DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **28**, Number **3**, November **2010**

doi:10.3934/dcds.2010.28.xx

pp. **1–XX**

A LIOUVILLE THEOREM FOR SOLUTIONS TO THE LINEARIZED MONGE-AMPERE EQUATION

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Abstract. We prove that global Lipschitz solutions to the linearized Monge-Ampere equation

$$L_\varphi u := \sum \varphi^{ij} u_{ij} = 0$$

must be linear in $2D$. The function φ is assumed to have the Monge-Ampere measure $\det D^2\varphi$ bounded away from 0 and ∞.

1. **Introduction.** In this paper we consider global C^2 solutions $u : \mathbb{R}^2 \to \mathbb{R}$ that satisfy certain types of degenerate elliptic equations

$$\sum a_{ij}(x) u_{ij} = 0 \quad \text{in } \mathbb{R}^2. \tag{1}$$

We are interested in equations (1) that appear as the linearized operator for the Monge-Ampere equation. We show that the only global Lipschitz solutions i.e

$$\|\nabla u(x)\|_{L^\infty(\mathbb{R}^2)} \le C. \tag{2}$$

must be linear.

For simplicity we assume throughout the paper that the coefficients a_{ij} are smooth and satisfy the ellipticity condition:

$$A(x) := (a_{ij}(x))_{ij} > 0.$$

We start by recalling two classical Liouville type theorems concerning global solutions of (1). The first is due to Bernstein (see [6], [1]) and asserts:

A global C^2 solution (in $\mathbb{R}^2$) which is bounded must be constant.

The result fails if one allows linear growth for u at ∞ as it can be seen from the following simple example

$$u(x) = \sqrt{1 + x_1^2} - \sqrt{1 + x_2^2},$$

for appropriate $A(x)$.

The second theorem states that *global solutions satisfying (1), (2) must be linear if the coefficients are uniformly elliptic i.e*

$$\lambda I \le A(x) \le \Lambda I, \qquad x \in \mathbb{R}^2.$$

This follows from the classical $C^{1,\alpha}$ interior estimates in $2D$ due to Morrey [7] and Nirenberg [8].

2000 *Mathematics Subject Classification.* Primary: 35J70, 35B65.
Key words and phrases. Monge-Ampere equation, Harnack inequality.
The author is supported by NSF grant 0701037.

In this short paper we prove a similar Liouville theorem for solutions to the linearized operator of the Monge-Ampere equation

$$\det D^2\varphi = f, \qquad \lambda \le f \le \Lambda. \tag{3}$$

Theorem 1.1. *Assume φ is a smooth convex function in $\mathbb{R}^2$ satisfying*

$$\lambda \le \det D^2\varphi \le \Lambda,$$

and denote by (φ^{ij}) the inverse matrix of $D^2\varphi$. If $u \in C^2$ is globally Lipschitz (i.e satisfies (2)) and solves

$$L_\varphi u := \sum \varphi^{ij}(x) u_{ij} = 0 \quad \text{in } \mathbb{R}^2, \tag{4}$$

then u is linear.

Equation (4) was studied by Caffarelli and Gutierrez in [3]. It appears for example in fluid meachanics (see [2], [4]), or in the affine maximal graph equation (see [10]) etc. The main result in [3] states that solutions of (4) satisfy the Harnack inequality in the sections of φ (see Section 2 for the precise statement). When dealing with the degenerate equation $L_\varphi u = 0$, the sections of φ play the same role as the euclidean balls do in the theory of uniformly elliptic equations.

Theorem 1.1 suggests that in $\mathbb{R}^2$, solutions of (4) satisfy stronger estimates than those obtained from Harnack inequality. In a forthcoming paper we intend to obtain interior $C^{1.\alpha}$ estimates for the equation $L_\varphi u = 0$ in $2D$.

