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NON-LOCAL MINIMAL SURFACES

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1. Introduction

In this paper we study the geometric properties, existence, regularity and related issues for a family of surfaces which are boundaries of sets minimizing certain integral norms. These surfaces can be interpreted as a non-infinitesimal version of classical minimal surfaces.

Our work is motivated by the structure of interphases that arise in classical phase field models when very long space correlations are present. Motion by mean curvature is obtained classically in two different ways. One way is as an asymptotic limit of phase field models involving a double well potential, that is as the steepest descent of the Ginzburg-Landau energy functional

$$\varepsilon \int |\nabla u|^2 dx + \frac{1}{\varepsilon} \int F(u) dx.$$

Another way is as a continuous limit of the following process (cellular automata, see [MBO]). Denote by χ_Ω the characteristic function of the set Ω and by $\mathcal{C}\Omega$ the complement of Ω. The surface $S_{k+1} = \partial\Omega_{k+1}$ at time $t_{k+1} = t_k + \delta$ is generated from $S_k = \partial\Omega_k$ by solving the heat equation

$$u_t - \triangle u = 0, \quad u(\cdot, 0) = u_k,$$

for a small interval of time ε, with initial data

$$u_k := \chi_{\Omega_k} - \chi_{\mathcal{C}\Omega_k}.$$

Thus $u(x, \varepsilon)$ is obtained by simply convolving u_k with the Gauss kernel

$$G_\varepsilon(x) = (4\pi\varepsilon)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\varepsilon}},$$

and define

$$\Omega_{k+1} = \{u(x, \varepsilon) > 0\}, \quad S_{k+1} = \partial\Omega_{k+1}.$$

If $\delta \sim \varepsilon^2$, S_k is a discrete approximation to motion by mean curvature at time $k\delta$ (see [E]). This can be thought as letting the two phases Ω and $\mathcal{C}\Omega$ mix for a short time ε and then segregate them according to density.

One example of long range correlation would consist in replacing the heat equation by a pure jump Levy process. The simplest and "more analytical family" of such processes is of course diffusion by fractional Laplace $(-\triangle)^\sigma$, $0 < \sigma < 1$. In this case we replace the gaussian above with the fundamental solution of

$$u_t + (-\triangle)^\sigma u = 0$$

which is of the form

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$$G(x,t) \sim \frac{t}{(|x|^2 + t^{\frac{1}{\sigma}})^{\frac{n+2\sigma}{2}}}.$$

If $\sigma \geq 1/2$ the process still converges to motion by mean curvature by taking the time step $\varepsilon \sim \delta^{2\sigma}$ for $\sigma > 1/2$ and $\varepsilon \sim \delta \log \delta$ for $\sigma = 1/2$. When $\sigma < 1/2$ the limiting model corresponds now to a non-local surface diffusion (see [CSo]). The normal velocity at a point $x_0 \in S$ satisfies

$$v(x_0) \sim \int_{\mathbb{R}^n} (\chi_\Omega(x) - \chi_{\mathcal{C}\Omega}(x))|x - x_0|^{-n-2\sigma} dx.$$

Going back to the phase field model, the $(-\triangle)^\sigma$ diffusion corresponds to the steepest descent for the energy

$$(1-\sigma)\int\int \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}} dxdy + \int F(u)dx,$$

that is the diffusion part of the energy is now the "σ fractional derivative" of u or the H^σ seminorm of u.

There is an extensive literature on the asymptotics for this problem (see for example [I, IPS, Sl]) but most mathematical results involve the hypothesis of finite first moments for the diffusion kernel, and that implies that the resulting interphase dynamics is still infinitesimal ($\sigma > 1/2$ in our discussion above).

In this paper we intend to study the "minimal surfaces" arising from the cases in which the surface evolution is non-local ($\sigma < 1/2$), i.e. surfaces $S = \partial\Omega$ whose Euler-Lagrange equation is

$$\int (\chi_\Omega(y) - \chi_{\mathcal{C}\Omega}(y))|y-x|^{-n-2\sigma} dy = 0 \qquad \text{for } x \in S.$$

Surprisingly such surfaces can be attained by minimizing the H^σ norm of the indicator function χ_Ω. Precisely, for $\sigma < 1/2$ and Ω reasonably smooth, $\|\chi_\Omega\|_{H^\sigma}$ becomes finite whereas for $\sigma = 1/2$ this is not true, i.e. we can not obtain classical minimal surfaces as sets minimizing an H^σ norm.

The main result of this paper is that S is a smooth hypersurface except for a closed singular set of $\mathcal{H}^{n-2}$ Hausdorff dimension. This parallels the classical minimal surface theory (the reader may find it useful to have it in mind, see for example [G]), except that we do not have in this paper the optimal dimension (in the classical minimal surface theory it is $n-8$).

Our main steps are

a) existence of minimizers and uniform positive density of Ω and $\mathcal{C}\Omega$

b) The Euler-Lagrange equation in the viscosity sense

c) Flatness implies $C^{1,\alpha}$ regularity

d) A monotonicity formula and existence of tangent cones

e) Existence of an "energy gap" between minimal cones and hyperplanes.

2. **Definitions, notations and main result**

As pointed out above, we will consider minimizers of the H^σ seminorm, $\sigma < 1/2$, of the characteristic function χ_E of a set E which is fixed outside a domain $\Omega \subset \mathbb{R}^n$,

$$\|\chi_E\|^2_{H^\sigma} = \int\int \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{n+2\sigma}} dxdy$$

$$= 2\int\int \frac{\chi_E(x)\chi_{\mathcal{C}E}(y)}{|x-y|^{n+2\sigma}} dxdy,$$

where $\mathcal{C}E$ denotes the complement of E.

We denote for simplicity

$$s := 2\sigma, \quad 0 < s < 1,$$

and

$$L(A,B) := \int\int \frac{1}{|x-y|^{n+s}} \chi_A(x)\chi_B(y) dxdy.$$

Clearly

$$L(A,B) \geq 0, \quad L(A,B) = L(B,A),$$

$$L(A_1 \cup A_2, B) = L(A_1, B) + L(A_2, B) \quad \text{for } A_1 \cap A_2 = \emptyset.$$

Definition 2.1. *(Local energy integral)* For a bounded set Ω, and for $E \subset \mathbb{R}^n$ we define

$$\mathcal{J}_\Omega(E) := L(E \cap \Omega, \mathcal{C}E) + L(E \setminus \Omega, \mathcal{C}E \cap \Omega)$$

to be the "Ω-contribution" for the $H^{s/2}$-norm of the characteristic function of E.

Definition 2.2. We say that E is a minimizer for $\mathcal{J}$ in Ω if for any set F with $F \cap (\mathcal{C}\Omega) = E \cap (\mathcal{C}\Omega)$ we have

$$\mathcal{J}_\Omega(E) \leq \mathcal{J}_\Omega(F).$$

Remarks. The set $E \cap (\mathcal{C}\Omega)$ plays the role of "boundary data" for $E \cap \Omega$.

If Ω is a bounded Lipschitz domain, then $inf \mathcal{J}_\Omega$ is bounded by $\mathcal{J}_\Omega(E \setminus \Omega) < \infty$.

A minimizer E of $\mathcal{J}_\Omega$ satisfies the following two conditions

$$L(A,E) - L(A, \mathcal{C}(E \cup A)) \leq 0 \qquad \text{if } A \subset \mathcal{C}E \cap \Omega \tag{2.1}$$

$$L(A, E \setminus A) - L(A, \mathcal{C}E) \geq 0 \qquad \text{if } A \subset E \cap \Omega. \tag{2.2}$$

Notice that the term $L(A,E)$ in (2.1) is finite since it is bounded by $\mathcal{J}_\Omega(E)$.

