m is even and $f^{(m)}(a) > 0$, then f has a local minimum at a,

(ii) if m is even and $f^{(m)}(a) < 0$, then f has a local maximum at a,

(iii) if m is odd and $f^{(m)}(a) \neq 0$, then f has neither a local minimum nor a local maximum at a.

Remark 8.49. This includes the classical result for $f \in C^2(I)$ that f has a local max/min at $a \in I$ if $f'(a) = 0$ and $f''(a) < 0$ / $f''(a) > 0$.

Proof. For x close to a by Taylor's formula of order m and center a

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \\ + \cdots + \frac{1}{m!}f^{(m)}(a)(x-a)^m + R_m(x,a), \end{aligned}$$

where the remainder $R_m(x,a)$ satisfies

$$\lim_{x \to a} \frac{R_m(x,a)}{(x-a)^m} = 0. \tag{15}$$

By assumption (14) we get that in fact

$$\begin{aligned} f(x) &= f(a) + \frac{1}{m!}f^{(m)}(a)(x-a)^m + R_m(x,a) \\ &= f(a) + (x-a)^m \left[\frac{1}{m!}f^{(m)}(a) + \frac{R_m(x,a)}{(x-a)^m}\right]. \end{aligned}$$

Now in the various cases we examine the sign of

$$(x-a)^m \left[\frac{1}{m!}f^{(m)}(a) + \frac{R_m(x,a)}{(x-a)^m}\right],$$

since this tells us about the sign of $f(x) - f(a)$.

(i) Let $\varepsilon = \frac{1}{2}\frac{1}{m!}f^{(m)}(a) > 0$. By (15), there exists $\delta > 0$ such that for all $x \in I$ with $|x-a| \leq \delta$ we have

$$\left|\frac{R_m(x,a)}{(x-a)^m} - 0\right| \leq \varepsilon \Longleftarrow -\varepsilon \leq \frac{R_m(x,a)}{(x-a)^m} \leq \varepsilon.$$

Hence,

$$\begin{aligned} \frac{1}{m!}f^{(m)}(a) + \frac{R_m(x,a)}{(x-a)^m} &\geq \frac{1}{m!}f^{(m)}(a) - \varepsilon \\ &= \frac{1}{m!}f^{(m)}(a) - \frac{1}{2}\frac{1}{m!}f^{(m)}(a) = \frac{1}{2}\frac{1}{m!}f^{(m)}(a) > 0 \end{aligned}$$

for all $x \in (a,b)$ with $|x-a| \leq \delta$. Since m is even, we have that $(x-a)^m > 0$ for $x \neq a$ so

$$\begin{aligned} f(x) &= f(a) + (x-a)^m \left[\frac{1}{m!} f^{(m)}(a) + \frac{R_m(x,a)}{(x-a)^m} \right] \\ &> f(a) + 0 \end{aligned}$$

for all $x \in I$ with $0 < |x-a| \leq \delta$, which shows that f has a local minimum at a.

(ii) Let $\varepsilon = -\frac{1}{2}\frac{1}{m!} f^{(m)}(a) > 0$. Argue similarly to the case above.

(iii) If m is odd then $(x-a)^m > 0$ if $x > a$ and so we proceed exactly as in the previous part to conclude that

$$f(x) > f(a)$$

for all $x \in I$ such that $a < x < a + \delta$. On the other hand, $(x-a)^m < 0$ if $x < a$ and so

$$f(x) < f(a)$$

for all $x \in I$ such that $a - \delta < x < a$. Thus f has neither a local minimum nor a local maximum at a.

$\square$

9 Riemann Integration of Functions on the Real Line

There are many different ways to define 'an integral', most prominently the Riemann integral and the Lebesgue integral. These differ in the way that the integral is understood, but both give the same results for continuous functions on bounded intervals and other 'nice' functions. The Lebesgue integral is more general. It can integrate more functions than the Riemann integral. We will work with the simpler concept of the Riemann integral. This is sufficient for many practical purposes. You will see the Lebesgue integral if you proceed with further courses in analysis or in probability and measure.

In rough terms the Riemann integral of a function $f : [a, b] \to \mathbb{R}$ is the 'area' bounded between the x-axis and the graph of f.

9.1 Definition via Upper and Lower Riemann Sums

The Riemann integral is based upon partitioning the interval of interest into segments and sampling the function on those segments. Formally, we partition the interval by:

Definition 9.1 (Partition/Dissection)**.** *Let $a < b$. A partition or dissection P of an interval $[a, b]$ if a finite sequence $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.*

We now sample the function at points in each partitioned segments to define two approximations of the area under the curve. One is an over estimation (the upper sum) and one is an under estimation (the lower sum).

Definition 9.2 (Upper/Lower Sums)**.** *Let $a < b$, P be a partition of $[a, b]$ and f be a function $f : [a, b] \to \mathbb{R}$. Let $u_i \doteq \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $l_i \doteq \inf_{x \in [x_{i-1}, x_i]} f(x)$. The upper sum with respect to P, denoted $U_P(f)$, is defined by*

$$U_P(f) \doteq \sum_{i=1}^{n} (x_i - x_{i-1}) u_i,$$

and the lower sum with respect to P, denoted $L_P(f)$, is defined by

$$L_P(f) \doteq \sum_{i=1}^{n} (x_i - x_{i-1}) l_i,$$

Remark 9.3. Clearly, $L_P(f) \leq U_P(f)$.

We want to take are partition finer and finer to approximate the area better.

Definition 9.4 (Refinement)**.** *If P_1 and P_2 are partitions of $[a, b]$. We say that P_2 refines P_1 if every point of P_1 is a point of P_2.*

Refining a partition decreases the upper sum and increases the lower sum.

Lemma 9.5. *If P_2 refines P_1, then $U_{P_2}(f) \leq U_{P_1}(f)$ and $L_{P_1}(f) \leq L_{P_2}(f)$.*

Proof. Suppose P_1 is given by $x_0 < x_1 < ... < x_n$. We will use induction on the number of points, i.e. if we can prove it for the addition of one point then we have shown it for the addition of finitely many points.

So suppose $z \in [a,b]$ is the added point. We have that $z \in (x_{i-1}, x_i)$ for some i. So,

$$U_{P_2}(f) - U_{P_1}(f) = (z - x_{i-1}) \sup_{x \in [x_{i-1}, z]} f(x) + (x_i - z) \sup_{x \in [z, x_i]} f(x) - (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x),$$
$$L_{P_2}(f) - L_{P_1}(f) = (z - x_{i-1}) \inf_{x \in [x_{i-1}, z]} f(x) + (x_i - z) \inf_{x \in [z, x_i]} f(x) - (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x),$$

Now,

$$\sup_{x \in [x_{i-1}, z]} f(x) \le \sup_{x \in [x_{i-1}, x_i]} f(x) = u_i, \qquad \sup_{x \in [z, x_i]} f(x) \le \sup_{x \in [x_{i-1}, x_i]} f(x) = u_i,$$
$$\inf_{x \in [x_{i-1}, z]} f(x) \ge \inf_{x \in [x_{i-1}, x_i]} f(x) = l_i, \qquad \inf_{x \in [z, x_i]} f(x) \ge \inf_{x \in [x_{i-1}, x_i]} f(x) = l_i.$$

Therefore,

$$U_{P_2}(f) - U_{P_1}(f) \le \Big[(z - x_{i-1}) + (x_i - z) - (x_i - x_{i-1})\Big] u_i = 0,$$
$$L_{P_2}(f) - L_{P_1}(f) \ge \Big[(z - x_{i-1}) + (x_i - z) - (x_i - x_{i-1})\Big] l_i = 0.$$

□

We now define two integrals based upon this refinement process. We take the smallest upper approximation via the partitioning process and the largest lower approximation via the partitioning process. When these agree we say that their common value is the integral of f:

Definition 9.6 (Upper/Lower/Riemann Integral)**.** *The upper integral of $f : [a,b] \to \mathbb{R}$ is*

$$\overline{\int_a^b} f(x)dx \doteq \inf_P (U_P(f)).$$

The lower integral of f is

$$\underline{\int_a^b} f(x)dx \doteq \sup_P (L_P(f)).$$

If these are equal, we say f is Riemann integrable and define

$$\int_a^b f(x)dx \doteq \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

as the Riemann integral of f.

Remark 9.7. Technically, we have defined Darboux integration, which can be shown to be equivalent Riemann integration. In the definition of Riemann integration, one does not take the $\inf f$ and $\sup f$ over the partitions but samples at a point in the subintervals. If the sum constructed in this way converges as one refines the partition, then you have a Riemann integrable function in the classical sense. Darboux's formulation is advantageous to apply in proofs. It is also easier to compute with, although, still difficult to use in practise.

Example 9.8. *Consider $f(x) = x$ on $[a, b]$. Let P be a partition $x_0 = a < x_1 < ... < x_n = b$. We have $u_i = x_i$ and $l_i = x_{i-1}$. So, the upper sum is*

$$U_P(f) = \sum_{i=1}^{n}(x_i - x_{i-1})x_i = \sum_{i=1}^{n}\Big(\frac{x_i^2}{2} - \frac{x_{i-1}^2}{2} + \frac{x_i^2}{2} - x_{i-1}x_i + \frac{x_{i-1}^2}{2}\Big),$$

which, you can check can be manipulated to

$$U_P(f) = \frac{1}{2}(b^2 - a^2) + \frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})^2.$$

Suppose, $\max_i(x_i - x_{i-1}) < \delta$ *then*

$$U_P(f) < \frac{1}{2}(b^2 - a^2) + \frac{\delta}{2}\sum_{i=1}^{n}(x_i - x_{i-1}) = \frac{1}{2}(b^2 - a^2) + \frac{\delta}{2}(b - a).$$

For any $\varepsilon > 0$, we can pick our partition so that $\delta = \frac{\varepsilon}{2(b-a)}$. Therefore,

$$U_P(f) < \frac{1}{2}(b^2 - a^2) + \frac{\varepsilon}{4} \implies \overline{\int_a^b} x dx < \frac{1}{2}(b^2 - a^2) + \varepsilon.$$

Similarly, one can show

$$\underline{\int_a^b} x dx > \frac{1}{2}(b^2 - a^2) - \varepsilon.$$

So, as $\varepsilon \to 0$, we get that the upper and lower integrals agree with a common value of $\frac{1}{2}(b^2 - a^2)$. So x is Riemann integrable on $[a, b]$ with Riemann integral

$$\int_a^b x dx = \frac{1}{2}(b^2 - a^2).$$

Is every function Riemann integrable? The short answer is no, not even every bounded function is Riemann integrable.

