Elliptic PDEs with fibered nonlinearities

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Abstract

In $\mathbb{R}^m \times \mathbb{R}^{n-m}$, endowed with coordinates $x = (x', x'')$, we consider bounded solutions of the PDE

$$\Delta u(x) = f(u(x))\chi(x').$$

We prove a geometric inequality, from which a symmetry result follows.

1 Introduction

In this paper we consider bounded weak solutions u of the equation

$$\Delta u(x) = f(u(x))\chi(x'), \tag{1}$$

where $f \in C^1(\mathbb{R})$, with $f' \in L^\infty(\mathbb{R})$, $\chi \in L^\infty_{\rm loc}(\mathbb{R}^m)$ and $x \in \Omega$, for some open set $\Omega \subseteq \mathbb{R}^n$.

In our notation, $x = (x', x'') = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.

We call the function $f(u)\chi(x')$ a "fibered nonlinearity". Its main feature is that when χ is constant, the equation in (1) boils down to the usual semilinar equation, while for nonconstant χ the nonlinearity changes only in dependence of a subset of variables.

In particular, χ is constant on the "vertical fibers" $\{x' = c\}$, thence the name of fibered nonlinearity for $f(u)\chi(x')$.

The basic model to have in mind is the case in which χ is the characteristic function of a ball. In this sense, (1) may be seen as an interpolation between standard semilinear PDEs and the ones driven by fractional operators (which correspond to PDEs in the halfspace, see [CS07]).

In this paper, inspired by similar results in the semilinear case (see [Far02, FSV08]) and in the fractional case (see [CSM05, SV08]), we prove a geometric inequality and a symmetry result.

We introduce some notation. Given a smooth function v, for any fixed $x' \in \mathbb{R}^m$ and $c \in \mathbb{R}$, we consider the level set

$$L_{c,x'}(v) := \Big\{ x'' \in \mathbb{R}^{n-m} \text{ s.t. } v(x', x'') = c \Big\}.$$

Fixed $x' \in \mathbb{R}^m$, we also define

$$G_{x'}(v) := \Big\{ x'' \in \mathbb{R}^{n-m} \text{ s.t. } \nabla_{x''} v(x', x'') \neq 0 \Big\}. \tag{2}$$

Notice that $L_{c,x'}(v)$ is a smooth $(n - m - 1)$-dimensional manifolds in the vicinity of the points of $G_{x'}(v)$, so we can introduce the principal curvatures on it, denoted by

$$\kappa_1(x', x''), \ldots, \kappa_{n-m-1}(x', x'').$$

We then define the full curvature $\mathcal{K}$ as

$$\mathcal{K}(x', x'') := \sqrt{\sum_{j=1}^{n-m-1} \big(\kappa_j(x', x'')\big)^2}.$$

At points $x'' \in G_{x'}(v)$, we also define ∇_L to be the tangential gradient along $L_{c,x'}$, with $c = v(x', x'')$, that is, for any $G : \mathbb{R}^n \to \mathbb{R}$ smooth in the vicinity of x'', we set

$$\nabla_L G(x', x'') := \nabla_{x''} G(x', x'') - \left(\nabla_{x''} G(x', x'') \cdot \frac{\nabla_{x''} v(x', x'')}{|\nabla_{x''} v(x', x'')|} \right) \frac{\nabla_{x''} v(x', x'')}{|\nabla_{x''} v(x', x'')|}.$$

Then, the following result holds:

Theorem 1. *Let u be a bounded weak solution of (1) in an open set $\Omega \subseteq \mathbb{R}^n$.*
Suppose that

$$\int_\Omega |\nabla \phi(x)|^2 + f'(u(x))\chi(x')\, \phi^2(x)\, dx \geqslant 0 \tag{3}$$

for any $\phi \in C_0^\infty(\Omega)$.
Then,

$$\int_{x' \in \mathbb{R}^m} \int_{x'' \in G_{x'}(u)} \phi^2 \left(|\nabla_{x''} u|^2 \mathcal{K}^2 + \big|\nabla_L |\nabla_{x''} u|\big|^2 \right) dx''\, dx' \leqslant \int_\Omega |\nabla_{x''} u|^2 |\nabla \phi|^2\, dx \tag{4}$$