Definition 2.3. If E satisfies (2.1) we say that E is a variational supersolution and if it satisfies (2.2) we say that E is a (variational) subsolution.

If E is both a variational subsolution and supersolution then it is a minimizer for $\mathcal{J}_\Omega$. Indeed, if $F \cap (\mathcal{C}\Omega) = E \cap (\mathcal{C}\Omega)$ and we denote by $A^+ = F \setminus E$, $A^- = E \setminus F$ we find

$$\mathcal{J}_\Omega(F) - \mathcal{J}_\Omega(E) = [L(A^-, E \setminus A^-) - L(A^-, \mathcal{C}E)] -$$

$$[L(A^+, E) - L(A^+, \mathcal{C}(E \cup A^+))] + 2L(A^-, A^+) \geq 0.$$

The main result of this paper can now be formulated as follows.

Theorem 2.4. Main theorem
If E minimizes J_{B_1}, then $\partial E \cap B_{1/2}$ is, to the possible exception of a closed set of finite $\mathcal{H}^{n-2}$ dimension, a $C^{1,\alpha}$ hypersurface around each of its points.

3. Existence and compactness of minimizers

In this section we prove some basic properties of minimizers.

Proposition 3.1. Lower semicontinuity of $\mathcal{J}$
If $\chi_{E_n} \to \chi_E$ in L^1_{loc} then

$$\liminf J_\Omega(E_n) \geq J_\Omega(E).$$

Proof: Recall that

$$L(A, B) = \int\int \frac{1}{|x-y|^{n+s}} \chi_A(x)\chi_B(y) dx dy.$$

It is clear that if $\chi_{A_n} \to \chi_A$, $\chi_{B_n} \to \chi_B$ in $L^1_{loc}(\mathbb{R}^n)$ then any sequence contains a subsequence, say n_k such that for a.e. (x, y)

$$\chi_{A_{n_k}}(x)\chi_{B_{n_k}}(y) \to \chi_A(x)\chi_B(y).$$

Fatou's lemma implies

$$\liminf_k L(A_{n_k}, B_{n_k}) \geq L(A, B).$$

□

Theorem 3.2. Existence of minimizers
Let Ω be a bounded Lipschitz domain and $E_0 \subset \mathcal{C}\Omega$ be a given set. There exists a set E, with $E \cap \mathcal{C}\Omega = E_0$ such that

$$\inf_{F \cap \mathcal{C}\Omega = E_0} \mathcal{J}_\Omega(F) = \mathcal{J}_\Omega(E).$$

Proof: The infimum is bounded since $\mathcal{J}_\Omega(E_0) < \infty$. Let F_n be a sequence of sets so that $\mathcal{J}_\Omega(F_n)$ converges to the infimum. The $H^{s/2}$ norms of the characteristic functions of $F_n \cap \Omega$ are bounded. Thus, by compactness, there is a subsequence that converges in $L^1(\mathbb{R}^n)$ to a set $E \cap \Omega$. Now the result follows from the lower semicontinuity.

□

Next we prove the following compactness theorem.

Theorem 3.3. *Assume E_n are minimizers for $\mathcal{J}_{B_1}$ and*

$$E_n \to E \quad in\ L^1_{loc}(\mathbb{R}^n).$$

Then E is a minimizer for $\mathcal{J}_{B_1}$ and

$$\lim_{n\to\infty} \mathcal{J}_{B_1}(E_n) = \mathcal{J}_{B_1}(E).$$

Proof: Assume $F = E$ outside B_1. Let

$$F_n := (F \cap B_1) \cup (E_n \setminus B_1),$$

then

$$\mathcal{J}_{B_1}(F_n) \geq \mathcal{J}_{B_1}(E_n).$$

It is easy to check that

$$|\mathcal{J}_{B_1}(F) - \mathcal{J}_{B_1}(F_n)| \leq L(B_1, (E_n \Delta E) \setminus B_1).$$

We denote

$$b_n := L(B_1, (E_n \Delta E) \setminus B_1)$$

and obtain

$$\mathcal{J}_{B_1}(F) + b_n \geq \mathcal{J}_{B_1}(E_n).$$

It suffices to prove that $b_n \to 0$. Then we will get

$$\mathcal{J}_{B_1}(F) \geq \limsup \mathcal{J}_{B_1}(E_n)$$

and the Theorem follows from the lower semicontinuity of $\mathcal{J}$:

$$\liminf \mathcal{J}_{B_1}(E_n) \geq \mathcal{J}_{B_1}(E).$$

Define now

$$a_n(r) := \mathcal{H}^{n-1}((E_n \Delta E) \cap \partial B_r),$$

we then obtain that for any $r_0 > 1$

$$b_n \leq C \int_1^{r_0} a_n(r)(r-1)^{-s} dr + C r_0^{-s}$$

where C is a universal constant. Since

$$\int_1^{r_0} a_n(r) dr \to 0, \qquad a_n(r) \leq C r_0^{n-1} \text{ for } r \leq r_0,$$

we find

$$\limsup b_n \leq C r_0^{-s},$$

which proves the theorem because r_0 is arbitrary.

$\square$

4. **Uniform density estimates**

Let E be a measurable set. We say that x belongs to the interior of E, (in the measure sense) if there exists $r > 0$ such that $|B_r(x) \setminus E| = 0$. We will always assume that the sets we consider, by possibly modifying them on a set of measure 0, contain their interior and do not intersect the interior of their complement.

In this case we see that $x \in \partial E$ if and only if for any $r > 0$, $|B_r(x) \cap E| > 0$ and $|B_r(x) \cap \mathcal{C}E| > 0$. Notice that ∂E is a closed set and the interior is an open set.

Theorem 4.1. Uniform density estimate
Assume E is a variational subsolution in Ω. There exists $c > 0$ universal (depending on n, s) such that if $x \in \partial E$ and $B_r(x) \cap E \subset \Omega$

$$|E \cap B_r(x)| \geq cr^n.$$

If E is a minimizer for $\mathcal{J}_\Omega$ then both E and $\mathcal{C}E$ satisfy the uniform density estimate. Theorem 4.1 is a consequence of the following lemma.

Lemma 4.2. *Assume E is a subsolution in B_1. There exists $c > 0$ universal such that, if $|E \cap B_1| \leq c$ then $|E \cap B_{1/2}| = 0$.*

Proof. For $r \in (0, 1]$, set

$$V_r = |E \cap B_r|, \quad a(r) = \mathcal{H}^{n-1}(E \cap \partial B_r).$$

We apply the Sobolev inequality

$$\|u\|_{L^p} \leq C\|u\|_{H^{s/2}}, \qquad p = \tfrac{2n}{n-s}$$

for $u = \chi_{E \cap B_r}$ and obtain

$$V_r^{\frac{n-s}{n}} \leq CL(A, \mathcal{C}A) \qquad \text{with } A := E \cap B_r.$$

From (2.2) we find

$$L(A, \mathcal{C}A) = L(A, \mathcal{C}E) + L(A, E \setminus A)$$

$$\leq 2L(A, E \setminus A) \leq 2L(A, \mathcal{C}B_r).$$

If $x \in A$ then

$$\int_{\mathcal{C}B_r} \frac{1}{|x-y|^{n+s}} dy \leq C \int_{r-|x|}^{\infty} \frac{1}{\rho^{n+s}} \rho^{n-1} d\rho \leq C(r - |x|)^{-s},$$

hence

$$L(A, \mathcal{C}B_r) = \int\int \frac{\chi_A(x)\chi_{\mathcal{C}B_r}(y)}{|x-y|^{n+s}} dxdy \leq C \int_0^r a(\rho)(r-\rho)^{-s}.$$

We conclude that

$$V_r^{\frac{n-s}{n}} \leq C \int_0^r a(\rho)(r-\rho)^{-s}.$$

Integrating the inequality above between 0 and t we find

$$\int_0^t V_r^{\frac{n-s}{n}} dr \le C t^{1-s} \int_0^t a(\rho) d\rho = C t^{1-s} V_t. \tag{4.1}$$

The proof is now of the standard De Giorgi iteration: set

$$t_k = \frac{1}{2} + \frac{1}{2^k}, \qquad v_k = V_{t_k};$$

notice that $t_0 = 1$ and $t_\infty = \frac{1}{2}$. Equation (4.1) yields

$$2^{-(k+1)} v_{k+1}^{\frac{n-s}{n}} \le C_0 v_k$$

with C_0 universal constant. This implies $v_k \to 0$ as $k \to \infty$ if $v_0 \le c$ with c universal, small enough.