Example 9.9. *The function*

$$f(x) \doteq \begin{cases} 1 & \text{if } x \in [0, 1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q}, \end{cases}$$

is called the Dirichlet function. Since we can always find a rational and irrational in every interval we have that

$$\inf_{x_{i-1} \le x \le x_i} f(x) = 0, \qquad \sup_{x_{i-1} \le x \le x_i} f(x) = 1,$$

and so for all $x_0 = 0 < \ldots < x_n = 1$ we have

$$L_P(f) = \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{x_{i-1} \le x \le x_i} f(x) = 0,$$
$$U_P(f) = \sum_{i=1}^{n}(x_i - x_{i-1})\, 1 = 1 - 0 = 1.$$

Therefore,

$$\underline{\int_0^1} f(x)\; dx = \sup 0 = 0, \quad \overline{\int_0^1} f(x)\; dx = \inf 1 = 1.$$

Remark 9.10. From the intuitive idea of an integral would think that the integral of the Dirichlet function should exist. Why? The integral is a continuous version of a sum, so an average of the function f can be contructed from an integral by dividing by the size of the domain. Here the domain is $[0,1]$, which has length 1 so the average of the function should be just the integral of f. The rationals are countable and irrationals are uncountable so you would expect that the average is 0, since there are far more irrationals that rationals. The Lebesgue integral fixes this issue and confirms our expectation that $\int_0^1 f d\mu = 0$.

9.2 Properties and Integrability

We start with a limiting criterion for integrability:

Proposition 9.11. *Let $f : [a,b] \to \mathbb{R}$. Then, f is Riemann integrable if, and only if, for any $\varepsilon > 0$ there exists a partition P such that $U_P(f) - L_P(f) < \varepsilon$.*

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. If f is Riemann integrable then there exist partitions P_1 and P_2 such that

$$U_{P_1}(f) < \int_a^b f(x)dx + \frac{\varepsilon}{2}, \qquad L_{P_2}(f) > \int_a^b f(x)dx - \frac{\varepsilon}{2}.$$

Let P be a common refinement of P_1 and P_2, i.e. a refinement that contains all points in P_1 and P_2. From lemma 9.5, that $U_P(f) \leq U_{P_1}(f)$ and $-L_P(f) \leq -L_{P_2}(f)$, which gives

$$U_P(f) - L_P(f) \leq U_{P_1}(f) - L_{P_2}(f) < \int_a^b f(x)dx + \frac{\varepsilon}{2} - \left[\int_a^b f(x)dx - \frac{\varepsilon}{2}\right] = \varepsilon.$$

($\Leftarrow$) Let $\varepsilon > 0$. By the definitions of inf and sup, for any partition $\check{P}$ we have

$$\overline{\int_a^b} f(x)dx = \inf_P(U_P(f)) \leq U_{\check{P}}(f), \qquad -\underline{\int_a^b} f(x)dx = -\sup_P(L_P(f)) \leq -L_{\check{P}}(f).$$

So,

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U_{\check{P}}(f) - L_{\check{P}}(f).$$

So, if $\check{P}$ is a partition where $U_{\check{P}}(f) - L_{\check{P}}(f) < \varepsilon$, then

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this gives that

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \to 0$$

as $\varepsilon \to 0$. Hence, f is Riemann integrable. □

Proposition 9.12 (Properties of the Riemann Integral)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be integrable and $\lambda, \mu \in \mathbb{R}$. Then,* *(i) the Riemann integral is linear, i.e. $\mu f + \lambda g$ is integrable with*

$$\int_a^b (\mu f + \lambda g)(x)dx = \mu \int_a^b f(x)dx + \lambda \int_a^b g(x)dx.$$

(ii) if g dominates f (i.e. $f(x) \le g(x)$ for all $x \in [a,b]$), then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

(iii) $|f|$ is integrable.

(iv) if $c \in (a,b)$, the restrictions of f to $[a,c]$ and $[c,b]$ are Riemann integrable and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(v) $f \cdot g$ is integrable.

Proof. We will prove the odd-numbered properties. Properties *(ii)* and *(iv)* are left as an exercise.

(i) Let P_1 and P_2 be partitions. Let P be a common refinement. Then,

$$\begin{aligned} U_P(\mu f + \lambda g) &= \sum_{i=1}^n (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} (\mu f(x) + \lambda g(x)) \\ &\le \sum_{i=1}^n (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1}, x_i]} \mu f(x) + \sup_{x \in [x_{i-1}, x_i]} \lambda g(x) \right) \\ &= \sum_{i=1}^n (x_i - x_{i-1}) \left(\mu \sup_{x \in [x_{i-1}, x_i]} f(x) + \lambda \sup_{x \in [x_{i-1}, x_i]} g(x) \right) \\ &= \mu U_P(f) + \lambda U_P(g) \end{aligned}$$

Hence, in particular, $U_P(\mu f + \lambda g) \le \mu U_P(f) + \lambda U_P(g) \le \mu U_{P_1}(f) + \lambda U_{P_2}(g)$, since P is a common refinement.

Therefore, we get

$$\overline{\int_a^b} (\mu f(x) + \lambda g(x))dx \le \mu \overline{\int_a^b} f(x)dx + \lambda \overline{\int_a^b} g(x)dx = \mu \int_a^b f(x)dx + \lambda \int_a^b g(x)dx$$

as both f and g are integrable. One proceeds similarly for the lower integral (noting that the inequality reverses, i.e. $\inf(\mu f + \lambda g) \ge \mu \inf f + \lambda \inf g$), which yields

$$\underline{\int_a^b} (\mu f(x) + \lambda g(x))dx \ge \mu \int_a^b f(x)dx + \lambda \int_a^b g(x)dx.$$

Since the upper integral is always greater or equal to the lower integral(see theorem 9.14 below), the above computation shows that they must be equal, with the common value $\mu \int_a^b f(x)dx + \lambda \int_a^b g(x)dx$ which finishes the proof.

(iii) The reverse triangle inequality gives

$$||f(x)| - |f(y)|| \le |f(x) - f(y)|.$$

Since for any $x \in [a,b]$ we have that $f(x) \le \sup f$, it follows that

$$\sup_x ||f(x)| - |f(y)|| = |\sup_x |f(x)| - |f(y)|| = \sup_x |f(x)| - |f(y)|.$$

Combining the two equations we get

$$\sup_x |f(x)| - |f(y)| \le \sup_x |f(x) - f(y)|.$$

Taking the supremum over y in the above inequality we get that

$$\sup_x |f(x)| - \inf_y |f(y)| \le \sup_y \sup_x |f(x) - f(y)|.$$

We look at the right-hand side of this inequality. First note that $\sup_x |f(x) - f(y)|$ will be equal to either $|\inf_x f(x) - f(y)|$ or $|\sup_x f(x) - f(y)|$, depending on which is larger. Then taking the supremum over y we always obtain $\sup_x f(x) - \inf_y f(y)$. Hence, we get that

$$\sup_x |f(x)| - \inf_y |f(y)| \le \sup_x f(x) - \inf_y f(y).$$

Finally, we use this inequality for any interval $[x_{i-1}, x_i]$ in the chosen partition P, which gives

$$\begin{aligned} U_P(|f|) - L_P(|f|) &= \sum_{i=1}^n (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1}, x_i]} |f(x)| - \inf_{y \in [x_{i-1}, x_i]} |f(y)| \right) \\ &\le \sum_{i=1}^n (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{y \in [x_{i-1}, x_i]} f(y) \right) \le U_P(f) - L_P(f). \end{aligned}$$

The result follows from the Riemann integrability criterion.

(v) As f and g are Riemann integrable, they are bounded, otherwise the upper sum (and upper integral) would diverge. Hence, there is $C > 0$ such that $|f(x)|, |g(x)| \le C$ for all $x \in [a,b]$. Let P be a partition, and by M_i and m_i denote the supremum and infimum of f on $[x_{i-1}, x_i]$, respectively. Similarly, denote by L_i and l_i the corresponding quantities for g.

For any points $u_i, v_i \in [x_{i-1}, x_i]$ we have

$$\begin{aligned} &\left| \sum_{i=1}^n (x_{i-1} - x_i) \left((f \cdot g)(v_i) - (f \cdot g)(u_i) \right) \right| \\ &\quad = \left| \sum_{i=1}^n (x_{i-1} - x_i) \big(f(v_i) \big(g(v_i) - g(u_i) \big) - \big(f(v_i) - f(u_i) \big) g(u_i) \big) \right| \\ &\quad \le \sum_{i=1}^n (x_{i-1} - x_i) \left(C(L_i - l_i) + (M_i - m_i)C \right) \\ &\quad = C\big(U_P(g) - L_P(g) + U_P(f) - L_P(f) \big). \end{aligned}$$

Since u_i, v_i were arbitrary it follows that

$$U_P(f \cdot g) - L_P(f \cdot g) \le C\big(U_P(f) - L_P(f) + U_P(g) - L_P(g) \big).$$

Let $\varepsilon > 0$. Since, f and g are integrable, we can find partitions P_1 and P_2 such that

$$U_{P_1}(f) - L_{P_1}(f) \le \frac{\varepsilon}{2C} \quad \text{and} \quad U_{P_2}(g) - L_{P_2}(g) \le \frac{\varepsilon}{2C}.$$

By P' denote a common refinement. Then by the above, we get

$$U_{P'}(f \cdot g) - L_{P'}(f \cdot g) \le C \left(\frac{\varepsilon}{2C} + \frac{\varepsilon}{2C} \right) = \varepsilon,$$

which finishes the proof by the Riemann integrability criterion. $\square$

Exercise 9.13. *Give an example of a bounded function $f : [a, b] \to \mathbb{R}$ such that $|f|$ is Riemann integrable over $[a, b]$, but f is not.*