for any $\phi \in C_0^\infty(\Omega)$.
More precisely,

$$\int_{x' \in \mathbb{R}^m} \int_{x'' \in G_{x'}(u)} \phi^2 \left(|\nabla_{x''} u|^2 \mathcal{K}^2 + \big|\nabla_L |\nabla_{x''} u|\big|^2 \right) dx''\, dx' + \int_\Omega \phi^2 \mathcal{S}\, dx \leqslant \int_\Omega |\nabla_{x''} u|^2 |\nabla \phi|^2\, dx, \tag{5}$$

for any $\phi \in C_0^\infty(\Omega)$, where

$$\mathcal{S} := \sum_{\substack{1 \leqslant i \leqslant m \\ m+1 \leqslant j \leqslant n}} u_{i,j}^2 - \big|\nabla_{x'} |\nabla_{x''} u|\big|^2. \tag{6}$$

We remark that (5) is indeed more precise (though more complicated) that (4), because $\mathcal{S} \geqslant 0$ (see (25) below for further details).

The result contained in Theorem 1 may be seen as an extension of similar results presented in [SZ98a, SZ98b] in the classical semilinear framework. We observe that (4) controls the tangential gradients and curvatures of level sets of stable solutions in terms of the gradient of the solution itself, at any slice of $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. This means that some geometric quantities of interest are bounded by an appropriate energy term.

Another way of looking at (4) is to consider it as a bound of the weighted L^2-norm of any test function by a weighted L^2-norm of its gradient, at any slice. In this sense, (4) is a kind of weighted Poincaré inequality.

Moreover, one can consider (4) an extension of the stability condition for minimal surfaces, in which the L^2-norm of any test function, weighted by the curvature, is controlled by the L^2-norm of the gradient (see, for instance, formula (10.20) in [Giu84]).

Using Theorem 1, we obtain the following symmetry result for entire solutions (i.e., solutions in the whole space) which have suitably low energy:

Theorem 2. *Let u be a bounded weak solution of (1) in the whole $\mathbb{R}^n$ and suppose that*

$$\partial_n u(x) > 0 \text{ for any } x \in \mathbb{R}^n. \tag{7}$$

Assume also that there exists $C > 0$ such that

$$\int_{B_R} |\nabla_{x''} u|^2\, dx \leqslant C R^2, \tag{8}$$

for any $R \geqslant C$.

Then, there exist $\omega \in \mathrm{S}^{n-m-1}$ *and* $u^\star : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ *such that*

$$u(x', x'') = u^\star(x', \omega \cdot x''), \tag{9}$$

for any $(x', x'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.

The idea of making use of Poincaré type inequalities on level sets to deduce suitable symmetries for the solutions has been also used in [Far02, CC06, FSV08]. The motivation for Theorem 2 comes from the many works recently appeared in connection with a conjecture of De Giorgi (see, for instance, [DG79, Mod85, BCN97, GG98, AC00, AAC01, Sav08, FV08]). In particular, Theorem 2 may be thought as an interpolation result between the classical symmetry for semilinear equation and the one for fractional operators (see [CSM05, SV08]), where, after a suitable extension (see [CS07]), the nonlinearity sits as a trace datum for the halfspace.

In fact, some involved technical arguments of [SV08] will become here more transparent. In Section 2, we recall some basic regularity theory for solutions of (1). Further motivation on assumptions (3) and (8) will be also given in Section 3. In Sections 4 and 5 we prove Theorem 1 and Theorem 2, respectively.