□

Corollary 4.3. Clean ball condition

Assume E is a minimizer for $\mathcal{J}_\Omega$, $x \in \partial E$ and $B_r(x) \subset \Omega$. There exist balls

$$B_{cr}(y_1) \subset E \cap B_r(x), \qquad B_{cr}(y_2) \subset \mathcal{C}E \cap B_r(x)$$

for some small $c > 0$ universal.

Proof. Assume $x = 0$ and $r = 1$. We decompose the space into cubes of size δ. We show that N_δ, the number of cubes that intersect $\partial E \cap B_1$, satisfies

$$N_\delta \le C\delta^{s-n}.$$

Let $Q_\delta \subset B_1$ be a cube such that $\partial E \cap Q_\delta \neq \emptyset$. From the density estimate,

$$|E \cap Q_{3\delta}|, |\mathcal{C}E \cap Q_{3\delta}| \ge c\delta^n$$

which implies

$$L(E \cap Q_{3\delta}, \mathcal{C}E \cap Q_{3\delta}) \ge c\delta^{n-s}.$$

Adding all these inequalities we obtain

$$L(E \cap B_1, \mathcal{C}E \cap B_1) \ge c_0 N_\delta \delta^{n-s}.$$

On the other hand, from minimality

$$L(E \cap B_1, \mathcal{C}E \cap B_1) \le L(E \cap B_1, \mathcal{C}E) \le L(E \cap B_1, \mathcal{C}B_1) \le C_0,$$

which proves the bound on N_δ.

Since $0 \in \partial E$, the density estimate implies that at least $c\delta^{-n}$ of the cubes from B_1 intersect $E \cap B_1$. Thus, if δ is chosen small universal, there exists a cube of size δ which is completely included in $E \cap B_1$.

□

Theorem 4.1 has the following (classical) corollary, useful in several places of the sequel.

Corollary 4.4. *(i) If E minimizes $\mathcal{J}_\Omega$ then*

$$\mathcal{H}^{n-s}(\partial E \cap \Omega) < \infty.$$

(ii) (Improvement of Theorem 3.3) Assume E_k are minimizers for $\mathcal{J}_{B_1}$ and

$$E_k \to E \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

For every $\varepsilon > 0$, ∂E_k is in an ε-neighborhood of ∂E as soon as n is large enough.

Proof. Fact (i) is straightforward from the proof of Corollary 4.3. Let us prove (ii): for this, assume the existence of a (possibly relabeled) sequence (x_k) and $\varepsilon_0 > 0$ such that

$$x_k \in \partial E_k \quad \text{and} \quad d(x_k, E) \geq \varepsilon_0.$$

By Theorem 4.1 we have

$$|E_k \backslash E| \geq |E_k \cap B_{\varepsilon_0/2}(x_k)| \geq c\varepsilon_0^n,$$

contradicting the L^1_{loc} convergence of E_k to E.

□

We will prove later that $\partial E \cap \Omega$ has in fact $n-1$ Hausdorff dimension.

5. The Euler-Lagrange equation in the viscosity sense

As we pointed out in the introduction, the Euler-Lagrange equation for $H^{s/2}$ minimization is the $(s/2)$-Laplacian. The theorem below can be thought as saying

$$\triangle^{s/2}(\chi_E - \chi_{\mathcal{C}E}) = 0 \quad \text{along } \partial E.$$

Theorem 5.1. *Assume E is a supersolution, $0 \in \partial E$ and the unit ball $B_1(-e_n)$ is included in E. Then*

$$\int_{\mathbb{R}^n} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x|^{n+s}} dx \leq 0.$$

In order to fix ideas, we prove first a comparison principle between ∂E and the hyperplane $\{x_n = 0\}$. The same techniques will be used in the proof of Theorem 5.1. More precisely, assume E is a minimizer in B_1 and $\{x_n \leq 0\} \setminus B_1 \subset E$. We want to show that $\{x_n \leq 0\} \subset E$. Define

$$A^- := \{x_n \leq 0\} \setminus E,$$

then from the minimality of E we obtain

$$0 \geq L(A^-, E) - L(A^-, \mathcal{C}(E \cup A^-)).$$

It is not obvious that we reach a contradiction if $|A^-| > 0$. We would like to consider another set as perturbation and make use of symmetry in order to obtain cancellations in the integrals.

For this let T be the reflection across $\{x_n = 0\}$ i.e. $T(x', x_n) = (x', -x_n)$ and let

$$A^+ = T(A^-) \setminus E.$$

Define

$$A = A^- \cup A^+,$$

and decompose it into two sets: A_1 which is symmetric with respect to $\{x_n = 0\}$ and the remaining part $A_2 \subset A^-$ i.e,

$$A_1 = A^+ \cup T(A^+), \quad A_2 = A^- \setminus T(A^+).$$

Finally, let F be the reflection of $\mathcal{C}(E \cup A)$, then from our hypothesis

$$F \subset \{x_n \le 0\} \subset E.$$

The minimality of E implies

$$0 \ge L(A,E) - L(A, \mathcal{C}(E \cup A)) = \sum (L(A_i, E) - L(A_i, \mathcal{C}(E \cup A)))$$

$$= L(A_1, E \setminus F) + L(A_2, E \setminus F) + (L(A_2, F) - L(T(A_2), F)).$$

All three terms are nonnegative and are 0 only if $|A_2| = 0$ and either $|A_1| = 0$ or $|E \setminus F| = 0$. At this point we remark that we can repeat the argument above for the hyperplane $\{x_n = -\varepsilon\}$ instead of $\{x_n = 0\}$ and in this case $|E \setminus F| > 0$. In conclusion we obtain $|A^-| = 0$ which proves the comparison principle.

We are now ready to prove Theorem 5.1. Again we consider symmetric sets as perturbations by using the radial reflection across a sphere. The proof is more involved since the cancellations have now error terms but they are balanced by using the positive density property.

Proof of Theorem 5.1: Without loss of generality assume that E contains $B_2(-2e_n)$. We will show

$$\limsup_{\delta \to 0} \int_{\mathbb{R}^n \setminus B_\delta} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x|^{n+s}} dx \le 0.$$

Fix $\delta > 0$ small, and $\varepsilon \ll \delta$.

We denote by d_x the distance from x to the sphere $\partial B_{1+\varepsilon}(-e_n)$.