Theorem 9.14. *Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be bounded. Then*

$$(b-a) \inf_{a \le x \le b} f(x) \le \underline{\int_a^b} f(x)\ dx \le \overline{\int_a^b} f(x)\ dx \le (b-a) \sup_{a \le x \le b} f(x). \tag{16}$$

Proof. We first prove that

$$\underline{\int_a^b} f(x)\ dx \le \overline{\int_a^b} f(x)\ dx.$$

Let P be the partition $x_0 < \ldots < x_n$ and Q be the partition $y_0 < \ldots < y_m$ of $[a, b]$ and let P' be the least common refinement, i.e. $P \cup Q$. By lemma 9.5, if we add a finite number of points to a partition, we increase the lower sum and decrease the upper sum. So,

$$L_P(f) \le L_{P'}(f) \le U_{P'}(f) \le U_Q(f). \tag{17}$$

Hence, $L_P(f) \le U_Q(f)$ for all partitions P and Q of $[a, b]$. Taking the supremum over all partitions P of $[a, b]$, we get

$$\underline{\int_a^b} f(x)\ dx = \sup_P L_P(f) \le U_Q(f)$$

for all partitions Q of $[a, b]$. Taking the infimum over all partitions Q of $[a, b]$, we get

$$\underline{\int_a^b} f(x)\ dx \le \inf_Q U(f, Q) = \overline{\int_a^b} f(x)\, dx.$$

The remaining inequalities in (16) follow from the fact that for f bounded, we have that

$$(b-a) \inf_{x \in [a,b]} f(x) \le L_P(f) \le U_P(f) \le (b-a) \sup_{x \in [a,b]} f(x).$$

□

Corollary 9.15. *Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Then*

$$(b-a) \inf_{a \le x \le b} f(x) \le \int_a^b f(x)\, dx \le (b-a) \sup_{a \le x \le b} f(x).$$

We have already seen in example that not every function is integrable; not even every bounded function is integrable! However, continuity can save the day! Here we will see an application of uniform continuity.

Theorem 9.16. *Every continuous function f on a closed and bounded interval $[a, b]$ is Riemann integrable.*

Proof. We are going to use proposition 9.11, i.e. we aim to show $U_P(f) - L_P(f) < \varepsilon$.

Let $\varepsilon > 0$. By theorem 7.36 we know that f is uniformly continuous. Hence, we can pick $\delta = \delta(\varepsilon)$ such that, whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}.$$

Let P be a partition of $[a, b]$ with $\max_i |x_i - x_{i-1}| < \delta$. This means that for each i, $|x_i - x_{i-1}| < \delta$. So, for each i, $u_i - l_i \leq \frac{\varepsilon}{2(b-a)}$. So when we construct the upper and lower Riemann sums we find,

$$U_P(f) - L_P(f) = \sum_{i=1}^{n}(x_i - x_{i-1})(u_i - l_i) \leq \frac{\varepsilon}{2}(b-a)\sum_{i}^{n}(x_i - x_{i-1}) = \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2} < \varepsilon.$$

□

Proposition 9.17. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function which is continuous on (a, b). Then f is Riemann integrable.*

Proof. Let's give ourselves an $\varepsilon > 0$. Boundedness means that $|f(x)| < C < \infty$ for all $x \in [a, b]$. Let's take a partition, P, $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of $[a, b]$ which satisfies that $|x_1 - x_0| < \frac{\varepsilon}{K}$ and $|x_n - x_{n-1}| < \frac{\varepsilon}{K}$ for $\infty > K > 0$ to be chosen. Since f is continuous on $[x_1, x_{n-1}]$, we know from theorem 9.16 that we can pick our paritition such that it contains a partition P' of $[x_1, x_{n-1}]$ satisfying $U_{P'}(f) - L_{P'}(f) < \frac{\varepsilon}{2}$. Now,

$$\begin{aligned} U_P(f) - L_P(f) &= U_{P'}(f) - L_{P'}(f) + (a - x_1)(u_1 - l_1) + (x_{n-1} - b)(u_n - l_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{K}2C + \frac{\varepsilon}{K}2C = \Big(\frac{1}{2} + \frac{4C}{K}\Big)\varepsilon. \end{aligned}$$

Pick $K \leq 4C$ and we're done. □

Definition 9.18 (Piecewise Continuity)**.** *A real-valued function f defined on $[a, b]$ is piecewise continuous if there is a partition of $[a, b]$ into finitely many intervals $I_1, \ldots, I_n$ (which can be points) on each of which f is continuous.*

Example 9.19. *The function $f : \mathbb{R} \to \mathbb{R}$, defined by*

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

is a piecewise continuous function.

Corollary 9.20. *Every bounded piecewise continuous function on $[a, b]$ is integrable*

Proof. Lets partition $[a, b]$ into intervals $I_1, ..., I_n$ on which f is continuous. Suppose I_j has end points x_{j-1} and x_j, i.e. I_j is $[x_{j-1}, x_j]$, $[x_{j-1}, x_j)$, $(x_{j-1}, x_j]$ or (x_{j-1}, x_j). Then f is bounded on $[x_{j-1}, x_j]$ and continuous on (x_{j-1}, x_j). So by proposition 9.17, f is integrable on $[x_{j-1}, x_j]$. The result now follows from linearity of the Riemann integral. □

Theorem 9.21. *Let $f : [a, b] \to \mathbb{R}$ be monotone. Then f is Riemann integrable.*

Proof. Without loss of generality we may assume that f is monotonically increasing (otherwise we can consider $-f$). By monotonicity, we know that $f(a) \leq f(b)$. If $f(a) = f(b)$ then $f(x) = f(a) = \text{const.}$ for all $x \in [a, b]$. Hence, f is Riemann integrable.

Assume now that $f(a) < f(b)$, let $\varepsilon > 0$ and assume we take a partition with

$$\max_i |x_i - x_{i-1}| < \varepsilon/2(f(b)-f(a)).$$

Since f is montonically increasing $u_i = f(x_i)$ and $l_i = f(x_{i-1})$. Therefore,

$$\begin{aligned} U_P(f) - L_P(f) &= \sum_i (x_i - x_{i-1})(f(x_i) - f(x_{i-1})) \\ &< \frac{\varepsilon}{2(f(b) - f(a))} \sum_i (f(x_i) - f(x_{i-1})) = \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

□

Remark 9.22. Note that our function is bounded since it is defined everywhere on the closed interval. If you relax this to a function on $(a, b]$, $[a, b)$ or (a, b) this theorem is not true, i.e. $\frac{1}{x}$.

The definitive criterion for Riemann integrability is the following (see further courses in analysis for more details):

Theorem 9.23. *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of its discontinuity points has Lebesgue measure zero.*

Here, by Lebesgue measure zero we mean the following.

Definition 9.24. *A set $E \subseteq \mathbb{R}$ has* ***Lebesgue measure zero*** *if for every $\varepsilon > 0$ there exists a countable family of open intervals (a_n, b_n) such that*

$$E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \textit{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) \leq \varepsilon.$$

Remark 9.25. Another issue with the Riemann integral is the way it interacts with limits. In particular, it is not necessarily, if you have a sequence $(f_n)_n$ of Riemann integrable functions that converges at every point to some function f, then f is Riemann integrable. For example, fix an enumeration of $\mathbb{Q} \cap [0, 1]$ and define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & x = q_i, \qquad \forall 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Now, f_n converges pointwise to the Dirichlet function, which not Riemann integrable. However,

$$\int_0^1 f_n(x)dx = 0 \quad \forall n \implies \lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

So,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

since the RHS doesn't exist.

9.3 The Fundamental Theorem of Calculus

Integration is effectively the inverse operation to differentiation. This is roughly speaking the content of the fundamental theorem of calculus. This theorem is very appropriately named. Almost always when one computes integrals by hand you invoke it; you use the fact that you know a function which differentiates to the function you are trying to integrate, i.e. there exists an antiderivative. The fundamental theorem is usually split into two parts, one involving differentiating an intergal and the other involving integrating a derivative. The first part of the fundamental theorem of calculus gives existence of antiderivatives for continuous functions. The second states that one can compute integrals in terms of antiderivatives.

Theorem 9.26 (The First Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. For $x \in [a, b]$ defined $F : [a, b] \to \mathbb{R}$ by*

$$F(x) \doteq \int_a^x f(t)dt.$$

Then F is (Lipschitz) continuous in $[a, b]$, differentiable in (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$.

Proof. To prove differentiability we have to consider the quotient

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\Big[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\Big] = \frac{1}{h}\int_x^{x+h} f(t)dt,$$

by the properties of the Riemann integral in proposition 9.12. We want to show that, for $x \in (a, b)$,

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

or equivalently that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $|h - 0| = |h| < \delta$ then,

$$\Big|\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\Big| < \varepsilon.$$

So, let $\varepsilon > 0$. Since f is continuous at $x \in (a, b)$, we can find $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Therefore, if $|h| = |x - (x+h)| < \delta$, then

$$\begin{aligned}\Big|\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\Big| &= \Big|\frac{1}{h}\int_x^{x+h} f(t) - f(x)dt\Big| \\ &\leq \frac{1}{|h|}\Big|\int_x^{x+h} |f(t) - f(x)|dt\Big| < \frac{1}{|h|}\Big|\int_x^{x+h} \varepsilon dt\Big| = \varepsilon.\end{aligned}$$

To prove continuity, we note that for all $x, y \in [a, b]$

$$|F(x) - F(y)| = \Big|\int_x^y f(t)dt\Big| \leq \Big|\int_x^y |f(t)|dt\Big| \leq \sup_{t\in[a,b]} |f(t)||x - y|,$$

where we use theorem 7.21. This shows that F is Lipschitz continuous and, therefore, continuous. ☐

Remark 9.27. You can relax continuity of f and just require Riemann integrability (see Tao). The proof is largely the same.