2 Regularity theory synopsis

We observe that if u is a bounded solution of (1) in a domain Ω, then, by Calderón-Zygmund theory,

$$u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n), \text{ for any } p \in (1, +\infty). \tag{10}$$

Consequently,

$$u_i \in L^\infty_{\mathrm{loc}}(\Omega), \text{ for any } 1 \leqslant i \leqslant n. \tag{11}$$

For any fixed $\epsilon \in (0, 1)$ and j, $m + 1 \leqslant j \leqslant n$, we define

$$v_\epsilon(x) := \frac{u(x + \epsilon e_j) - u(x)}{\epsilon}.$$

We point out that v_ϵ is an incremental quotient in the x''-variables. Therefore,

$$\Delta v_\epsilon(x) = \frac{\chi(x')}{\epsilon}\Big(f(u(x + \epsilon e_j)) - f(u(x))\Big). \tag{12}$$

Notice that, given any $R > 0$, if $x \in B_R$, then

$$\left|\frac{\chi(x')}{\epsilon}\Big(f(u(x + \epsilon e_j)) - f(u(x))\Big)\right| \leqslant \|\chi\|_{L^\infty(B_R)}\|f'\|_{L^\infty(\mathbb{R})}\|u_j\|_{L^\infty(B_{R+1})},$$

so the right hand side of (12) is in $L^\infty_{\mathrm{loc}}(\Omega)$, with norm independent of ϵ, thanks to (11).

Consequently, Calderón-Zygmund theory gives that, for any $R > 0$ and $p \in (1, +\infty)$,

$$\|v_\epsilon\|_{W^{2,p}(B_R)} \leqslant C(R, p),$$

for some $C(R, p) > 0$, not depending on ϵ.

Therefore (see, for instance, Theorem 3 on page 277 of [Eva98]),

$$u_j \in W^{2,p}_{\mathrm{loc}}(\Omega), \text{ for any } p \in (1, +\infty) \text{ and any } j = m + 1, \ldots, n. \tag{13}$$

3 Motivating assumptions (3) and (8)

The goal of this section is to give some geometric condition which is sufficient for (3) and (8) to hold. We begin with a classical observation. For $t_o \in \mathbb{R}$ fixed, we set

$$F(t) := \int_{t_o}^{t} f(s)\,ds. \tag{14}$$

Given an open set $\Omega \subseteq \mathbb{R}^n$, we also define

$$\mathcal{E}_\Omega(v) := \int_\Omega \frac{|\nabla u(x)|^2}{2} + F(v(x))\chi(x')\,dx.$$

It is customary to say that u is a local minimizer for $\mathcal{E}_\Omega$ if its energy increases under compact perturbations, that is, if for any bounded open set $U \subset \Omega$ we have that $\mathcal{E}_U(u)$ is well-defined and finite, and

$$\mathcal{E}_U(u+\phi) \geqslant \mathcal{E}_U(u)$$

for any $\phi \in C_0^\infty(U)$.

Lemma 3. *If u is a local minimizer in some domain Ω, it satisfies (1) and (3) in Ω.*

The proof of Lemma 3 is standard and it follows simply by looking at the first and second variation of $\epsilon \mapsto \mathcal{E}_U(u+\epsilon\phi)$, which is minimal at $\epsilon = 0$:

$$0 = \frac{d}{d\epsilon}\mathcal{E}_U(u+\epsilon\phi)\Big|_{\epsilon=0} \leqslant \frac{d^2}{d\epsilon^2}\mathcal{E}_U(u+\epsilon\phi)\Big|_{\epsilon=0}.$$

We now remark that monotonicity in one direction implies stability:

Lemma 4. *Let u be a weak solution of (1) in Ω and suppose that $\partial_n u > 0$ in Ω. Then, (3) holds.*

The method of the proof of Lemma 4 is also classical (see, for instance [AAC01]): we fix $\phi \in C_0^\infty(\Omega)$, we define $\psi := \phi^2/u_n$, and we use (1), (10), (13) and Cauchy inequality to obtain

$$\begin{aligned}
& \int_\Omega |\nabla\phi|^2 + f'(u)\chi(x')\phi^2 \\
=\ & \int_\Omega |\nabla\phi|^2 + \frac{\phi^2|\nabla u_n|^2}{u_n^2} - \frac{\phi^2|\nabla u_n|^2}{u_n^2} + f'(u)\chi(x')u_n\psi \\
\geqslant\ & \int_\Omega 2\frac{\phi\nabla\phi\cdot\nabla u_n}{u_n} - \frac{\phi^2|\nabla u_n|^2}{u_n^2} + \Big(f(u)\chi(x')\Big)_n\psi \\
=\ & \int_\Omega \nabla\psi\cdot\nabla u_n + \Delta u_n\psi \\
=\ & 0.
\end{aligned}$$

This proves Lemma 4.