Let T be the radial reflection with respect to the sphere $\partial B_{1+\varepsilon}(-e_n)$ in the annulus $1 - 2\delta < d_x < 1 + 2\delta$ i.e,

$$\frac{x + Tx}{2} + e_n = (1+\varepsilon)\frac{x + e_n}{|x + e_n|},$$

and notice that

$$|DT(x)| \le 1 + 3d_x, \quad |T(x) - T(y)| \ge (1 - 3\max\{d_x, d_y\})|x - y|.$$

We define various sets:

$$A^- := B_{1+\varepsilon}(-e_n) \setminus E$$

$$A^+ := T(A^-) \setminus E, \quad A := A^- \cup A^+.$$

We decompose A into two disjoint sets A_1 and A_2, with $A_1 = T(A_1)$,

$$A = A_1 \cup A_2, \quad A_1 := T(A^+) \cup A^+, \quad A_2 \subset A^- \subset B_{1+\varepsilon}(-e_n).$$

and define

$$F := T(B_\delta \cap \mathcal{C}(E \cup A)).$$

It is easy to check that

$$F \subset B_{1+\varepsilon}(-e_n) \setminus A^- \subset E \cap B_\delta.$$

We have

$$L(A, E) - L(A, \mathcal{C}(E \cup A) =$$

$$[L(A, E \setminus B_\delta) - L(A, \mathcal{C}E \setminus B_\delta)] + [L(A, F) - L(A, T(F))] + L(A, (E \cap B_\delta) \setminus F)$$

$$:= I_1 + I_2 + I_3 \leq 0.$$

Since $I_3 \geq 0$ we obtain

$$I_1 + I_2 \leq 0.$$

We estimate I_1 by using that $A \subset B_{2\sqrt{\varepsilon}}$, thus

$$\left| \frac{1}{|A|} I_1 - \int_{\mathbb{R}^n \setminus B_\delta} \frac{\chi_E - \chi_{\mathcal{C}E}}{|y|^{n+s}} dy \right| \leq C \int_{\mathbb{R}^n \setminus B_\delta} \frac{\sqrt{\varepsilon}}{|y|^{n+s+1}} dy$$

$$\leq C \varepsilon^{1/2} \delta^{-1-s}. \tag{5.1}$$

To estimate I_2 we write

$$I_2 = [L(A_1, F) - L(A_1, T(F))] + [L(A_2, F) - L(A_2, T(F)].$$

By changing the variables $x \to Tx$, $y \to Ty$ we have

$$L(A_1, T(F)) = \int \int \chi_{A_1}(x) \chi_F(y) \frac{|DT(x)||DT(y)|}{|Tx - Ty|^{n+s}} dxdy$$

$$\leq \int \int \chi_{A_1}(x) \chi_F(y) \frac{1 + C \max\{d_x, d_y\}}{|x - y|^{n+s}} dxdy.$$

Also by changing $y \to Ty$ we find

$$L(A_2, T(F)) = \int\int \chi_{A_2}(x)\chi_F(y)\frac{|DT(y)|}{|x - Ty|^{n+s}}dxdy$$

$$\leq \int\int \chi_{A_2}(x)\chi_F(y)\frac{1 + Cd_y}{|x - y|^{n+s}}dxdy$$

and we have used that

$$|x - y| \leq |x - Ty| \quad \text{for } x, y \in B_{1+\varepsilon}. \tag{5.2}$$

We conclude that

$$-I_2 \leq C\int\int \chi_A(x)\chi_F(y)\frac{\max\{d_x, d_y\}}{|x - y|^{n+s}}dxdy.$$

We estimate the contribution in the integral above for x outside $B_{1+\varepsilon}(-e_n)$, i.e. $x \in A^+$, by changing $x \to Tx$ and using (5.2)

$$\int\int \chi_{A^+}(x)\chi_F(y)\frac{\max\{d_x, d_y\}}{|x - y|^{n+s}}dxdy$$

$$\leq \int\int \chi_{A^-}(x)\chi_F(y)\frac{\max\{d_x, d_y\}|DT(x)|}{|Tx - y|^{n+s}}dxdy$$

$$\leq 2\int\int \chi_{A^-}(x)\chi_F(y)\frac{\max\{d_x, d_y\}}{|x - y|^{n+s}}dxdy.$$

Hence

$$-I_2 \leq C\int\int \chi_{A^-}(x)\chi_F(y)\frac{\max\{d_x, d_y\}}{|x - y|^{n+s}}dxdy.$$

For fixed $x \in A^-$

$$\int_{B_\delta \setminus B_{2d_x}(x)} \frac{\max\{d_x, d_y\}}{|x - y|^{n+s}}dy \leq C\int_{2d_x}^{2\delta} \frac{r}{r^{n+s}}r^{n-1}dr \leq C\delta^{1-s}.$$

When $y \in B_{2d_x}(x)$,

$$\max\{d_x, d_y\} \leq 3d_x \leq 3\varepsilon,$$

thus

$$-I_2 \leq C\delta^{1-s}|A| + C\varepsilon\int\int \frac{\chi_{A^-}(x)\chi_F(y)}{|x - y|^{n+s}}dxdy$$

$$= C\delta^{1-s}|A| + C\varepsilon L(A^-, F). \tag{5.3}$$

We will prove the following lemma:

Lemma 5.2. *There exists a sequence of* $\varepsilon \to 0$ *such that*

$$\varepsilon L(A^-, F) \leq C\varepsilon^\eta |A^-|$$

where η *is such that* $0 < \eta < 1 - s$.

Now the proof of the theorem follows. Indeed, since

$$I_1/|A| \leq -I_2/|A|$$

we let $\varepsilon \to 0$ and use (5.1), (5.3) and the lemma to conclude

$$\int_{\mathbb{R}^n \setminus B_\delta} \frac{\chi_E - \chi_{\mathcal{C}E}}{|y|^{n+s}} dy \leq C\delta^{1-s}.$$

□

Proof of Lemma 5.2: We use (2.1) for A^- and find

$$L(A^-, F) \leq L(A^-, E) \leq L(A^-, \mathcal{C}(E \cup A) \leq L(A^-, \mathcal{C}(B_{1+\varepsilon}(-e_n)).$$

If $x \in B_{1+\varepsilon}(-e_n)$ then

$$\int_{\mathcal{C}B_{1+\varepsilon}(-e_n)} \frac{1}{|x-y|^{n+s}} dy \leq C \int_{d_x}^\infty \frac{1}{r^{n+s}} r^{n-1} dr \leq C d_x^{-s}.$$

We denote

$$a(r) := \mathcal{H}^{n-1}(\mathcal{C}(E \cap \partial B_{1+r}(-e_n))),$$

and prove that for a sequence of $\varepsilon \to 0$

$$\varepsilon \int_0^\varepsilon a(r)(\varepsilon - r)^{-s} dr \leq \varepsilon^\eta \int_0^\varepsilon a(r) dr.$$

Assume by contradiction that for all ε small we have the opposite inequality i.e.

$$\int_0^\varepsilon a(r)(\varepsilon - r)^{-s} dr > \varepsilon^{\eta - 1} \int_0^\varepsilon a(r) dr.$$

Integrating in ε between 0 and λ we find

$$\lambda^{1-s} \int_0^\lambda a(r) dr \geq c(s, \eta) \lambda^\eta \int_0^{\lambda/2} a(r) dr$$

hence, for any fixed $M > 0$

$$\int_{\lambda/2}^\lambda a(r) dr \geq M \int_{\lambda/4}^{\lambda/2} a(r) dr$$

provided that λ is small. Writing this inequality for $\lambda = 2^{-k}$, $k \geq k_0$ we obtain

$$|\mathcal{C}E \cap B_{1+2^{-k}}(-e_n)| = \int_0^{2^{-k}} a(r) dr \leq (M/2)^{k_0 - k}$$

for all $k \geq k_0$.