Theorem 1.1 can be proved in fact in a more general form, where the coefficient matrix $A(x)$ is "uniformly elliptic" with respect to the inverse of $D^2\varphi$ i.e.

$$c(D^2\varphi)^{-1} \le A(x) \le C(D^2\varphi)^{-1}, \qquad 0 < c < C,$$

and the Monge-Ampere measure

$$\det D^2\varphi = \mu$$

satisfies a standard doubling condition (see conditions 0.3-0.4 of [3]). In this setting, the Liouville theorem for uniformly elliptic equations mentioned above appears as a consequence of Theorem 1.1 by taking $\varphi(x) := |x|^2$.

The proof of Theorem 1.1 follows the same strategy as the proof of Bernstein theorem for elliptic equations in $2D$. If u is a solution to (1) and is not linear, then any tangent plane splits the graph of u into at least 4 unbounded connected components. Then we apply the Harnack inequality of Caffarelli and Gutierrez in certain nondegenarate directions and obtain a contradiction. Similar ideas have been used in [9], [5] for other degenerate equations.

2. **Geometry of sections and Harnack inequality.**

Let $S_h(x)$, the section of φ at the point x and of height $h > 0$, be defined as

$$S_h(x) := \{y \in \mathbb{R}^2 | \quad \varphi(y) < \varphi(x) + \nabla\varphi(x)\cdot(y-x) + h\}.$$

We list below the key properties (see for example [3]) for the sections of a convex function φ that satisfies

$$\det D^2\varphi = f, \qquad \lambda \le f \le \Lambda.$$

a) $S_h(x)$ is convex, and if $h \le t$ then $S_h(x) \subset S_t(x)$.

b) To each section $S_h(x)$ we can associate an ellipse

$$E_h(x) = A_h(x)B_1, \qquad \text{with } A_h(x) \text{ a symmetric matrix}$$

such that

$$E_h(x) \subset S_h(x) - x \subset C\, E_h(x),$$

with the constant C depending only on λ ,Λ. In what follows we denote by

$$|E_h(x)| \text{– the ratio between the longest and the shortest axis of } E_h(x) \tag{5}$$
$$\xi_h(x) \text{– the direction of the longest axis of } E_h(x).$$

c) If $x_1 \in S_h(x_0)$ then

$$S_h(x_0) \subset S_{C_1 h}(x_1) \subset S_{C_2 h}(x_0),$$

with C_1, C_2 depending only on λ, Λ.

d) If $M > 1$ then

$$S_{Mh}(x) - x \subset M(S_h(x) - x) \subset S_{C(M)h} - x,$$

for some constant $C(M)$ depending on λ, Λ and M.

Caffarelli and Gutierrez proved in [3] the Harnack inequality for solutions of the linearized operator

$$L_\varphi u = \sum \varphi^{ij} u_{ij}.$$

Precisely, if $u \geq 0$ in $S_h(x_0)$ then

$$\inf_{S_{h/2}(x_0)} u \geq c \sup_{S_{h/2}(x_0)} u,$$

with $c > 0$ a small constant depending only on λ, Λ. We need the following weak Harnack inequality for supersolutions which was proved also in [3] (Theorem 2).

Theorem. *If $L_\varphi(u) \leq 0$ and $u \geq 0$ in $S_h(x_0)$ then,*

$$\inf_{S_{h/2}(x_0)} u \geq c \inf_{S_{h/4}(x_0)} u$$

with $c > 0$ a small constant depending on λ, Λ.

Applying the theorem repeatedly we see that for any $\tau \leq 1/4$,

$$\inf_{S_{h/2}(x_0)} u \geq c(\tau) \inf_{S_{h\tau}(x_0)} u \tag{6}$$

with $c(\tau) > 0$ depending also on τ.