We now give a sufficient condition for (8) to hold:

Lemma 5. *Let $t_o := -1$ in (14).*
Assume that $F(t) \geqslant 0$ for any $t \in \mathbb{R}$, and $F(+1) = 0$.
Suppose also that $\chi \in L^\infty(\mathbb{R}^m)$ and $\chi(x') \geqslant 0$ for any $x' \in \mathbb{R}^m$.
Let u be a local minimum in the whole $\mathbb{R}^n$, with $|u(x)| \leqslant 1$ for any $x \in \mathbb{R}^n$.

Then, there exists $C > 0$ such that

$$\int_{B_R} |\nabla u|^2 \, dx \leqslant C R^{n-1}, \tag{15}$$

for any $R > 1$.

In particular, if also $n \leqslant 3$, then (8) holds.

The proof of Lemma 5 is also standard (see, for instance, [AAC01]): first of all, by (11),

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leqslant C_0, \tag{16}$$

for a suitable universal constant $C_0 > 0$.

We then take $R > 1$, $h \in C^\infty(B_R)$, with $h = -1$ in B_{R-1}, $h = 1$ on ∂B_R and $|\nabla h| \leqslant 10$. We set $v(x) := \min\{u(x), v(x)\}$ and we deduce from the minimality of u that

$$\int_{B_R} |\nabla u|^2 \, dx \leqslant \mathcal{E}_{B_R}(u) \leqslant \mathcal{E}_{B_R}(v) = \int_{B_R \setminus B_{R-1}} |\nabla u|^2 + |\nabla h|^2 + \|\chi\|_{L^\infty(\mathbb{R}^n)} \|F\|_{L^\infty([-1,1])} \, dx,$$

which implies (15) via (16), thus proving Lemma 5.

We would like to remark that the fibered Allen-Cahn nonlinearity $\chi(x')f(u) = \chi(x')(u^3 - u)$ may be considered as the typical example satisfying the assumptions of Lemma 5.

Below is another criterion for obtaining (8):

Lemma 6. *Let u be a bounded weak solution of (1) in the whole $\mathbb{R}^n$ and suppose that there exist $C_0 > 0$ and $\sigma \in [1, 2]$ such that*

$$\int_{x' \in B_R} |\chi(x')| \, dx' \leqslant C_0 R^{m-\sigma}, \tag{17}$$

for any $R \geqslant C_0$.

Then, there exists $C_1 > 0$ for which

$$\int_{x \in B_R} |\nabla u(x)|^2 \, dx \leqslant C_1 R^{n-\sigma}, \tag{18}$$

for any $R \geqslant C_1$.

In particular, (8) holds

(P1) if $n \leqslant 3$ and χ is bounded and supported in $\{|x_1'| \leqslant C_2\}$,

(P2) or if $m \geqslant 2$, $n \leqslant 4$ and χ is bounded and supported in $\{|x_1'| + |x_2'| \leqslant C_2\}$,

for some $C_2 > 0$.

The last claim in Lemma 6 plainly follows from (18) (taking $\sigma := 1$ in case (P1) holds and $\sigma := 2$ in case (P2) holds), so we focus on the proof of (18).

For this, we take $R \geqslant \max\{C_0, 1\}$ and we define

$$M := 1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{|r| \leqslant \|u\|_{L^\infty(\mathbb{R}^n)}} |f(r)|.$$

We choose $\tau \in C_0^\infty(B_{2R}, [0, 1])$, with $\tau = 1$ in B_R and $|\nabla \tau| \leqslant 10/R$.