On the other hand, positive density of the complement at 0 gives

$$|\mathcal{C}E \cap B_{1+2^{-k}}(-e_n)| \geq |\mathcal{C}E \cap B_{2^{-k}}| \geq c2^{-nk}$$

and we reach a contradiction if we choose $M > 2^{n+1}$.

□

Some consequences of the Euler-Lagrange equation are the following.

Corollary 5.3. *a) If $E \cap \mathcal{C}\Omega$ is contained in the strip $\{a \leq x_n \leq b\}$, then E is contained in the same strip.*

b) Hyperplanes are local minimizers.

c) If $x_0 \in \partial\Omega \cap \partial E$ and $\mathcal{C} \cap E$ has an interior tangent ball at x_0, then E is a viscosity supersolution at x_0.

Finally, an important observation is the following comparison tool.

Lemma 5.4. *Let E_δ be the δ neighborhood of E, i.e.*

$$E_\delta = \{x \mid dist(x, E) \leq \delta\}.$$

Then if $x_0 \in \partial E_\delta$ realizes its distance at $y_0 \in \partial E$ then E has at y_0 an external tangent ball and

$$\int_{\mathbb{R}^n} \frac{\chi_{E_\delta} - \chi_{\mathcal{C}E_\delta}}{|x - x_0|^{n+s}} dx \geq \int_{\mathbb{R}^n} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x - y_0|^{n+s}} dx$$

in the principal value sense.

In particular, if E is a viscosity solution at y_0 then E_δ is a viscosity subsolution at x_0.

The proof is straightforward after translating E_δ by $y_0 - x_0$.

This lemma can be applied to prove for instance that minimizers are graphs under appropriate geometric conditions.

6. **Improvement of flatness**

In this section we prove the following theorem, in the spirit of the regularity theorem of de Giorgi for classical minimal surfaces [G], Chap. 8:

Theorem 6.1. *Assume E is minimal in B_1. There exists $\varepsilon_0 > 0$ depending on s and n such that if*

$$\partial E \cap B_1 \subset \{|x \cdot e_n| \leq \varepsilon_0\}$$

then $\partial E \cap B_{1/2}$ is a $C^{1,\gamma}$ graph in the e_n direction.

As a consequence we obtain

Corollary 6.2. *If ∂E has at x_0 a tangent ball $B_r \subset E$, then ∂E is a $C^{1,\gamma}$ surface in a neighborhood of x_0.*

Because the case of nonlocal minimal surfaces contains difficulties on its own, it is useful to recall the main steps of the method of [S].

6.1. **Minimal surfaces.** . Let us define the flatness of a cylinder to be the ratio between its height and the diameter of the base. It is well-known that Theorem 6.1 reduces to proving an improvement of flatness theorem of the type

Theorem 6.3. *Assume E is minimal in B_1. There exists three reals: $\eta_0 > 0$, $\varepsilon_0 > 0$, $q_0 \in (0,1)$ and an orthonormal basis $(\tilde{e}_i)_{1\le i\le n}$ such that*

$$\partial E \cap B_1 \subset \{|x \cdot e_n| \le \varepsilon\}, \qquad \varepsilon \le \varepsilon_0$$

then

$$\partial E \cap B_{\eta_0} \subset \{|x \cdot \tilde{e}_n| \le q_0\eta_0\varepsilon\}$$

.

In other words, $\partial E \cap B_{\eta_0}$ can be included in a cylinder of flatness $q_0\varepsilon$. The iteration of this theorem produces the $C^{1,\gamma}$ regularity in a neighborhood of 0.

The main tool is a Harnack inequality

Theorem 6.4. [S] *Assume E minimal in B_1, and $0 \in \partial E$. There is $\varepsilon_0 > 0$ and $\nu \in (0,1)$ such that, if*

$$\partial E \cap B_1 \subset \{|x \cdot e_n| \le \varepsilon\}, \qquad \varepsilon \le \varepsilon_0$$

then

$$\partial E \cap \{|x'| \le \frac{1}{2}\} \subset \{|x \cdot \tilde{e}_n| \le (1-\nu)\varepsilon\}$$

.

Here we have denoted by $x = (x', x_n)$ the generic point of $\mathbb{R}^n$.

Assume now the existence of a sequence of minimal sets E_m and a sequence ε_m going to 0 such that $0 \in \partial E_m$ and

$$\partial E \cap B_1 \subset \{|x \cdot e_n| \le \varepsilon_m\}$$

and none of the sets E_m satisfies the conclusion of Theorem 6.3. Iterate Theorem 6.4, at the k^{th} iteration ∂E_m is in a cylinder whose base has diameter 2^{-k} and height $(1-\nu)^k\varepsilon_m$. The assumptions of Theorem 6.4 cease to be valid when $2^k(1-\nu)^k\varepsilon_m$ becomes of the order of ε_0, hence

$$k \sim \frac{1}{\log 2(1-\nu)} \log \frac{\varepsilon_0}{\varepsilon_m}.$$

Consider the vertical dilations of E_m:

$$E_m^* = \{(x', \frac{x_n}{\varepsilon_m}), \quad (x', x_n) \in \partial E_m.$$

As a consequence of the above considerations, the intersection of ∂E_m^* with the vertical line $\{x' = 0\}$ converges to $\{0\}$. The same operation may be done for any other point $(x', x_n) \in \partial E_m$, provided that ∂E_m has been suitably translated. The end result is that, in $B_{1/2}$, the sequence ∂E_m^* converges to the graph of a function $\{(x', v(x')), \ |x'| \le \frac{1}{2}\}$. Moreover, v is Hölder: indeed, the first k for which x_1' and x_2' cease to be in the same cylinder $\{|x'| \le 2^{-k}\}$ is $k \sim \dfrac{\log|x_1' - x_2'|}{\log 2}$, and the oscillation of v in the corresponding cylinder -normalized by ε_m - is $(1-\nu)^k$. Hence the oscillation of v is of order $|x'_1 - x'_2|^\alpha$,

$$\alpha = \frac{|\log(1-\nu)|}{\log 2}.$$

On the other hand, the (signed) distance function to ∂E_m, denoted by $d_m(x)$ (with the convention that $d_m < 0$ if $x \in E_m$ can be computed as

$$d_m(x) = \varepsilon_m v(x') - x_n + o(\varepsilon_m),$$

moreover it satisfies

Theorem 6.5. [1]. *d_m is harmonic, in the viscosity sense, on ∂E_m.*

This means that, if we touch ∂E_m from above or below by quadratic graphs, the corresponding inequalities hold. An easy limiting procedure yields

$$-\Delta_{x'} v = 0 \quad \text{in } \{|x'| \leq \frac{1}{2}\},$$

in the viscosity sense. This implies in turn that v is harmonic in the classical sense in $\{|x'| \leq \frac{1}{2}\}$, hence smooth. In particular, its graph can be included in a cylinder of arbitrary flatness μ around 0. However, recall that the sequence of dilations E^*_m converges to $(x', v(x'))$, hence can be included in a cylinder of flatness, say, 2μ around 0 for n large enough. This is a contradiction, and Theorem 6.3 is proved.

6.2. **The proof of Theorem 6.1: linear equations.** . We are going to follow the same strategy as above: consider a sequence of thinner and thinner nonlocal minimal sets, prove that their dilations converge to some Hlder graph with the aid of a - yet to prove - Harnack inequality, and finally translate a viscosity relation - here, Theorem 5.1 - into a linear viscosity relation in one less dimensions, in order to prove further regularity for the limiting graph v. Here is a preliminary result for global solutions to the linear equation $\triangle^\sigma u = 0$, $0 < \sigma < 1$.