Theorem 9.28 (The Second Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$. Then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let P be a partition $a = x_0 < x_1 < ... < x_n = b$. We can write

$$F(b) - F(a) = \sum_{k=1}^{n} [F(x_i) - F(x_{i-1})].$$

By the mean value theorem 8.24, in each (x_{i-1}, x_i) there exists a c_i such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1}).$$

So,

$$F(b) - F(a) = \sum_i f(c_i)(x_i - x_{i-1}),$$

which shows that

$$L_P(f) \leq F(b) - F(a) \leq U_P(f).$$

since we know $l_i \leq f(c_i) \leq u_i$. Since P was arbitrary, by the squeeze theorem 4.14,

$$F(b) - F(a) = \int_a^b f(t)dt.$$

□

Remark 9.29. The function F is often called an antiderivative of f: a differentiable function F whose derivative is equal to the original function f.

9.4 Techniques

The integral of a continuous function over a bounded interval always exists. That does not mean that it is always easily computed. In this section, we will show a few common techniques for actually computing complicated integrals.

9.4.1 Integration by Parts

As a corollary of the fundamental theorem of calculus, we have the formula for integration by parts. Arguably, integration by parts is the most important technique in the analysis of partial differential equations.

Corollary 9.30 (Integration by Parts)**.** *Let $f, g : [a, b] \to \mathbb{R}$ functions such that f' and g' exist and are Riemann integrable. Then*

$$\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)dx$$

Proof. By proposition 9.12, $f' \cdot g + f \cdot g'$ is integrable (f and g are continuous). Note that the function $h = f \cdot g$ is an antiderivative of $f' \cdot g + f \cdot g'$ by the product rule. By the fundamental theorem of calculus applied to h, we have

$$f(b)g(b) - f(a)g(a) = h(b) - h(a) = \int_a^b h'(x)dx = \int_a^b \Big(f'(x)g(x) + f(x)g'(x)\Big)dx.$$

□

9.4.2 A Digression on PDE

Solving PDE can be hard. Integration by parts is a very important tool. For example, suppose you wanted solve the PDE

$$\partial_x u + \partial_t u = 0$$

for some $u(t, x)$ on $(t, x) \in [0, T] \times [a, b]$ (given some initial conditions). We need to think about what it means to solve a PDE and what properties a solution to such an equation should have. A basic thing one would require is that the problem is *well-posed*

1. there exists a solution in a suitable function space,
2. the solution is unique (this is not a strict requirement, you can instead try to classify solutions),
3. the solution depends continuously on initial data, since in physical problems one would expect the solution to change only a little if you change the data.

However, we have still skirted the issue of what a solution really means. A natural requirement of a solution to PDE, which involves k^{th}-derivatives, would be to have a solution which is in C^k. This is typically what is called 'classical' solution to such a PDE. In the above example you could ask for the solution to be C^1 in time and space, i.e. first partial derivatives exist and are continuous.

In most cases, a direct argument to show existence of a classical solution to PDE is very hard. Moreover, a classical solution may not be the physical phenonena one wishes to describe. For example, many equations designed to model fluid dynamics (the above equation is an example of one) need to be flexible enough to allow for shocks formation, which is a curve of discontinuity in u.

Let's reformulate the problem slightly and try to weaken our notion of solution by separating the regularity of the solution from the existence of a solution. Let $v(t, x)$ be a C^1 function which vanishes on the boundary of $[0, T] \times [a, b]$, we will call this a test function. Then,

$$\int_0^T \int_a^b (\partial_x u + \partial_t u)v dt dx = 0. \tag{18}$$

Assuming we can change the order of integration, using IBP and using the boundary values of v we find

$$\int_0^T \int_a^b (\partial_x v + \partial_t v)u dt dx = 0. \tag{19}$$

So if u satisfies the original PDE (18) then u satifies (19) for any test function. Notice that for u to solve (19), u does not even have to be differentiable. In this case u would be called a weak solution to (18).

The common story in PDE is that it's easier to establish the existence of weak solutions to PDE than classical solutions. This is usually for functional analytic reasons. A standard tactic in establishing existence of solutions is finding approximate solutions in some function space and showing they converge to a true solution. Since the spaces of functions we use to define the notion of weak solution typically have nicer convergence properties than classical one.

One many now ask about the regularity of such solutions. It turns out that, once one establishes existence of a weak solution, in some space of functions, there is a large number of standard techniques to try to show that a weak solution is in fact classical, i.e. in our case C^1.

9.4.3 Substitution

Next we discuss *integration by substitution*, also referred to as change of variables.

Theorem 9.31 (Change of Variables)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $\phi : [c, d] \to [a, b]$ be continuously differentiable such that $\phi(c) = a$ and $\phi(d) = b$. Then*

$$\int_a^b f(x)dx = \int_c^d f(\phi(t))\phi'(t)dt.$$

Proof. Let $F : [a, b] \to \mathbb{R}$ and $G : [c, d] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(y)dy, \qquad G(z) = (F \circ \phi)(z) = F(\phi(z)).$$

By the first Fundamental Theorem of Calculus and the chain rule we have

$$F'(x) = f(x), \qquad G'(z) = F'(\phi(z))\phi'(z) = f(\phi(z))\phi'(z).$$

Therefore,

$$\begin{aligned}\int_c^d f(\phi(z))\phi'(z)dz &= \int_c^d G'(z)dz = G(d) - G(c) = F(\phi(d)) - F(\phi(c)) \\ &= F(b) - F(a) = \int_c^d F'(x)dx = \int_c^d f(x)dx.\end{aligned}$$

□

9.5 Improper Integrals

The Riemann integral can be slightly modified to allow for unboundedness (or ill-definition) of the function and unbounded domain of integration (or both). This is the topic of improper integrals.

If we return to the definition of the Riemann integral of a function $f : [a, b] \to \mathbb{R}$. Note that we actually need f to be bounded above on $[a, b]$. Otherwise, for any partition P there will be a subinterval $[x_{i-1}, x_i]$ on which f is unbounded above. This will give an infinite contribution to the upper sum and, therefore, the standard Riemann integral is undefined. (Analogously, we need f to be bounded below). However, you can make sense of Riemann integrals of some unbounded functions by taking a limit of Riemann integrals where f is bounded. Let's do an example:

Example 9.32. *Let's try to compute the integral of $f(x) = x^{-\frac{1}{a}}$ $(a > 1)$ on $[0, 1]$, where we define $f(0) = 0$ (in fact we can define $f(0)$ to be value including $\pm\infty$ or we can leave it undefined). The Riemann integral of this function in the sense we have discussed thus far is not defined. If we have some partition P with $0 = x_0 < x_1 < ... < x_n = 1$ then*

$$\sup_{x\in[0,x_1]} f(x) = \infty \qquad \inf_{x\in[0,x_1]} f(x) = 0.$$

*Clearly, the upper sum does not converge since it has an infinite contribution. However, intuitively the length of $[0, x_1]$ decreases linearly, i.e. faster than $x_1^{-1/a}$ 'blows up'. So, you might hope to make sense of area under the curve since the width of the first segment decreases more rapidly than the function diverges. To deal with this situation we can relax our notion of Riemann integral and **define***

$$\int_0^1 \frac{1}{x^{\frac{1}{a}}} dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x^{\frac{1}{a}}} dx.$$

An antiderivative F for $x^{-\frac{1}{a}}$ is $F(x) = \frac{a}{a-1} x^{\frac{a-1}{a}}$. So, by the Fundamental Theorem of Calculus,

$$\int_0^1 \frac{1}{x^{\frac{1}{a}}} dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x^{\frac{1}{a}}} dx = \lim_{\varepsilon\to 0} [F(1) - F(\varepsilon)] = \frac{a}{a-1} - \lim_{\varepsilon\to 0} \frac{a\varepsilon^{\frac{a-1}{a}}}{a-1} = \frac{a}{a-1} < \infty$$

Another situation we can envisage is of the infinite domain of integration. Here the standard Riemann integral fails to be well-defined because any partition will have a segment of infinite length. Again, you can make sense of Riemann integrals of functions on unbounded domains by taking a limit of Riemann integrals where the domain is bounded. Again, let's do an example:

Example 9.33. *Let's try to compute the integral of $f(x) = x^{-a}$ $(a > 1)$ on $[1, \infty)$. The Riemann integral of this function in the sense we have discussed thus far is not defined. If we have some partition P with $1 = x_0 < x_1 < ... <$ '$x_n = \infty$' then*

$$x_n - x_{n-1} = \infty.$$

*Clearly, the upper/lower sum do not converge since they have an infinite contribution. However, intuitively the length of $[x_{n-1}, x_n)$ increases linearly as $x_n \to \infty$, i.e. slower than ${x_n}^{-a}$ 'decays'. So, you might hope to make sense of area under the curve since the width of the final segment increases less rapidly than the function decays. To deal with this situation we can relax our notion of Riemann integral and **define***

$$\int_1^\infty \frac{1}{x^a} dx = \lim_{\varepsilon\to\infty} \int_1^\varepsilon \frac{1}{x^a} dx.$$

An antiderivative F for x^{-a} is $F(x) = \frac{1}{1-a}x^{1-a}$. So, by the Fundamental Theorem of Calculus,

$$\int_1^\infty \frac{1}{x^a}dx = \lim_{\varepsilon\to\infty}\int_1^\varepsilon \frac{1}{x^a}dx = \lim_{\varepsilon\to\infty}[F(\varepsilon) - F(1)] = \lim_{\varepsilon\to\infty}\frac{1}{1-a}\varepsilon^{1-a} - \frac{1}{1-a} = \frac{1}{a-1} < \infty.$$

10 Sequences of Functions

In this last section, we apply what we discovered above to the metric space of continuous functions and give a short outline of more advanced results. It is a very successful strategy in mathematics to think about functions in two different ways – first, as a function which takes a set X to a set Y, and second as an element of a (typically infinite-dimensional) space of functions. So a function is a point in a space where each element is a function. This may seem overly theoretical, but has proven quite helpful in the last century. This point of view is the focus of *functional analysis*. We will only scratch the surface of the subject.