Then, using (1) and Cauchy inequality,

$$\begin{aligned} \int_{B_{2R}} \tau^2 |\nabla u|^2 &= \int_{B_{2R}} \nabla u \cdot \nabla(\tau^2 u) - 2\tau u \nabla u \cdot \nabla \tau \\ &\leqslant \int_{B_{2R}} \left| f(u)\chi(x')\tau^2 u \right| + 2M\tau |\nabla u| \, |\nabla \tau| \\ &\leqslant \int_{B_{2R}} M^2 |\chi(x')| + \frac{1}{2}\tau^2 |\nabla u|^2 + 2M^2 |\nabla \tau|^2. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\int_{B_R}\tau^2|\nabla u|^2 &\leqslant \frac{1}{2}\int_{B_{2R}}\tau^2|\nabla u|^2 \\
&\leqslant 2M^2\int_{x\in B_{2R}}|\chi(x')|+|\nabla\tau(x)|^2\,dx \\
&\leqslant \bar{C}M^2\Big(R^{n-m}\int_{x'\in B_{2R}}|\chi(x')|\,dx'+\int_{x\in B_{2R}}\frac{1}{R^2}\,dx\Big) \\
&\leqslant \tilde{C}M^2\big(R^{n-\sigma}+R^{n-2}\big),
\end{aligned}$$

for suitable $\tilde{C}$, $\bar{C}>0$, where (17) has been taken into account.
This proves (18) and Lemma 6.

4 Proof of Theorem 1

Given a matrix $M=\{M_{i,j}\}_{1\leqslant i,j\leqslant N}\in\mathrm{Mat}(N\times N)$, we now set

$$|M|:=\sqrt{\sum_{1\leqslant i,j\leqslant N}M_{i,j}^2}. \tag{19}$$

Fixed $x'\in\mathbb{R}^m$, we also define

$$\Omega_{x'}:=\big\{x''\text{ s.t. }(x',x'')\in\Omega\big\}$$

and

$$H_{x'}(u):=\Big\{x''\in\Omega_{x'}\text{ s.t. }\nabla_{x''}u(x',x'')=0\Big\}. \tag{20}$$

Observe, from (2), that

$$\Omega_{x'}\;=\;G_{x'}(u)\,\cup\,H_{x'}(u)\,,\qquad\text{and the union is disjoint.} \tag{21}$$

Now, let $\phi\in C_0^\infty(\Omega)$.
From (1), (10), (13) and (19),

$$\begin{aligned}
&\int_\Omega f'(u)\chi(x')|\nabla_{x''}u|^2\phi^2=\int_\Omega\nabla_{x''}\Big(f(u)\chi(x')\Big)\cdot\nabla_{x''}u\phi^2 \\
&=\int_\Omega\nabla_{x''}(\Delta u)\cdot\nabla_{x''}u\phi^2=\sum_{j=m+1}^n\int_\Omega\Delta u_ju_j\phi^2 \\
&=-\sum_{j=m+1}^n\int_\Omega\nabla u_j\cdot\nabla(u_j\phi^2)=-\sum_{j=m+1}^n\int_\Omega\Big(|\nabla u_j|^2\phi^2+u_j\nabla u_j\cdot\nabla\phi^2\Big) \\
&=-\int_\Omega|D^2_{x''}u|^2\phi^2-\sum_{\substack{1\leqslant i\leqslant m\\ m+1\leqslant j\leqslant n}}\int_\Omega u_{i,j}^2\phi^2-\frac{1}{2}\int_\Omega\nabla|\nabla_{x''}u|^2\cdot\nabla\phi^2.
\end{aligned} \tag{22}$$

We now test (3) against the function $|\nabla_{x''}u|\phi$: this function is Lipschitz continuous, due to (13), so we can plug it inside (3), via an easy density argument.
We obtain

$$\begin{aligned}
0&\leqslant\int_\Omega\Big|\nabla\big(|\nabla_{x''}u|\phi\big)\Big|^2+f'(u)\chi(x')|\nabla_{x''}u|^2\phi^2 \\
&=\int_\Omega|\nabla_{x''}u|^2|\nabla\phi|^2+\big|\nabla|\nabla_{x''}u|\big|^2\phi^2+\frac{1}{2}\nabla|\nabla_{x''}u|^2\cdot\nabla\phi^2+f'(u)\chi(x')|\nabla_{x''}u|^2\phi^2.
\end{aligned} \tag{23}$$