Before we state the next lemmas we introduce the following notation. We define $\mathcal{A}_\delta$ as the set

$$\mathcal{A}_\delta := \{(x,t)| \quad diam\, S_t(x) \geq \delta, \quad S_t(x) \subset B_{1/\delta}\}. \tag{7}$$

Lemma 2.1. *Let $S_h(0)$ be the maximal section at 0 which is included in B_1. Assume that (see (5))*

$$|E_h(0)| \leq M,$$

for some constant M. Then

$$|E_t(x)| \leq C(M,\delta) \qquad \forall (x,t) \in \mathcal{A}_\delta,$$

with $C(M,\delta)$ a large constant depending on M, δ, λ, Λ.

Proof. Since $S_h(0)$ is the maximal section included in B_1 and satisfies $|E_h(0)| \le M$ we see from property b) that for a small constant $c(M)$ depending on M (and on λ, Λ)

$$B_c \subset S_h(0) \subset B_1.$$

Then by property d) we can find a large constants $C_1(M,\delta)$ such that

$$B_{2/\delta} \subset S_{C_1h}(0) \subset B_{C_1}.$$

Now by c), there exist $C_2(M,\delta)$, $C_3(M,\delta)$ such that for any $x \in B_{1/\delta}$,

$$B_1 \subset S_{C_2h}(x) - x \subset B_{C_3}.$$

Now, we use d) and find $c_1(M,\delta)$, $c_2(M,\delta)$ small such that

$$B_{c_1} \subset S_{c_2h}(x) - x \subset B_{\delta/2}.$$

This shows, by property a), that any section $S_t(x)$ with $x \in B_{1/\delta}$ and $diam\, S_t(x) \le \delta$ contains a ball of radius c_1 in the interior, and the conclusion of the lemma follows easily. □

Lemma 2.2. *Let $S_h(0)$ be the maximal section at 0 included in B_1, and assume $|E_h(0)| \le M$. Let u be defined on $B_r(x)$ for some $x \in B_1$ and $\delta \le r \le 1$.*

If $u \ge 0$ in $B_r(x)$ and

$$L_\varphi u \le 0,$$

then

$$\inf_{B_{r/2}(x)} u \ge c(M,\delta) \inf_{B_{r/4}(x)} u \tag{8}$$

with $c(M,\delta)$ a small positive constant depending on M, δ, λ, Λ.

Proof. It suffices to show that if $u \ge 0$ in $B_\delta(x)$ then

$$\inf_{B_\eta(x)} u \ge c(M,\delta) \inf_{B_{\eta/2}(x)} u,$$

for some $\eta(M,\delta)$ small.

By Lemma 2.1 there exists $\eta(M,\delta)$ small and a section $S_t(x)$ with $(x,t) \in \mathcal{A}_\delta$ such that

$$B_\eta \subset S_{t/2}(x) - x \subset S_t(x) - x \subset B_\delta.$$

By property d), we can find $\tau(M,\delta) > 0$ such that

$$S_{\tau t}(x) - x \subset B_{\eta/2}.$$

Now the weak Harnack inequality (6) applied to u in S_t gives

$$\inf_{B_\eta(x)} u \ge \inf_{S_{t/2}(x)} u \ge c(\tau) \inf_{S_{\tau t}(x)} u \ge c(\tau) \inf_{B_{\eta/2}(x)} u.$$

□

Lemma 2.3. *Let $S_h(0)$ be the maximal section included in B_1 and assume*

$$|E_h(0)| \ge M.$$

There exists $\sigma(M,\delta)$ such that for all $(x,t) \in \mathcal{A}_\delta$ (see (5), (7))

$$|E_t(x)| \ge \sigma^{-1}, \qquad \angle(\xi_t(x), \xi_h(0)) \le \sigma$$

and $\sigma(M,\delta) \to 0$ as $M \to \infty$.

Here $\angle(\xi_1, \xi_2)$ denotes the angle ($\in [0, \pi/2]$) between the lines of directions ξ_1 and ξ_2.