10.1 Pointwise and Uniform Convergence

So far we have talked about convergence sequences in metric spaces. A natural direction to explore is convergence of a sequence of functions. To this end, let A be a set and $(f_n)_n$ a sequence of functions $f_n : A \to \mathbb{R}$. We now ask what could we mean by convergence for such a sequence? Well there is an obvious candidate: for each $x \in A$, $(f_n(x))_n$ is a sequence in the reals. So we get a notion of convergence induced by convergence in $\mathbb{R}$. This is pointwise convergence:

Definition 10.1 (Pointwise Convergence)**.** *Let A be a set and suppose $(f_n)_n$ is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. We say that f_n converges pointwise to f on A if $f(x) = \lim_{n\to\infty} f_n(x)$ at each $x \in A$.*

Remark 10.2. Note that this only requires that at each $x \in A$ the following condition holds: for all $\varepsilon > 0$, there exists $N(\varepsilon, x) \in \mathbb{N}$ such that, if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$. Note the dependence of N on $x \in A$.

The most natural question to ask next is: can we deduce the properties of f from f_n? For example, if f_n are continuous, is f? The answer is generally no, pointwise convergence does not preserve properties.

Example 10.3. *Let $f_n : [-1, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^{1/2n+1}$. The pointwise limit is:*

$$f_n(x) \to f(x) \doteq \begin{cases} 1 & x \in (0, 1] \\ 0 & x = 0 \\ -1 & x \in [-1, 0). \end{cases}$$

Each f_n is continuous but f is clearly discontinuous.

The notion of convergence that preserves properties of the sequence is uniform convergence:

Definition 10.4 (Uniform Convergence)**.** *Let A be a set and let $f_n, f : A \to \mathbb{R}$ for all $n \in \mathbb{N}$. We say that $(f_n)_n$ converges uniformly to f on A, if for all $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for all $n \geq N$ and for all $x \in A$, we have $|f_n(x) - f(x)| < \varepsilon$. Equivalently, $(f_n)_n$ converges uniformly to f on A, if for all $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for all $n \geq N$, we have*

$$\sup_{x\in A} |f_n(x) - f(x)| < \varepsilon.$$

Remark 10.5. It is easy to convince yourself that $f_n \to f$ uniformly $\implies$ $f_n \to f$ pointwise. This is useful for finding the uniform limit, if it exists.

Pointwise convergence does not imply uniform convergence:

Example 10.6. *Let's consider to* $f_n(x) = \operatorname{sign}(x)|x|^{1/2n+1}$ *on* $[-1, 1]$*. The pointwise convergence is*

$$f_n(x) \to f(x) \doteq \begin{cases} 1 & x \in (0,1] \\ 0 & x = 0 \\ -1 & x \in [-1, 0). \end{cases}$$

For $(f_n)_n$ *to converge uniformly it would have to converge to this pointwise limit. Note that* $f_n(1/2^{2n+1}) = 1/2$ *and* $f(1/2^{2n+1}) = 1$*. So we can find an* $x = 1/2^{2n+1}$ *such that* $|f_n(x) - f(x)| = \frac{1}{2}$*. So if* $\varepsilon < 1/2$ *then this demonstates the violation of the definition of uniform convergence.*

Now that we have introduced uniform convergence we can show that this is the correct notion to preserve continuity:

Theorem 10.7. *Let* (X, d) *be a metric space and* $A \subseteq X$*. Let* $(f_n)_n$ *be a sequence of functions from* $f_n : A \to \mathbb{R}$ *and* $f : A \to \mathbb{R}$*. Suppose* $f_n \to f$ *uniformly. If all* f_n *are continuous at* $x \in A$ *then* f *is continuous at* x*.*

Proof. Let $\varepsilon > 0$. By uniform convergence, we can choose $N = N(\varepsilon) \in \mathbb{N}$ such that, for all $n \geq N$,

$$\sup_{x \in A} |f_n(x) - f(x)| < \varepsilon.$$

We also know that f_N is continuous at $x \in A$. So this means there exists a $\delta > 0$ such that if $d(x, y) < \delta$ and $y \in A$, then

$$|f_N(x) - f_N(y)| < \varepsilon.$$

So, now we can combine these two facts with the triangle inequality to show that if $y \in A$ and $|x - y| < \delta$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &\leq 3\varepsilon. \end{aligned}$$

Note that the essential requirement of uniform convergence is used for the last term involving y. (It is also use for the first term but pointwise convergence at x would be enough here). $\square$

10.2 Weierstrass Approximation of Continuous Functions

The idea of approximations in analysis is very important. The theorem we aim at here is known as the (Stone–)Weierstrass approximation theorem. It effectively says that polynomial functions can be used to approximate continuous functions $C^0([a, b], \mathbb{R})$.

Theorem 10.8 (Weierstrass Approximation Theorem)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then there exists a sequence of polynomials $(p_n)_n$ such that $p_n \to f$ uniformly on $[a, b]$.*

Lemma 10.9. *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1, \qquad \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{k}{n}\right)^2 x^k (1-x)^{n-k} = \frac{x(1-x)}{n}$$

Proof. The first can be verifed via the binomial theorem:

$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k.$$

If you let $t = \frac{x}{1-x}$ you find

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{-k}.$$

Differentiating and multiplying by t gives

$$nt(1+t)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k t^k = \sum_{k=0}^{n} \binom{n}{k} k t^k.$$

Therefore,

$$\frac{xn}{(1-x)^n} = \sum_{k=0}^{n} \binom{n}{k} k x^k x^{-k} \implies x = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k}.$$

Once more, differentiating and multiplying by t gives

$$n(n-1)t^2(1+t)^{n-2} = \sum_{k=1}^{n} \binom{n}{k} k(k-1) t^k = \sum_{k=0}^{n} \binom{n}{k} k(k-1) t^k.$$

So,

$$x^2 = \sum_{k=0}^{n} \binom{n}{k} \frac{k(k-1)}{n(n-1)} x^k (1-x)^{n-k}.$$

So,

$$x^2 = \sum_{k=0}^{n} \binom{n}{k} x^2 x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x \frac{k}{n} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{k(k-1)}{n(n-1)} x^k (1-x)^{n-k}.$$

So,

$$\sum_{k=0}^{n} \binom{n}{k} x^2 x^k (1-x)^{n-k} - 2 \sum_{k=0}^{n} \binom{n}{k} x \frac{k}{n} x^k (1-x)^{n-k} = -x^2,$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{k(k-1)}{n^2} + \frac{k}{n^2}\Big] x^k (1-x)^{n-k} = \frac{n-1}{n} x^2 + \frac{1}{n} x.$$

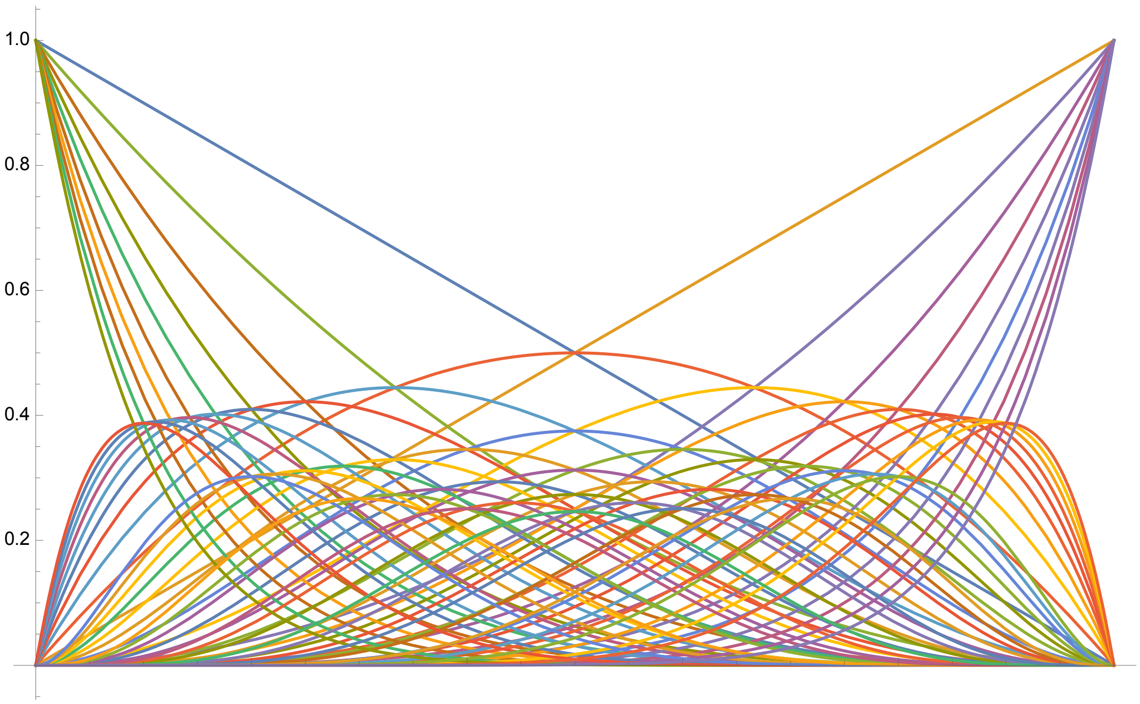
Therefore,

$$\sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{k}{n}\right)^2 x^k (1-x)^{n-k} = \left(1 - \frac{1}{n}\right) x^2 + \frac{1}{n} x - x^2 = \frac{x(1-x)}{n}.$$

$\square$

Proof of Theorem 10.8. We will prove this on $[0,1]$ for simplicity, the following proof can be adapted to $[a,b]$ by a change of variables.