If u is a function such that

$$\int \frac{|u|}{(1+|x|^2)^{\frac{n+2\sigma}{2}}} < \infty, \tag{6.1}$$

$\triangle^\sigma u$ is defined as

$$\triangle^\sigma u(y) = \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}} dx.$$

The integral above is convergent in the principal value sense if there exists a smooth tangent function that touches u by above (or below) at y.

We recall the notion of viscosity solutions (see [CSi2]).

Definition 6.6. The continuous function u satisfies

$$\triangle^\sigma u \leq f \quad \text{in } B_1$$

in the viscosity sense (u is a supersolution) if the inequality holds at all points $y \in B_1$ where u admits a smooth tangent function by below.

Similarly, one can define the notion of subsolution. If u is both a supersolution and a subsolution we say that u is a viscosity solution.

In [CSi2] it was proved that if

$$\triangle^\sigma u = f \quad \text{in } B_1$$

in the viscosity sense then

$$\|u\|_{C^{2,\gamma}(B_{1/2})} \leq C(\|f\|_{C^{1,1}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}), \tag{6.2}$$

for γ, C depending only on n and σ.

If $\sigma > 1/2$ then (6.1) is satisfied for functions $u(x)$ that grow at infinity at most like $|x|^{1+\alpha}$, $0 < \alpha < 2\sigma - 1$.

Proposition 6.7. *Let $\sigma > 1/2$, and assume*

$$|u(x)| \leq 1 + |x|^{1+\alpha}, \qquad 0 < \alpha < 2\sigma - 1,$$

and

$$\triangle^\sigma u = 0 \quad \text{in } \mathbb{R}^n$$

in the viscosity sense. Then u is linear.

Proof. The function

$$v(x) := u(x)\chi_{B_2}(x)$$

satisfies

$$\triangle^\sigma v(x) = f(x) \qquad \text{in } B_1$$

with

$$\|f\|_{C^{1,1}(B_1)}, \quad \|v\|_{L^\infty} \leq C(\alpha).$$

From (6.2) we obtain

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C(\alpha).$$

This estimate holds also for the rescaled functions

$$u_R(x) := \frac{u(Rx)}{R^{1+\alpha}}, \qquad R \geq 1,$$

since they satisfy the same hypotheses as u. This gives

$$|\nabla u(Rx) - \nabla u(0)| \leq C(\alpha) R^{\alpha-1} |Rx| \qquad \text{if } |x| \leq 1/2$$

which implies that $\nabla u(x_0) = \nabla u(0)$ for any $x_0 \in \mathbb{R}^n$.

□

Remark. The proposition is valid also for $0 < \sigma \leq 1/2$ except that we have to replace $\triangle^\sigma u = 0$ with "$\nabla \triangle^\sigma u = 0$" that is

$$\triangle^\sigma u(y) - \triangle^\sigma u(z) \leq 0, \quad y, z \in \mathbb{R}^n$$

whenever we can touch u by below at y and by above at z with smooth functions.

6.3. **Improvement of flatness and Proof of Theorem 6.1.** The proof of the Harnack inequality goes by contradiction: if our minimal set cannot be included in cylinder of a lesser height as we approch 0, we contradict the viscosity relation. However, this relation is nonlocal, in contrast with what happens for classical minimal surfaces. A possible remedy to this is to work at an intermediate scale: $|x'| \sim a_m^{-1}$, $x_m \sim \varepsilon_m a_m$, where $a_m \to +\infty$ and $\varepsilon_m a_m \to 0$. However, we would in the end obtain a graph $(x', v(x'))$ with no control on v. This is not desirable, since we wish to prove that v is $\dfrac{1+s}{2}$-harmonic. But then we need a control on v at infinity.

This is why we have to prove a special type of improvement of flatness for non-local minimal surfaces. Here it is below, and Theorem 6.1 follows easily.

Theorem 6.8. *Assume ∂E is minimal in B_1 for $H^{s/2}$, $s < 1$, and fix $0 < \alpha < s$. There exists k_0 depending on s, n and α such that if $0 \in \partial E$ and*

$$\partial E \cap B_{2^{-k}} \subset \{|x \cdot e_k| \le 2^{-k(\alpha+1)}, |e_k| = 1\}, \quad \textit{for } k = 0, 1, 2, \dots, k_0$$

then there exist vectors e_k for all $k \in \mathbb{N}$ for which the inclusion above remains valid.

Rescaling by a factor 2^k, the situation above can be described as follows. There exists k_0 depending on s, n, α, such that if for some $k \ge k_0$

$$\partial E \cap B_{2^l} \subset \{|x \cdot e_l| \le 2^l 2^{\alpha(l-k)}\}, \quad |e_l| = 1, \quad \forall l \ge 0,$$

then the inclusion holds also for $l = -1$, i.e.

$$\partial E \cap B_{1/2} \subset \{|x \cdot e_{-1}| \le 2^{-1} 2^{-\alpha(k+1)}\}.$$

In other words, if $\partial E \cap B_{2^l}$ has $2^{\alpha(l-k)}$ flatness all the way to B_1, it also has it for $B_{1/2}$, and we may dilate and repeat the same argument from there on. Note that for $l > k$ the flatness condition becomes trivial, and for $k = k_0$ we can attain that condition if we start with a very flat solution in B_1 (with $2^{-(\alpha+1)k_0}$ flatness) and we dilate it by a factor 2^{k_0}. The idea of the proof is then by compactness: if not, we will take a sequence E_m of solutions for $m \to \infty$, make a vertical dilation E_m^* and show that there is a subsequence converging to the graph of a continuous function u which solves $\triangle^{(1+s)/2} u = 0$. In order to do that we need first a rough “Harnack type” inequality that will provide the continuity of u. For a similar compactness argument see [S].

Lemma 6.9. *Assume that for some large k, ($k > k_1$)*

$$\partial E \cap B_1 \subset \{|x_n| \le a := 2^{-k\alpha}\}$$

and

$$\partial E \cap B_{2^l} \subset \{|x \cdot e_l| \le a 2^{l(1+\alpha)}\}, \quad l = 0, 1, \dots, k.$$

Then either

$$\partial E \cap B_\delta \subset \{\frac{x_n}{a} \le 1 - \delta^2\}$$

or

$$\partial E \cap B_\delta \subset \{\frac{x_n}{a} \ge -1 + \delta^2\},$$

for δ small, depending on s, n, α.

The hypothesis above can be interpreted in the following way. We are not only requiring flatness of $\partial E \cap B_1$ of order $a = 2^{-k\alpha}$ but also flatness for all diadic balls B_{2^l} of order $a2^{l\alpha}$, from B_1 to B_{2^k}, i.e. until flatness becomes of order one.