By comparing (22) and (23), and recalling (6), we see that

$$\int_\Omega \phi^2 \left(|D^2_{x''}u|^2 - \big|\nabla_{x''}|\nabla_{x''}u|\big|^2 + \mathcal{S}\right) \leqslant \int_\Omega |\nabla_{x''}u|^2|\nabla\phi|^2. \tag{24}$$

Fixed $x' \in \mathbb{R}^m$, let now $W(x'') := |\nabla_{x''}u(x',x'')|$. Since $W \in W^{1,p}_{\rm loc}(\Omega_{x'})$ for a.e. x', because of (13), we deduce from Stampacchia's Theorem (see, for instance, Theorem 6.19 of [LL97]) that $\nabla_{x''}W(x'') = 0$ for a.e. $x'' \in \{W = 0\}$.

Analogously, if $W^{(j)}(x'') := u_j(x',x'')$, with $m+1 \leqslant j \leqslant n$, one obtains from Stampacchia's Theorem that $W^{(j)}_i(x'') = 0$ for a.e. $x'' \in \{W^{(j)} = 0\}$.

Therefore, recalling (20), for a.e. $x' \in \mathbb{R}^m$ and a.e. $x'' \in H_{x'}(u)$, we have that

$$0 = |\nabla_{x''}W(x'')| = W^{(j)}_i(x'') \qquad \text{for any } \ m+1 \leqslant i,j \leqslant n,$$

or, simply,

$$0 = \big|\nabla_{x''}|\nabla_{x''}u(x',x'')|\big| = |D^2_{x''}u|.$$

Hence, recalling formula (21) here and formula (2.1) of [SZ98a],

$$\begin{aligned}
&\int_\Omega \phi^2 \left(|D^2_{x''}u|^2 - \big|\nabla_{x''}|\nabla_{x''}u|\big|^2\right)\\
=\ &\int_{x'\in\mathbb{R}^m}\int_{x''\in\Omega_{x'}} \phi^2 \left(|D^2_{x''}u|^2 - \big|\nabla_{x''}|\nabla_{x''}u|\big|^2\right)\\
=\ &\int_{x'\in\mathbb{R}^m}\int_{x''\in G_{x'}(u)} \phi^2 \left(|D^2_{x''}u|^2 - \big|\nabla_{x''}|\nabla_{x''}u|\big|^2\right)\\
=\ &\int_{x'\in\mathbb{R}^m}\int_{x''\in G_{x'}(u)} \phi^2 \left(|\nabla_{x''}u|^2\mathcal{K}^2 + \big|\nabla_L|\nabla_{x''}u|\big|^2\right).
\end{aligned}$$

From this and (24), we conclude that (5) holds true.

Now we claim that

$$\begin{aligned}
&\mathcal{S} \geqslant 0 \text{ a.e.,}\\
&\text{and if equality holds at points of } \{\nabla_{x''}u \neq 0\},\\
&\text{then } \nabla_{x''}u_i \text{ is either } 0\\
&\text{or parallel to } \nabla_{x''}u, \text{ for any } 1 \leqslant i \leqslant m.
\end{aligned} \tag{25}$$

Notice that (5) and (25) imply (4), thus the proof of (25) would end the proof of Theorem 1.

In order to prove (25), we focus on points $(x'x'') \in \{|\nabla_{x''}u| \neq 0\}$, since for almost any other point Stampacchia's Theorem gives that $\mathcal{S} \geqslant 0$.