We let $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. These are known as the Bernstein polynomials and for each x satisfy lemma 10.9. Here is a plot of the first 65, i.e. for $n \in \{1, ..., 10\}$:



Define

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x).$$

These polynomials chop up the domain $[0,1]$ and sample the function at $\frac{k}{n}$ then multiply it by a Bernstein polynomial. The claim is that it is these p_n that converge to f uniformly, i.e. for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in [0,1]} |p_n(x) - f(x)| < \varepsilon.$$

By lemma 10.9, we have

$$p_n(x) - f(x) = \sum_{k=1}^{n} \left[f\left(\frac{k}{n}\right) - f(x) \right] b_{n,k}(x).$$

Hence, by the triangle inequality

$$|p_n(x) - f(x)| \leq \sum_{k=1}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x).$$

Now f is continuous and $[0,1]$ is a compact set, so f is uniform continuous. Therefore, $\forall \varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|x - y| < \delta$ then

$$|f(x) - f(y)| < \varepsilon.$$

So we can divide up the sum in the following way,

$$|p_n(x) - f(x)| \leq \sum_{\substack{k=1 \\ |x-\frac{k}{n}|<\delta}}^{n} \left|f\Big(\frac{k}{n}\Big) - f(x)\right| b_{n,k}(x) + \sum_{\substack{k=1 \\ |x-\frac{k}{n}|\geq\delta}}^{n} \left|f\Big(\frac{k}{n}\Big) - f(x)\right| b_{n,k}(x).$$

The first term on the RHS is fine by uniform continuity. The second term can be dealt with as follows. A continuous function on a compact set is bounded, i.e. $\sup_x |f(x)| = K < \infty$. Second,

$$\sum_{\substack{k=1 \\ |x-\frac{k}{n}|\geq\delta}}^{n} b_{n,k}(x) = \sum_{\substack{k=1 \\ |x-\frac{k}{n}|\geq\delta}}^{n} 1 \times b_{n,k}(x) \leq \sum_{\substack{k=1 \\ |x-\frac{k}{n}|\geq\delta}}^{n} \frac{1}{\delta^2}\Big(x - \frac{k}{n}\Big)^2 b_{n,k}(x)$$

since

$$\left|x - \frac{k}{n}\right| \geq \delta \implies \frac{|x-\frac{k}{n}|^2}{\delta^2} \geq 1.$$

Now, we can use lemma 10.9 to deduce that

$$\sum_{\substack{k=1 \\ |x-\frac{k}{n}|\geq\delta}}^{n} b_{n,k}(x) \leq \frac{x(1-x)}{\delta^2 n} \leq \frac{1}{\delta^2 n}.$$

Combining, we find

$$|p_n(x) - f(x)| \leq \varepsilon + \frac{2K}{\delta^2 n} < 2\varepsilon.$$

if we choose $N > \frac{2K}{\delta^2 \varepsilon}$. $\square$

Remark 10.10. In 1937, Stone isolated the algebraic/functional structure of the polynomials on $[a, b]$ to extend the Weierstrass approximation theorem to an arbitrary metric space. The analogous theorem, known as the Stone–Weierstrass approximation theorem, roughly states that if a subfamily of continuous functions on a compact metric space X forms an algebra (with some further assumptions) then it approximates $C^0(X, \mathbb{R})$. This is really beyond the scope of this course, one should see a course on functional analysis.

10.3 Banach Spaces

Definition 10.11 (Banach Space). *A normed vector space $(X, ||\cdot||)$ is said to be complete if every Cauchy sequence in X converges in x (under the metric induced by the norm).*

Remark 10.12. Since the metric is implicitly defined when you have a norm, the standard defintion of a Banach space is as a complete normed space.

Example 10.13. *Any finite dimensional normed space is complete. This follows from showing that there exists an isomorphism to $\mathbb{R}^n$ with the usual norm*

$$||x||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example 10.14 (ℓ^p Spaces)**.** *Recall that a sequence in $\mathbb{R}$ is construct from a map from $\mathbb{N}$ to $\mathbb{R}$, $x : \mathbb{N} \to \mathbb{R}$. Let $\mathbb{R}^{\mathbb{N}}$ denote the space of sequences, i.e.*

$$x \in \mathbb{R}^{\mathbb{N}} \implies x = (x_n)_n.$$

Now denote a sequence in $\mathbb{R}^{\mathbb{N}}$ as $(x^{(n)})_n$. This means for each n, $x^{(n)}$ is a sequence:

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots).$$

We can define a family of norms on $\mathbb{R}^{\mathbb{N}}$ indexed by $p \in [1, \infty)$

$$||x||_p = \Big(\sum_{i=1}^{\infty} |x_i|^p\Big)^p, \qquad ||x||_\infty = \sup_n |x_n|$$

From these we can define the sequence normed spaces ℓ^p constructed from the $\mathbb{R}^{\mathbb{N}}$ by

$$\ell^p = \{x \in \mathbb{R}^{\mathbb{N}} : ||x||_p < \infty\}.$$

These turn out to be Banach spaces for all $p \in [1, \infty]$.

Example 10.15 (c, c_0 and c_{00})**.** *One can define the space of convergence sequences:*

$$c = \{x \in \mathbb{R}^{\mathbb{N}} : \exists l \in \mathbb{R} \ s.t. \ x_n \to l\}$$

We know that convergence sequences are bounded, therefore, $c \subseteq \ell^\infty$. It turns out that c is also a Banach space under $|| \cdot ||_\infty$ since one can show it is a closed subspace of ℓ^∞.

We can restrict our attention to convergence sequences which converge to 0*:*

$$c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : x_n \to 0\} \subseteq c.$$

Again this turns out to be a Banach space.

On the contrary if we restirct our attention to sequences which are eventually 0*:*

$$c_{00} = \{x \in \mathbb{R}^{\mathbb{N}} : x_n = 0 \quad \forall n \geq N(x) \in \mathbb{N}\} \subseteq c_0.$$

However, this is not a Banach space with respect to $|| \cdot ||_\infty$ since the sequence $(x^{(n)})_n$ defined through

$$x^{(n)} = \Big(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, 0, \dots\Big)$$

is Cauchy but does not converge to an element of c_{00}. Instead, it converges to an element of c_0.

We will now show that the space of continuous functions

Theorem 10.16. *The space of bounded real valued functions*

$$\mathcal{B}([a, b], \mathbb{R}) \doteq \{f : [a, b] \to \mathbb{R} : \sup_{x \in [a,b]} |f(x)| < \infty\}$$

with the supremum norm

$$|| \cdot ||_\infty \doteq \sup_{x \in [a,b]} | \cdot |.$$

is a Banach space.

Proof. Let $(f_n)_n \subseteq \mathcal{B}([a,b],\mathbb{R})$ be a Cauchy-sequence with respect to the metric:

$$d_\infty(f,g) = ||f-g||_\infty = \sup_{x\in[a,b]} |f(x)-g(x)|,$$

i.e. for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in[a,b]} |f_n(x) - f_m(x)| = d(f_n, f_m) < \varepsilon \quad \text{for all } n, m \geq N\,.$$

Therefore, given $x \in [a,b]$ it holds that

$$|f_n(x) - f_m(x)| \leq \sup_{x\in[a,b]} |f_n(x) - f_m(x)| = d(f_n, f_m) < \varepsilon \quad \text{for all } n, m \geq N\,.$$

This implies that for each $x \in \mathbb{R}$ $(f_n(x))_n \subseteq \mathbb{R}$ is a Cauchy-sequence. This implies that $f\colon [a,b] \to \mathbb{R}$ given by

$$f(x) \doteq \lim_{n\to\infty} f_n(x)$$

is a well-defined real-valued function by completeness of $\mathbb{R}$. This is our candidate limit in $||\cdot||_\infty$.

We now show that f is bounded. Fix $\varepsilon = 1$ then there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < 1 \quad \text{for all } n, m \geq N\,.$$

Note that for $\kappa \in \mathbb{R}$

$$||f_n(x) - \kappa| - |f(x) - \kappa|| \leq |(f_n(x) - \kappa) - (f(x) - \kappa)| \implies \lim_{n\to\infty} |f_n(x) - \kappa| = |f(x) - \kappa|,$$

or in otherwords,

$$\lim_{n\to\infty} |f_n(x) - \kappa| = \Big| \lim_{n\to\infty} f_n(x) - \kappa \Big|.$$

Thus,

$$\begin{aligned} |f(x)| &= |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \\ &= \lim_{n\to\infty} |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + \|f_N\|_\infty < \infty\,. \end{aligned}$$

Finally, we show convergence in norm, i.e. that $d(f_n, f) \to 0$ as $n \to \infty$. To this end, let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $d(f_n, f_m) < \varepsilon/2$ for all $n, m \geq N$. Then for all $n \geq N$ we have

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \lim_{m\to\infty} d_\infty(f_n, f_m) \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{for all } x \in [a,b]\,.$$

Thus, for all $n \geq N$ there holds

$$d_\infty(f_n, f) = \sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon\,.$$

This shows that $d_\infty(f_n, f) \to 0$ as $n \to \infty$. □

Corollary 10.17. *The space* $(C^0([a,b]), ||\cdot||_\infty \doteq \sup_{x\in[a,b]} |\cdot|)$ *is a Banach space.*

Proof. We note that every continuous function on a compact set is bounded. Hence, every Cauchy sequence converges in $\mathcal{B}([a,b],\mathbb{R})$ with $\|\cdot\|_\infty$, i.e uniformly. By theorem 10.7, if f_n are all continuous then f is. □

10.4 Sequential Compactness, Convergent Subsequences and Arzelà–Ascoli

We have seen that if f_n is continuous and $f_n \to f$ uniformly, then f is continuous. Moreover, that if $f_n \in C^0$ is Cauchy then f_n converges to an element of C^0. What can we say about an arbitrary sequence? Under what conditions do sequences have convergent subsequences? In $\mathbb{R}$, we found that if a sequence that lies in closed and bounded set A then it has a convergent subsequence in A, i.e. a set A being closed and bounded gave sequential compactness (see 6.59). This relied upon Bolzano-Weierstrass: a bounded sequence in $\mathbb{R}$ always has a convergent subsequence in $\mathbb{R}$. The closedness then told us the limit was in A. The issue in function spaces is that closed and boundedness does not imply sequential compactness. Indeed, one has the following theorem:

Theorem 10.18 (Riesz Theorem). *A normed space $(X, ||\cdot||)$ is finite dimensional if and only if the closed unit ball in X is compact. In particular, the closed unit ball of any infinite-dimensional normed space is not compact.*