Proof. If $y \in \partial E \cap B_{1/2}$, the non-local contribution to the Euler-Lagrange equation is

$$\left| \int_{\mathbb{R}^n \setminus B_{1/2}(y)} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-y|^{n+s}} \, dx \right| \leq C \int_{1/2}^{2^{k-1}} \frac{ar^{n-1+\alpha}}{r^{n+s}} \, dr + C \int_{2^{k-1}}^{\infty} \frac{r^{n-1}}{r^{n+s}} \, dr$$

$$\leq C(a + 2^{-ks}) \leq C(n,s,\alpha)a. \tag{6.3}$$

Let us recall now that E contains $\{x_n < -a\} \cap B_1$ and assume that it contains more than half of the measure of the cylinder

$$D := \{|x'| \leq \delta\} \times \{|x_n| \leq a\}.$$

Then we show that E must contain

$$\{x_n \geq (-1+\delta^2)a\} \cap B_\delta.$$

Indeed, if the conclusion does not hold then, when we slide by below the parabola

$$x_n = -\frac{a}{2}|x'|^2$$

we touch ∂E at a first point $y \in \partial E$ with

$$|y'| \leq 2\delta, \quad |y_n| \leq 2a\delta^2.$$

Denote by P the subgraph of the tangent parabola to ∂E at y. We write

$$\int_{B_{1/2}(y)} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-y|^{n+s}} \, dx = \int_{B_{1/2}(y)} \frac{\chi_P - \chi_{\mathcal{C}P}}{|x-y|^{n+s}} \, dx + \int_{B_{1/2}(y)} \frac{\chi_{E \setminus P}}{|x-y|^{n+s}}$$

$$= I_1 + I_2.$$

If $a \leq \delta$ we estimate

$$I_1 \geq -C \int_0^{1/2} \frac{ar^n}{r^{n+s}} \, dr \geq -C(n,s)a,$$

and, since $E \setminus P$ contains more than 1/4 of the measure of the cylinder D,

$$I_2 \geq Ca\delta^{n-1}/(4\delta)^{n+s} \geq C(n)\delta^{-1-s}a.$$

If $\delta > 0$ is chosen small depending on n, s and α (and $k_1(\delta)$ large so that $a \leq \delta$) then

$$\int_{\mathbb{R}^n} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-y|^{n+s}} \, dx > 0$$

and we contradict the Euler-Lagrange equation at y.

$\square$

As k becomes much larger than k_1, we can once again apply Harnack inequality several times. Indeed, after a dilation of factor $1/\delta = 2^{m_0}$ we have that in B_1, ∂E is included in a cylinder of flatness $(a(1-\delta^2/2)/\delta$. Clearly, as we double the balls the flatness $a(r)$ gets multiplied at most by a factor 2^α as long as $a(r) \le 1$, hence we satisfy again the hypothesis of the lemma. We can apply Harnack inequality as long as the flatness of the inner cylinder remains less than δ, thus we can apply it roughly $c|\log a|$ times. As a consequence we obtain compactness of the sets

$$\partial E^* := \{(x', \frac{x_n}{a})| \quad x \in \partial E\},$$

as $a \to 0$.

More precisely, we consider minimal surfaces ∂E with $0 \in \partial E$, for which there exists k such that

$$\partial E \cap B_1 \subset \{|x_n| \le a := 2^{-k\alpha}\}$$

and

$$\partial E \cap B_{2^l} \subset \{|x \cdot e_l| \le a 2^{l(1+\alpha)}\}, \quad l = 0, 1, \ldots, k.$$

Lemma 6.10. *If E_m is a sequence of minimal sets with $a_m \to 0$ there exists a subsequence m_k such that*

$$\partial E^*_{m_k} \to (x', u(x'))$$

uniformly on compact sets, where $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is Holder continuous and

$$u(0) = 0, \quad |u| \le C(1 + |x|^{1+\alpha}).$$

Proof. From the discussion above when $x' \in B_1$, ∂E^*_m is included between the graphs of

$$\pm C \max\{b_m^\gamma, |x'|^\gamma\}$$

with $b_m \to 0$ as $m \to \infty$. This statement remains valid if we translate the origin at some other point $x_0 \in \partial E^*_m$ with $|x'_0| \le 1/2$. Thus, by the Theorem of Arzela-Ascoli, we can find a subsequence of ∂E^*_m's that converges uniformly in $B_{1/2}$ to the graph of a Holder continuous function.

The same analysis can be done in larger and larger balls since we can estimate for fixed l the angle between e_l and e_{l+1} by the flatness coefficient $2^{\alpha(l-m)}$. Thus we obtain the uniform convergence on compact sets of $\partial E^*_{m_k}$ to the graph of u. Clearly, $u(0) = 0$ and there exist $p_k \in \mathbb{R}^{n-1}$, $p_0 = 0$, such that

$$|u(x') - p_k \cdot x'| \le 2^{k(1+\alpha)} \quad \text{in } B_{2^k}, \text{ for all } k \ge 0.$$

We see that

$$|p_{k+1} - p_k| \le C 2^{k\alpha},$$

thus

$$|p_k| \le C 2^{k\alpha},$$

which implies the growth condition on u.

□

Lemma 6.11. *The limit function u satisfies*

$$\triangle^{\frac{s+1}{2}} u = 0 \quad \textit{in the viscosity sense in } \mathbb{R}^{n-1},$$

and therefore is linear.

Proof. Assume $\varphi + |x'|^2$ is a smooth tangent function that touches u by below, say for simplicity, at the origin. We can find ∂E minimal, a small such that ∂E is included in a $a\varepsilon$ neighborhood of $(x', au(x'))$ for $|x'| \le R$ and ∂E is touched by below at x_0, $|x_0'| \le \varepsilon$ by a vertical translation of $a\varphi$.

From the Euler-Lagrange equation

$$\frac{1}{a}\int_{\mathbb{R}^n} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-x_0|^{n+s}}\, dx \le 0.$$

We estimate this integral in terms of the function u by integrating on square cylinders with center x_0, i.e.

$$D_r := \{|x' - x_0'| < r, \quad |(x-x_0)\cdot e_n| < r\}.$$

Fix δ small and R large, and assume $a, \varepsilon \ll \delta$. In D_δ we use that E contains the subgraph P of a translation of $a\varphi$ thus,

$$\frac{1}{a}\int_{D_\delta} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-x_0|^{n+s}}\, dx \ge \frac{1}{a}\int_{D_\delta} \frac{\chi_P - \chi_{\mathcal{C}P}}{|x-x_0|^{n+s}}\, dx$$

$$\ge -\frac{1}{a}\int_0^{2\delta} \frac{C(\varphi) a r^n}{r^{n+s}}\, dr = -C(\varphi)\delta^{1-s}.$$

If

$$x \in A := (D_R \setminus D_\delta) \cap \{(x-x_0)\cdot e_n \le C(R)a\}$$

we have

$$\left| \frac{1}{|x-x_0|^{n+s}} - \frac{1}{|x'-x_0'|^{n+s}} \right| \le C(\delta, R) a^2,$$

and we estimate

$$\frac{1}{a}\int_{D_R \setminus D_\delta} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-x_0|^{n+s}}\, dx = \frac{1}{a}\int_A \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-x_0|^{n+s}}\, dx$$

$$= \frac{1}{a}\int_{B_R \setminus B_\delta} \frac{a2(u(x') - u(x_0') + O(\varepsilon))}{|x'-x_0'|^{n+s}}\, dx' + O(a^2)$$

$$= 2\int_{B_R \setminus B_\delta} \frac{u(x') - u(x_0')}{|x'-x_0'|^{n+s}}\, dx' + O(\varepsilon) + O(a^2).$$

In $\mathbb{R}^n \setminus D_R$ we estimate as in (6.3)

$$\frac{1}{a}\left| \int_{\mathcal{C}D_R} \frac{\chi_E - \chi_{\mathcal{C}E}}{|x-x_0|^{n+s}} \right| \le \frac{1}{a}\left(\int_{R/2}^{\infty} \frac{a r^{n+\alpha-1}}{r^{n+s}}\, dr + C a^{1+\eta} \right)$$

$$\le C R^{\alpha - s} + C a^{\eta}.$$

We let $\varepsilon, a \to 0$ and find from the Euler-Lagrange equation

$$\int_\delta^R \frac{u(x') - u(0)}{|x'|^{n+s}}\, dx' \le C(\delta^{1-s} + R^{\alpha-s}).$$

We obtain the desired result as $\delta \to 0, R \to \infty$. $\square$

Proof of Theorem 6.8

Assume by contradiction that Theorem 6.8 does not hold. Then we can find a sequence of minimal surfaces ∂E_m with $a_m \to 0$ such that they satisfy the hypothesis of Lemma 6.10 but each ∂E_m cannot be included in a cylinder of flatness $2^{-\alpha} a_m$ in $B_{1/2}$. This contradicts the fact that there exists a subsequence $\partial E^*_{m_k}$ that converges uniformly on compact sets to a linear function passing through the origin.