Then, if $|\nabla_{x''}u| \neq 0$,

$$\begin{aligned}
\big|\nabla_{x'}|\nabla_{x''}u|\big|^2 &= \frac{1}{|\nabla_{x''}u|^2}\sum_{i=1}^m\Big(\sum_{j=m+1}^n u_j u_{i,j}\Big)^2\\
&= \frac{1}{|\nabla_{x''}u|^2}\sum_{i=1}^m\Big|\nabla_{x''}u\cdot\nabla_{x''}u_i\Big|^2\\
&\leqslant \sum_{i=1}^m\Big|\nabla_{x''}u_i\Big|^2\\
&= \sum_{\substack{1\leqslant i\leqslant m\\ m+1\leqslant j\leqslant n}} u^2_{i,j},
\end{aligned}$$

where Cauchy inequality was used.

This proves (25) and concludes the proof of Theorem 1.

5 Proof of Theorem 2

As usual, we denote by χ_A the characteristic function of a set A.

By Lemma 4, we have that (3) is implied by (7). In particular, in the assumptions of Theorem 2, we have that $G_{x'}(u) = \mathbb{R}^{n-m}$.

Following [Far02, FSV08], now we perform the choice of an appropriate test function in Theorem 1. For this, given $R > 1$, we define

$$\phi(x) := \chi_{B_{\sqrt{R}}}(x) + \frac{2\,\log(R/|x|)}{\log R}\chi_{B_R \setminus B_{\sqrt{R}}}(x).$$

We observe that

$$|\nabla \phi(x)| = \frac{2\,\chi_{B_R \setminus B_{\sqrt{R}}}(x)}{|x|\ \log R}. \tag{26}$$

Since ϕ is Lipschitz continuous, we can use it as a test function in (5), up to a density argument. Therefore, we obtain from (5), (25) and (26) that

$$\begin{aligned} &\int_{x' \in \mathbb{R}^m \cap B_{\sqrt{R}/2}} \int_{x'' \in B_{\sqrt{R}/2}} \Big(|\nabla_{x''} u|^2 \mathcal{K}^2 + \big|\nabla_L |\nabla_{x''} u|\big|^2 \Big) + \int_{B_R} \mathcal{S} \\ &\qquad \leqslant \frac{4}{(\log R)^2} \int_{B_R \setminus B_{\sqrt{R}}} \frac{|\nabla_{x''} u|^2}{|x|^2}\,dx. \end{aligned} \tag{27}$$

We define

$$E(R) := \int_{B_R} |\nabla_{x''} u|^2\,dx.$$

We now employ Fubini's Theorem to observe that

$$\begin{aligned} &\int_{B_R \setminus B_{\sqrt{R}}} \frac{|\nabla_{x''} u|^2}{2|x|^2}\,dx - \frac{1}{2R^2} E(R) \\ &\leqslant \int_{B_R \setminus B_{\sqrt{R}}} \frac{|\nabla_{x''} u|^2}{2|x|^2}\,dx - \frac{1}{2R^2} \int_{B_R \setminus B_{\sqrt{R}}} |\nabla_{x''} u|^2\,dx \\ &= \int_{B_R \setminus B_{\sqrt{R}}} |\nabla_{x''} u|^2 \Big(\int_{|x|}^R s^{-3}\,ds \Big)\,dx \\ &\leqslant \int_{\sqrt{R}}^R \left(\int_{B_s \setminus B_{\sqrt{R}}} |\nabla_{x''} u|^2 s^{-3}\,dx \right) ds \\ &\leqslant \int_{\sqrt{R}}^R s^{-3} E(s)\,ds. \end{aligned} \tag{28}$$

From (8), (27) and (28),

$$\begin{aligned} &\int_{x' \in \mathbb{R}^m \cap B_{\sqrt{R}/2}} \int_{x'' \in B_{\sqrt{R}/2}} \Big(|\nabla_{x''} u|^2 \mathcal{K}^2 + \big|\nabla_L |\nabla_{x''} u|\big|^2 \Big) + \int_{B_R} \mathcal{S} \\ &\qquad \leqslant \frac{8}{(\log R)^2} \Big(R^{-2} E(R) + \int_{\sqrt{R}}^R s^{-3} E(s)\,ds \Big) \leqslant \frac{\bar{C}}{\log R}, \end{aligned}$$

as long as $R \geqslant C$, for suitable C, $\bar{C} > 0$.