Proof. See a course in functional analysis. See below for $(C^0([a,b]), ||\cdot||_\infty)$. □

Example 10.19. *Let's show this in $C^0([-1,1], ||\cdot||_\infty)$. Consider the closed unit ball at the origin:*

$$\overline{B}_1(0) \doteq \{f \in C^0([-1,1]) : ||f||_\infty \leq 1\}.$$

This set is bounded by definition. To see that it is closed, lets show that $C^0([-1,1]) \setminus \overline{B}_1(0)$ is open. Take a function $f \in C^0([-1,1]) \setminus \overline{B}_1(0)$. Then

$$\sup_{x\in[-1,1]} |f(x)| > 1$$

so there exists $x_0 \in [-1,1]$ such that $|f(x_0)| > 1$. Set $\varepsilon = \frac{|f(x_0)|-1}{2} > 0$ and observe that for $g \in B_\varepsilon(f)$ we have

$$\begin{aligned}
\sup_{x\in[-1,1]} |g(x)| = \sup_{x\in[-1,1]} |f(x) - (f(x) - g(x))| &\geq \sup_{x\in[-1,1]} \big||f(x)| - |g(x) - f(x)|\big| \\
&\geq \sup_{x\in[-1,1]} \big(|f(x)| - |g(x) - f(x)|\big) \\
&\geq \sup_{x\in[-1,1]} (|f(x)| - \varepsilon) \geq |f(x_0)| - \varepsilon > 1
\end{aligned}$$

so $(\overline{B}_1(0))^c$ is open and thus $\overline{B}_1(0)$ is closed. The sequence of functions $f_n \in B$

$$f_n(x) = \begin{cases} -1 & x \leq -\frac{1}{n} \\ nx & -\frac{1}{n} < x < \frac{1}{n} \\ 1 & x \geq \frac{1}{n} \end{cases}$$

does not have a subsequence which converges in $C^0([-1,1])$ under $||\cdot||_\infty$ since the pointwise limit is dicontinuous, and if a uniform limit were to exist, it would be continuous and agree with the pointwise limit.

We would like to restore sequential compactness in some way, since it is a very useful tool for extracting convergence subsequences. There are two options (which turn out to be related):

1. (Not Examinable). Weaken the topology on the normed space. Now this may sound a little vague, so let's elaborate. On a normed space $(X, ||\cdot||)$ we can induce a metric $d(\cdot,\cdot) \doteq ||\cdot - \cdot||$. This defines our open and closed sets, i.e. our 'normal' topology! This is sometimes called the 'strong' or norm topology. One can, however, define different notions of topology. One can induce a topology on X from the dual normed space X' of continuous linear function(al)s that map $X \to \mathbb{R}$, which has natural notion of norm given by

$$||\Phi||_{X'} = \sup_{\substack{x\in X,\\ ||x||_X=1}} |\Phi(x)|.$$

   One then defines a notion of *weak* convergence of $x_n \in X$ to $x \in X$, denoted $x_n \rightharpoonup x$, by

$$x_n \rightharpoonup x \iff |\Phi(x_n - x)| \to 0 \qquad \forall \Phi \in X'.$$

   This turns out to be profitable in restoring compactness and preserving properties under (weak) convergence.

2. Restrict our attention to a subset of the closed unit ball. This is the rough idea of the Arzelà–Ascoli theorem for $C^0([a,b], ||\cdot||_\infty)$.

We now aim to state and prove the Arzela–Ascoli theorem. One should think of this theorem as an analogy to Bolzano–Weierstrass for sequence of functions. Namely, it gives necessary and sufficient conditions for when a sequence of functions on a compact metric space has a uniformly convergent subsequence. Whilst it's statement may appear slightly opaque, it sees application in many areas of analysis. A concrete example is in the Peano existence theorem for ordinary differential equations. To state it we need to discuss what it means for families of functions to be comparibly continuous.

Definition 10.20 (Pointwise Equicontinuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces and let $(f_n)_n$ be a sequence of functions $f_n : X \to Y$. The sequence $(f_n)_n$ is pointwise equicontinuous at $x \in X$ if for all $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon) > 0$ such that if $d_X(x, y) < \delta$ then $d_Y(f_n(x), f_n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.*

Again we can uniformise by taking out the dependence on the point:

Definition 10.21 (Uniform Equicontinuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subseteq X$ and let $(f_n)_n$ be a sequence of functions $f_n : A \to Y$. The sequence $(f_n)_n$ is uniformly equicontinuous on A if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $d_X(x, y) < \delta$ then $d_Y(f_n(x), f_n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.*

Remark 10.22. One can replace this sequential definition with simply a family of functions.

Remark 10.23. We have talked about many notions of continuity at this stage, it is appropriate to take stock and compare. So, what is the difference between continuity, uniform continuity, pointwise equicontinuity and uniform equicontinuity? In simple terms all that differs is what δ depends on. For a sequence of functions $(f_n)_n$:

- continuity, $\delta = \delta(\varepsilon, x, n)$.
- uniform continuity, $\delta = \delta(\varepsilon, n)$.
- pointwise equicontinuity, $\delta = \delta(\varepsilon, x)$.
- uniform equicontinuity, $\delta = \delta(\varepsilon)$.

Lemma 10.24. *Let (X, d_X) and (Y, d_Y) be metric spaces and $K \subseteq X$ a compact subset. Let $(f_n)_n$ be a sequence of functions $f_n : K \to Y$. Then if f is pointwise equicontinuous it is uniformly equicontinuous.*

Proof. Let $\varepsilon > 0$. Our sequence of functions $(f_n)_n$ is pointwise equicontinuous. So, for each $x \in A$, there exists $\delta(\varepsilon, x) > 0$ such that if $d_X(x, y) < \delta$ then

$$d_Y(f(x), f(y)) < \varepsilon.$$

Now, $\{B_{\frac{\delta(\varepsilon,x)}{2}}(x)\}_{x\in K}$ forms an open cover of K. So there exists a finite subcover

$$B_{\frac{\delta(\varepsilon,x_1)}{2}}(x_1), \dots, B_{\frac{\delta(\varepsilon,x_N)}{2}}(x_N).$$

Set $\delta = \frac{1}{2}\min_i\{\delta(\varepsilon, x_i)\}$ and suppose $d_X(x, y) < \delta$. Then there exists an i such that $x \in B_{\frac{\delta(\varepsilon,x_i)}{2}}(x_i)$. Moreover, $y \in B_{\delta(\varepsilon,x_i)}(x_i)$ since

$$d_X(y, x_i) \leq d_X(x, y) + d_X(x_i, x) < \delta(\varepsilon, x_i).$$

So, $x, y \in B_{\delta(\varepsilon,x_i)}(x_i)$ which implies

$$d_Y(f_n(x), f_n(y)) \leq d_Y(f_n(x_i), f_n(y)) + d_Y(f_n(x_i), f_n(x)) < 2\varepsilon$$

by pointwise equicontinuity at x_i.

□

We need one more definition before stating the theorem, a notion of boundedness thats uniform in x and n. Let's state three definitions to try and make this clear.

Definition 10.25 (Boundedness of a Function)**.** *A function $f : A \to \mathbb{R}$ is bounded on A if there exists $C \in \mathbb{R}$ such that* $\sup_{x\in A} |f(x)| < C$.

Now if you have a sequence of functions $(f_n)_n$ you can ask for boundedness of each f_n via the above definition but you can ask for a different notion of boundedness, pointwise boundedness, i.e. boundedness with respect to the index n.

Definition 10.26 (Pointwise Boundedness)**.** *A sequence of functions $(f_n)_n$ with $f_n : A \to \mathbb{R}$ is pointwise bounded if for each $x \in A$, there exists $M(x) \in \mathbb{R}_+$ such that*

$$\sup_n |f_n(x)| < M(x).$$

Remark 10.27. This says that at each $x \in A$ the sequence $(x_n = f_n(x))_n$ is bounded.

We also want to uniformise this in x for a sequence of functions.

Definition 10.28 (Uniform Boundedness)**.** *Let (X, d_X) be a metric space, $A \subseteq X$ and let $(f_n)_n$ be a sequence of functions $f_n : X \to \mathbb{R}$. The sequence $(f_n)_n$ is uniformly bounded on A if there exists $C \in \mathbb{R}$ (independent of n) such that*

$$\sup_n \left(\sup_{x \in A} f_n(x) \right) < C.$$

Remark 10.29. If you are concerned with the order of sup. One can think of $(f_n(x))_n$ as a double indexed sequence (a_{mn}) for which you have the following estimates

$$a_{mn} \leq \sup_n a_{mn} \implies \sup_m a_{mn} \leq \sup_m \sup_n a_{mn} \implies \sup_n \sup_m a_{mn} \leq \sup_m \sup_n a_{mn}.$$

You can swap m and n in this estimate and, therefore, $\sup_n \sup_m a_{mn} = \sup_m \sup_n a_{mn}$.

Lemma 10.30. *Let (X, d) is a metric space, $K \subseteq X$ is compact and $(f_n)_n$ be a sequence of functions which are pointwise equicontinuous. If $(f_n)_n$ is pointwise bounded then $(f_n)_n$ is uniformly bounded on K.*

Proof. We have that for each $x \in A$, there exists $M(x) \in \mathbb{R}_+$ such that

$$\sup_n |f_n(x)| < M(x).$$

Our sequence $(f_n)_n$ is pointwise equicontinuous. This means for each $x \in K$ and $\forall \varepsilon > 0$, there exists $\delta(x, \varepsilon) > 0$ such that if $d_X(x, y) < \delta$ then $\sup_n |f_n(x) - f_n(y)| < \varepsilon$.