Proof. We need to show that, for δ fixed,

$$\inf_{(x,t)\in\mathcal{A}_\delta} |E_t(x)| \to \infty, \qquad \sup_{(x,t)\in\mathcal{A}_\delta} \angle(\xi_t(x), \xi_h(0)) \to 0 \qquad \text{as } M \to \infty.$$

From Lemma 2.1 it follows that if $|E_t(x)| \le N$ for some $(x,t) \in \mathcal{A}_\delta$, then

$$|E_h(0)| \le C(\delta, N).$$

This shows that

$$\inf_{(x,t)\in\mathcal{A}_\delta} |E_t(x)| \to \infty \quad \text{as } M \to \infty.$$

Now assume that for some $(x,t) \in \mathcal{A}_\delta$

$$\angle(\xi_t(x), \xi_h(0)) \ge \sigma_0 > 0.$$

Let x^* be the point of intersection of the line passing through x and of direction $\xi_t(x)$ with the line passing through 0 and direction $\xi_h(0)$. Clearly $|x^*| \le C(\delta, \sigma_0)$. Moreover by the properties c) and d), there exists a section $S_{t^*}(x^*)$ with

$$S_h(0) \subset S_{t^*}(x^*), \quad S_t(x) \subset S_{t^*}(x^*), \qquad diam\, S_{t^*}(x^*) \le C(\delta, \sigma_0).$$

Since $S_{t^*}(x^*)$ contains 2 segments of length δ at an angle σ_0, it contains also a small ball of radius $c(\delta, \sigma_0)$, hence

$$|E_{t^*}(x^*)| \le C(\sigma_0, \delta).$$

By Lemma 2.1, this implies that $|E(0,h)| \le C(\sigma_0, \delta)$. In conclusion

$$\angle(\xi_t(x), \xi_h(0)) \to 0 \quad \text{as } M \to \infty.$$

$\square$

3. **Proof of Theorem 1.1.**

Without loss of generality assume $u(0) = 0$ and $\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \le 1$. Let

$$K := \nabla u(\mathbb{R}^2).$$

We need to show that K consists of a single point. As in the proof of the theorem of Bernstein, the key will be to use the following $2D$ theorem.

Theorem. *Assume $u \in C^2(\mathbb{R}^2)$ satisfies $L_\varphi u = 0$. If x is a nondegenerate point i.e $D^2u(x) \ne 0$, then the set*

$$\{y \in \mathbb{R}^2 | \quad u(y) > u(x) + \nabla u(x) \cdot (y - x)\}$$

contains at least two disconnected unbounded components that have x as a boundary point.

From now on we assume by contradiction that u is not linear.

First we remark that the set of nondegenerate points is dense in $\mathbb{R}^2$. Indeed, otherwise $D^2u = 0$ in a neighborhood, and by unique continuation (since $\varphi \in C^\infty$) $D^2u = 0$ in whole $\mathbb{R}^2$ and we reach a contradiction.

Clearly, the images of the gradients of these nondegenerate points form a dense open subset of K.

Let R_n be a sequence converging to ∞ and let

$$u_n(x) := \frac{u(R_n x)}{R_n}$$

represent the corresponding rescalings of u. The functions u_n satisfy

$$L_{\varphi_n} u_n = 0, \qquad \varphi_n(x) := \frac{\varphi(R_n x)}{R_n^2}.$$

The function φ_n also satisfies (3), and its sections are obtained by $1/R_n$-dilations of the original sections of u. Denote

$$e_n := |E_{h_n}(0)|$$

where $S_{h_n}(0)$ is the maximal section of u included in B_{R_n}. We distinguish 2 cases:

1) There exists a sequence of $R_n \to \infty$ such that e_n remains bounded;

2) $e_n \to \infty$ as $R_n \to \infty$.

We show that we reach a contradiction in both cases.

Case 1. By assumption, there exists M such that $|e_n| \le M$ for all n. Without loss of generality we can assume that

$$u_n \to u^* \quad \text{uniformly on compact sets.}$$

From Lemma 2.2, each u_n satisfies the weak Harnack inequality (8), thus the same inequality holds for u^* if $u^* \ge 0$ in $B_r(x)$.