By taking R arbitrarily large, we thus deduce that

$$\mathcal{K} = 0 = \big|\nabla_L |\nabla_{x''} u|\big| \text{ for any } x' \in \mathbb{R}^m \text{ and } x'' \in \mathbb{R}^{n-m}, \tag{29}$$

and that

$$\mathcal{S} = 0. \tag{30}$$

From (29) and Lemma 2.11 of [FSV08] (applied here at any fixed x'), we infer that, for any fixed $x' \in \mathbb{R}^m$, the function $x'' \mapsto u(x', x'')$ is a function of one variable, up to rotation.
That is, for any fixed $x' \in \mathbb{R}^m$, there exists $\omega(x') \in \mathrm{S}^{n-m-1}$ and $u^\star : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ such that

$$u(x', x'') = u^\star(x', \omega(x') \cdot x''), \tag{31}$$

for any $(x', x'') \in \mathbb{R}^n$.
We now claim that

$$\nabla_{x''} u(x', x'') \text{ is parallel to } \omega(x'). \tag{32}$$

To check this, we let $\eta(x') \in \mathrm{S}^{n-m-1}$ be orthogonal to $\omega(x')$ and we exploit (31) to get that

$$u(x', x'' + t\eta(x')) = u^\star(x', \omega(x') \cdot x''),$$

and so, by differentiating with respect to t, we see that

$$\nabla_{x''} u(x', x'') \cdot \eta(x') = 0.$$

This proves (32).
In the light of (32), we now write

$$\nabla_{x''} u(x', x'') = c(x', x'')\omega(x'), \tag{33}$$

for some $c(x', x'') \in \mathbb{R}$.
In fact, (7) and (33) imply that

$$c(x', x'') \neq 0 \text{ for all } (x', x'') \in \mathbb{R}^n. \tag{34}$$

Moreover, (13) and (33) give that

$$\text{the map } (x', x'') \mapsto c(x', x'')\omega(x') \text{ belongs to } W^{2,n+1}_{\mathrm{loc}}(\mathbb{R}^n) \subset C^1(\mathbb{R}^n). \tag{35}$$

Hence,

$$\Big(c(x', x'')\omega(x')\Big)_i = \nabla_{x'} u_i(x', x''), \tag{36}$$

for any $1 \leqslant i \leqslant n$.
Since

$$c^2(x', x'') = \big(c(x'x'')\omega(x')\big) \cdot \big(c(x'x'')\omega(x')\big),$$

we deduce from (33) and (35) that $c^2 \in C^1(\mathbb{R}^n)$.
As a consequence of this and of (34), we have that

$$c \in C^1(\mathbb{R}^n). \tag{37}$$

This, (33) and (34) thus imply that

$$\omega \in C^1(\mathbb{R}^m). \tag{38}$$

In particular,

$$0 = \left(\frac{1}{2}\right)_i = \left(\frac{\omega(x') \cdot \omega(x')}{2}\right)_i = \omega_i(x') \cdot \omega(x'), \tag{39}$$

for any $1 \leqslant i \leqslant m$.
From (25), (30), (32) and (36), we have that $\nabla_{x''} u_i(x', x'')$ is parallel to $\nabla_{x''} u(x', x'')$, that is

$$\Big(c(x', x'')\omega(x')\Big)_i = k^{(i)}(x', x'')\omega(x'),$$

for some $k^{(i)}(x', x'') \in \mathbb{R}$.

Then, making use of (39) twice, the latter equation gives that

$$\begin{aligned} 0 &= k^{(i)}(x', x'')\omega(x') \cdot \omega_i(x') \\ &= \Big(c(x', x'')\omega(x')\Big)_i \cdot \omega_i(x') \\ &= c(x', x'')\omega_i(x') \cdot \omega_i(x'), \end{aligned}$$

for any $1 \leqslant i \leqslant m$.

Consequently, from (34), we conclude that $\omega_i(x') = 0$ for any $1 \leqslant i \leqslant m$, and so $\omega(x')$ is constant, say

$$\omega(x') = \omega.$$

This and (31) give (9), thus ending the proof of Theorem 2.

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