Let $\varepsilon = 1$, therefore, there exists $\delta(x) = \delta(x, 1) > 0$ such that if $d_X(x, y) < \delta$ then

$$\sup_n |f_n(x) - f_n(y)| < 1.$$

Now, $\{B_{\delta(x)}(x)\}_{x \in K}$ cover K so there exists a finite subcover,

$$B_{\delta(x_1)}(x_1), \ldots, B_{\delta(x_N)}(x_N).$$

For each $x \in K$ there exists i such that $x \in B_{\delta(x_i)}(x_i)$. This means

$$||f_n(x)| - |f_n(x_i)|| \leq |f_n(x) - f_n(x_i)| < 1.$$

This gives that

$$|f_n(x)| < 1 + |f_n(x_i)| \leq 1 + M(x_i) \leq 1 + \max_i M(x_i).$$

Since our choices are independent of x, and this last estimate is independent of x, we can take the $\sup_n \sup_x$ to give the result. □

Theorem 10.31 (Arzelà–Ascoli)**.** *Let (X, d) is a metric space, $K \subseteq X$ is compact and $(f_n)_n$ be a sequence of functions $f_n \in C^0(K, \mathbb{R})$. If $(f_n)_n$ is pointwise bounded and pointwise equicontinuous on K, then $(f_n)_n$ contains a subsequence which converges uniformly, i.e. it converges in the metric*

$$d_\infty(f, g) \doteq ||f - g||_\infty = \sup_{x \in K} |f(x) - g(x)|$$

on $C^0(K, \mathbb{R})$.

The proof has a few ingredients:

1. Density of the rationals. (In generality, you need a countable dense subset or, in fancier terms, K has to be a seperable. It turns out that if K is a compact metric space then it is seperable.)
2. Bolzano–Weierstrass, in particular, the implication that $(\mathbb{R}, |\cdot|)$ is a complete metric space.
3. A diagonal argument: this is a way of extracting a convergent subsequence.
4. Equicontinuity: comparible continuity of our functions.
5. Heine–Borel: closed and bounded in $(\mathbb{R}, |\cdot|)$ implies sequential compactness.

Proof. We will prove this for $K = [a, b]$ with $a < b$ with the usual metric. Note that by lemmas 10.30 and 10.24, we may assume that $(f_n)_n$ are uniformly bounded and uniformly equicontinuous on $[a, b]$.

Let's sketch the proof:

1. We can reduce the problem to exhibiting a Cauchy subsequence. Since $C^0([a, b])$ with d_∞ is complete.
2. We can extract pointwise convergence of a subsequence of $(f_n)_n$ on $\mathbb{Q}\cap[a, b]$, which can be done by countability, Bolzano-Weierstrass (since our functions are uniformly bounded) and a diagonal argument (essentially to pick out the correct subsequence). We can now use the density of the rationals to extend to all $x \in [a, b]$.
3. We can use uniform equicontinuity in conjunction with compactness of $[a, b]$ and density of the rationals to cover $[a, b]$ with a *finite collection* of open balls centred at rationals $x_1, ..., x_N$ of a size $\delta > 0$ compatible with uniform continuity.
4. Combining point 3 with pointwise convergence let us extract uniform convergence on a finite number of rationals x_k (whose δ balls cover our space). In particular, the sequence is (uniformly) Cauchy on this finite number of rationals.
5. Any $x \in [a, b]$ is in one of the $B_\delta(x_k)$ balls. One can then bridge Cauchy-ness at x_k to Cauchy-ness at x via uniform equicontinuity and the triangle inequality.

Enumerate $\mathbb{Q} \cap [a, b]$ by $\mathbb{Q} \cap [a, b] = (x_k)_{k\in\mathbb{N}}$ and evaluate the functions f_n on (x_k) to create a countable set $(f_n(x_k))_{k,n\in\mathbb{N}}$. By uniform boundedness of $(f_n)_n$, the real sequence $(f_n(x_1))_{n\in\mathbb{N}}$ is bounded. By the Bolzano-Weierstrass Theorem there exists a subsequence $(f_{1,n})_{n\in\mathbb{N}} \subseteq (f_n)_{n\in\mathbb{N}}$ such that $(f_{1,n}(x_1))_{n\in\mathbb{N}}$ converges. Now, the real sequence $(f_{1,n}(x_2))_{n\in\mathbb{N}}$ is bounded so there exists a converging subsquence $(f_{2,n}(x_2))_{n\in\mathbb{N}}$. Note that the subsequence $(f_{2,n})_n$ converges both at x_1 and x_2. Proceeding in this way, we obtain a countable collection of subsequences of the original sequence,

$$(f_n)_n \supseteq (f_{1,n})_n \supseteq (f_{2,n})_n \supseteq \cdots.$$

which we can organise instrutively as a infinite array:

$$\begin{matrix} f_{1,1} & f_{2,1} & f_{3,1} & \cdots \\ f_{1,2} & f_{2,2} & f_{3,2} & \cdots \\ f_{1,3} & f_{2,3} & f_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}.$$

The sequence in the n-th column converges at the point $x_1, \dots, x_n$ and each column is a subsequence of the previous one.
Claim: The diagonal sequence $(h_n \doteq f_{n,n})_{n\in\mathbb{N}}$ is a subsquence of the original sequence $(f_n)_{n\in\mathbb{N}}$ that converges pointwise at each $\mathbb{Q} \cap [a,b]$.

Let $\varepsilon > 0$. By uniform equicontinuity of $(f_n)_n$ there exists a $\delta(\varepsilon) > 0$ such that, if $|x-y| < \delta$ then

$$\sup_n |f_n(x) - f_n(y)| < \frac{\varepsilon}{3} \implies \sup_{\substack{x,y\in[a,b] \\ |x-y|<\delta}} \sup_n |h_n(x) - h_n(y)| < \frac{\varepsilon}{3}$$

By the density of the rationals, the balls $\{B_\delta(x_k)\}_{k\in\mathbb{N}}$ are an open cover of $[a,b]$. Since $[a,b]$ is compact (by Heine-Borel), there exists $N(\delta) \in \mathbb{N}$ such that

$$[a,b] \subseteq \bigcup_{k=1}^{N(\delta)} B_\delta(x_k).$$

Our subsequence $(h_n)_n$ converges pointwise for all x_k. So, in particular, $(h_n)_n$ converges pointwise for all x_k with $k = 1, \dots, N(\delta)$. This means the sequence is Cauchy: $\exists M \in \mathbb{N}$ such that

$$|h_n(x_k) - h_m(x_k)| < \frac{\varepsilon}{3} \quad \text{for all } k = 1, \dots, N(\delta) \text{ and for all } n, m \geq M\,.$$

Now let $x \in [a,b]$ be arbitrary. Since we have an open cover, $x \in B_\delta(x_k)$ for some $k \in \{1, \dots, N(\delta)\}$. Therefore, for all $n, m \geq M$ we have

$$|h_n(x) - h_m(x)| \leq |h_n(x) - h_n(x_k)| + |h_n(x_k) - h_m(x_k)| + |h_m(x_k) - h_m(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

by the triangle inequality, uniform equicontinuity and that $(h_n)_m$ is Cauchy on $\mathbb{Q} \cap [a,b]$. Since $x \in [a,b]$ was arbitrary and none of our choices depend on x, we have that for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $n, m \geq M$ we have

$$\sup_{x\in[a,b]} |h_n(x) - h_m(x)| < \varepsilon.$$

Thus $(h_n)_n \subseteq (f_n)_n$ is a Cauchy sequence with respect to d_∞. As $(C^0([a,b]), d_\infty)$ is complete, the sequence $(h_n)_n$ converges with respect to d_∞.

Proof of Claim: Fix $k \in \mathbb{N}$. Since $(f_{k,n}(x_k))_{n\in\mathbb{N}}$ converges, there exists $y_k \in \mathbb{R}$ such that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$|f_{k,n}(x_k) - y_k| < \varepsilon \quad \text{for all } n \geq N.$$

Now, for $n \geq M = \max\{k, N\}$, we have that $(h_n)_{n\geq M} \subseteq (f_{k,n})_{n\in\mathbb{N}}$. Therefore, $h_n = f_{k,m_n}$ with $m_n \geq n$. So,

$$|h_n(x_k) - y_k| = |f_{k,m_n}(x_k) - y_k| < \varepsilon \quad \text{for all } n \geq N\,.$$

□

One can reformulate the Arzelà-Ascoli theorem as a Heine-Borel-type theorem for $C^0([a, b])$ with d_∞.

Theorem 10.32 (Arzelà-Ascoli)**.** *If a subset $A \subseteq (C^0([a, b]), d_\infty)$ is closed, uniformly bounded and uniformly equicontinuous, then it is compact.*

The converse is also true:

Theorem 10.33 (Converse of Arzelà-Ascoli)**.** *If a subset $A \subseteq (C^0([a, b]), d_\infty)$ is compact, then A is closed, uniformly bounded and uniformly equicontinuous.*

Proof. Let $\varepsilon > 0$ and let $A \subseteq (C^0([a, b]), d_\infty)$ be compact. Note that closedness and uniform boundedness follows from our discuss on general metric spaces. Compact always implies closed and (uniform) boundedness for metric spaces, see theorem 6.58.

Recall compactness gives that every open cover has a finite subcover. In particular,

$$\bigcup_{f \in A} B_{\frac{\varepsilon}{3}}(f)$$

is an open cover of A. Therefore, compactness of A means there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$A \subseteq \bigcup_{k=1}^{N(\varepsilon)} B_{\frac{\varepsilon}{3}}(f_k),$$

for some $f_1, \ldots, f_N \in A$.

Each f_k is continuous on $[a, b]$, so, in particular, f_k is uniformly continuous on $[a, b]$. Therefore, for each k, there exists $\delta(k) > 0$ such that $|f_k(x) - f_k(y)| < \varepsilon/3$ if $|x - y| < \delta$. Note that since $k \in \{1, \ldots, N(\varepsilon)\}$, one can pick $\delta' = \min_k \delta(k)$ to achieve uniform equicontinuity for $(f_k)_{k=1}^N$.

Let $f \in A$ be arbitrary and choose $k \in \{1, \ldots, N(\varepsilon)\}$ such that $f \in B_{\varepsilon/3}(f_k)$. Then, for $x, y \in [a, b]$ such that $|x - y| < \delta$, we have that

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.$$

This shows that A is uniformly equicontinuous. $\square$