Lemma 3.1. *Let $\nu \in \mathbb{R}^2$, $|\nu| = 1$ be a unit direction, and assume*

$$\min_{p \in \bar{K}} \nu \cdot p$$

is achieved for $p = p_\nu \in \bar{K}$. Then

$$u^*(t\nu) = t\nu \cdot p_\nu$$

either for all $t \ge 0$ or for all $t \le 0$.

Proof. The equation is invariant under addition with linear functionals, thus we may assume for simplicity that $\nu = e_2$ and $p_\nu = 0$, that is

$$K \subset \{x \cdot e_2 \ge 0\}, \quad 0 \in \bar{K}.$$

This implies that the functions u, u_n, and u^* are all increasing in the e_2 direction and that there exists a sequence of nondegenerate points for u whose gradients approach 0. By passing if necessary to a subsequence we may assume that there exists $x_n \to 0$ with $\nabla u_n(x_n) = \nabla u(R_n x_n) \to 0$ and x_n is a nondegenerate point for u_n. Define

$$l_n(x) := u_n(x_n) + \nabla u_n(x_n) \cdot (x - x_n),$$

then clearly $l_n \to 0$ uniformly on compact sets. By the theorem above, the set $\{u_n > l_n\}$ contains at least 2 unbounded connected components that have x_n as a boundary point.

Since u^* is increasing in the e_2 direction, it suffices to show that either $u^*(e_2) = 0$ or $u^*(-e_2) = 0$. Assume by contradiction that

$$u^*(-e_2) < 0, \quad u^*(e_2) > 0.$$

Then we can find δ (depending on u^*) and a rectangle

$$\mathcal{R} := [-2\delta, 2\delta] \times [-1, 1]$$

such that u^* is positive on the top of $\mathcal{R}$ and negative on the bottom. This implies that for all n large, u_n is positive on the top of $\mathcal{R}$ and negative on the bottom. We conclude that the set $\{u_n > l_n\}$ has an unbounded connected component U that does not intersect the top or the bottom of, say the rectangle

$$\mathcal{R}_1 := [\delta, 2\delta] \times [-1, 1],$$

but intersects both lateral sides of $\mathcal{R}_1$. Let P be a nonintersecting polygonal line included in U which connects the lateral sides. This polygonal line splits $\mathcal{R}_1$ into two disjoint domains $\mathcal{R}_1^+$ (containing the top) and $\mathcal{R}_1^-$.

From each u_n we create a supersolution $\tilde{u}_n : \mathcal{R}_1 \to \mathbb{R}$ to $L_{\varphi_n} \tilde{u}_n \le 0$ as follows.

First we replace u_n by l_n in the set U. Clearly the new function is a supersolution. Then we modify this function to be equal to l_n in $\mathcal{R}_1^-$.

Notice that $\tilde{u}_n$ converges uniformly in $\mathcal{R}_1$ to

$$(u^*)^+ := \max\{u^*, 0\}.$$

Since $\tilde{u}_n$ satisfies the weak Harnack inequality of Lemma 2.2, we see that the same conclusion holds for $(u^*)^+$ as well. This implies that $(u^*)^+ > 0$ in the interior of $\mathcal{R}_1$. On the other hand $(u^*)^+ = 0$ in a neighborhood of the bottom of $\mathcal{R}_1$ since u^* is negative there. We reached a contradiction and the lemma is proved. □

By the lemma above, $u^*(x) = p_\nu \cdot x$ (and $u(x) = p_{-\nu} \cdot x$) on at least half of the line $t\nu$. Since the set K has nonempty interior, by the definition of p_ν we have

$$\nu \cdot (p_\nu - p_{-\nu}) < 0,$$

which implies that $u^*(t\nu)$ is linear both for $t \ge 0$ and $t \le 0$ but with different slopes. We conclude that u^* is homogenous of degree one.

Since u^* is continuous, homogenous of degree 1 but not linear, we can easily find a ball $B_r(x)$ and a linear function l such that $u^* - l \ge 0$ in $B_r(x)$, $(u^* - l)(x) = 0$ but $u^* - l$ is not identically 0 in $B_r(x)$. This contradicts weak Harnack inequality for $u^* - l$, and concludes Case 1.

Case 2. By passing to a subsequence, we can assume that the directions

$$\xi_n := \xi_{h_n}(0) \to e_2 \quad \text{as } n \to \infty,$$

and as before $u_n \to u^*$ uniformly on compact sets. First we show that u^* satisfies weak Harnack inequality in the e_2 direction.

Lemma 3.2. *Assume*

$$u^*(x + te_2) \ge 0, \quad \textit{for all } |t| \le r,$$

for some $x \in B_1$ and $0 < r \le 1$. Then

$$\inf_{|t| \le \frac{r}{2}} u^*(x + te_2) \ge c \inf_{|t| \le \frac{r}{4}} u(x + te_2), \tag{9}$$

where $c > 0$ depends only on λ, Λ.

Proof. It suffices to prove (9) with $r/2$ replaced by ηr and $r/4$ by $\eta r/2$ with η a small constant depending on λ, Λ.

Since $|\nabla u| \le 1$, u_n and u^* are Lipschitz functions with Lipschitz constant 1. By hypothesis, $u^* \ge 0$ on the segment $[x - re_2, x + re_2]$, hence

$$u^* + 2\varepsilon > 0 \quad \text{on} \quad \mathcal{R} = [-\varepsilon, \varepsilon] \times [x - re_2, x + re_2],$$

and the same inequality holds for u_n for all n large.

Let $S_{t_n}(x)$ be the maximal section of u_n at x which is included in $\mathcal{R}$. ¿From the hypotheses $e_n \to \infty$, $\xi_n \to e_2$ and Lemma 2.3 we see that

$$2r + 2\varepsilon \geq diam\, S_{t_n}(x) \geq 2r$$

for all large n. From the properties of sections we see that there exist constants η, τ small such that $S_{t_n/2}(x)$ contains a segment of length 2η centered at x and

$$S_{\tau t_n}(x) \subset [-\varepsilon, \varepsilon] \times [x - \frac{\eta}{2} e_2, x + \frac{\eta}{2} e_2].$$

We apply weak Harnack inequality (6) for $u_n + 2\varepsilon$ in $S_{t_n}(x)$ and use that u_n is Lipschitz to obtain

$$\inf_{|t| \leq \eta r} u_n(x + te_2) \geq c \inf_{|t| \leq \frac{\eta}{2} r} u_n(x + te_2) - C\varepsilon.$$

The lemma is proved by letting $n \to \infty$ and then $\varepsilon \to 0$. $\square$

Lemma 3.3. *If $\nu = \pm e_2$ we have*

$$u^*(t\nu) = t\nu \cdot p_\nu$$

either for all $t \geq 0$ or $t \leq 0$.

Proof. The proof is essentially identical to the proof of Lemma 3.1. We need to remark that the supersolutions $\tilde{u}_n$ obtained from u_n are uniformly Lipschitz. Hence, as in the proof of Lemma 3.3 above, the weak Harnack inequality for $\tilde{u}_n$ implies the weak Harnack inequality for their limit $(u^*)^+$ in the e_2 direction. This gives that $(u^*)^+ > 0$ in $\mathcal{R}_1$ and we reach a contradiction as before. $\square$

Now we are ready to reach a contradiction in Case 2.

The previous lemma implies (as in Case 1) that on the line te_2 the function u^* is linear on both half lines $t \geq 0$ and $t \leq 0$, but with different slopes. This contradicts that $u^* - l$ satisfies weak Harnack inequality in the e_2 direction for an appropriate linear function l.

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Received March 2010; revised April 2010.

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