$\square$

7. **The extension problem**

The purpose of this section is to extend to our surfaces ∂E the classical monotonicity formula for minimal surfaces:

If ∂E is a minimal surface then

$$J(r) = \frac{Area(\partial E \cap B_r)}{r^{n-1}}$$

is a monotone function of r. The function $J(r)$ is constant for cones and attains its minimum for a hyperplane, with a gap between the hyperplane and all other minimal cones.

The quantity that we will consider is somewhat related to the local energy of E in the ball of radius r, but we need to go to an extension in one extra variable to define it.

Let u be a function in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|^2)^{\frac{n+s}{2}}}\, dx < \infty.$$

We consider the extension $\tilde{u}$ of u,

$$\tilde{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}, \qquad \mathbb{R}^{n+1}_+ = \{(x,z), x \in \mathbb{R}^n, z \geq 0\}$$

which solves

$$\begin{cases} div(z^a \nabla \tilde{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on } \{z=0\} \end{cases} \tag{7.1}$$

with

$$a = 1 - s.$$

The function $\tilde{u}$ can be computed explicitly,

$$\tilde{u}(\cdot, z) = P(\cdot, z) * u, \quad P(x,z) = c_{n,a} \frac{z^{1-a}}{|x^2+z^2|^{\frac{n+1-a}{2}}}.$$

In [CSi1] it was shown that

$$\lim_{z\to 0} -z^a \tilde{u}_z(\cdot, z) = c'_{n,a}(-\Delta)^{s/2} u$$

in the sense of distributions. When $u \in C_0^\infty(\mathbb{R}^n)$ it can be checked directly that

$$\int_{\mathbb{R}^{n+1}_+} z^a|\nabla \tilde{u}|^2\,dxdz = \int_{\{z=0\}} (-z^a\tilde{u}_z)\tilde{u}\,dx$$
$$= \tilde{c}_{n,a}\int\int \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy = \tilde{c}_{n,a}\|u\|_{H^{s/2}(\mathbb{R}^n)}.$$

By approximation, this equality holds for all functions $u \in H^{s/2}$ that are compactly supported. As in Definition 2.1, we introduce the local contribution of the $H^{s/2}$ seminorm of u in B_1 i.e.

$$J_r(u) := \int\int \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\chi_{B_r}(x)(\chi_{B_r}(y) + 2\chi_{\mathbb{R}^n\setminus B_r}(y))\,dxdy.$$

Notice that if $u, v \in H^{s/2}$ and $u = v$ outside B_r then

$$\|u\|^2_{H^{s/2}} - \|v\|^2_{H^{s/2}} = J_r(u) - J_r(v).$$

If $u = \chi_E - \chi_{\mathcal{C}E}$ then

$$J_r(u) = 2\mathcal{J}_{B_r}(E).$$

Proposition 7.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^{n+1}$ and denote*

$$\Omega_0 := \Omega \cap \{z = 0\} \subset \mathbb{R}^n, \qquad \Omega_+ := \Omega \cap \{z > 0\}.$$

a) If $\Omega_0 \subset\subset B_1$ then

$$\int_{\Omega_+} z^a|\nabla\tilde{u}|^2 \le CJ_1(u) \tag{7.2}$$

with C depending on Ω.

b) If $B_1 \subset\subset \Omega_0$ and u is bounded in $\mathbb{R}^n$ then

$$J_1(u) \le C(1 + \int_{\Omega_+} z^a|\nabla\tilde{u}|^2).$$

with C depending on Ω, $\|u\|_{L^\infty}$.

Proof. a) Without loss of generality we can assume that $\int_{B_1} u = 0$. Then

$$\int \frac{u(x)^2}{(1+|x|^2)^{\frac{n+s}{2}}}dx \le \int\int \frac{(u(x)-u(y))^2}{(1+|x|^2)^{\frac{n+s}{2}}}\frac{\chi_{B_1}(y)}{|B_1|}dydx \le CJ_1(u),$$

and by Holder inequality

$$\int \frac{|u(x)|}{(1+|x|^2)^{\frac{n+s}{2}}}dx \le CJ_1(u)^{1/2}.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a cutoff function such that $\varphi = 1$ in Ω_0 and it is compactly supported in B_1. We write

$$u = u\varphi + u(1-\varphi) = u_1 + u_2,$$

and clearly $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$. Since u_1 is compactly supported we have

$$\int_{\mathbb{R}^{n+1}_+} z^a|\nabla\tilde{u}_1|^2 = \tilde{c}_{n,a}\|u_1\|_{H^{s/2}} = \tilde{c}_{n,a}J_1(u_1) \le CJ_1(u).$$

If $(x,z) \in \Omega_+$ then

$$z^a|\nabla \tilde{u}_2(x,z)| \le C \int \frac{|u_2(y)|}{(1+|y|^2)^{\frac{n+s}{2}}} dy \le C J_1(u)^{1/2},$$

hence

$$\int_{\Omega_+} z^a |\nabla \tilde{u}_2|^2 \le C J_1(u)$$

which proves a).

b) Since $s<1$ we have

$$\int\int \frac{(u(x)-u(y))^2}{|x-y|^{n+s}} \chi_{B_1}(x)\chi_{\mathcal{C}B_1}(y) dx dy \le C\|u\|^2_{L^\infty}.$$

Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a cutoff function supported in Ω such that $\varphi = 1$ in $B_1 \cap \{z=0\}$. Denote by $v(x) = \varphi(x,0)u(x)$. Then

$$\begin{aligned} 1 + \int_{\Omega_+} z^a |\nabla \tilde{u}|^2 &\ge c \int z^a |\nabla(\varphi \tilde{u})|^2 \\ &\ge c \int z^a |\nabla \tilde{v}|^2 \ge c J_1(v). \end{aligned}$$

We obtained the desired result since

$$J_1(\varphi u) \ge \int\int \frac{(u(x)-u(y))^2}{|x-y|^{n+s}} \chi_{B_1}(x)\chi_{B_1}(y) dx dy.$$

□

Remark 1: If $\bar{v}$ is a function defined in a bounded Lipschitz domain $\Omega \in \mathbb{R}^{n+1}_+$ with

$$\int_\Omega z^a |\nabla \bar{v}|^2 < \infty$$

then, from Holder's inequality,

$$\int_\Omega |\nabla \bar{v}| < \infty$$

hence we can define the trace of $\bar{v}$ on $\partial\Omega$. Clearly, the trace of $\tilde{u}$ on Ω_0 equals u.

Remark 2: Assume $\bar{v}$ is compactly supported in Ω and has trace v on Ω_0. Then

$$\int_{\Omega^+} z^a |\nabla \bar{v}|^2 \ge \int_{\mathbb{R}^{n+1}_+} z^a |\nabla \tilde{v}|^2.$$

To see this we denote by $\bar{v}_k$ the solution to equation (7.1) in B_k^+ which has trace v on $\{z=0\}$ and 0 on $\partial B_k \cap \{z>0\}$. Extend $\bar{v}_k$ to be 0 outside B_k^+, then for large k

$$\int_{\Omega^+} z^a |\nabla \bar{v}|^2 \ge \int z^a |\nabla \bar{v}_k|^2.$$

It can be checked that $\nabla \bar{v}_k$ converges to $\nabla \tilde{v}$ in $L^2(z^a dx dz)$ and we obtain the result as $k \to \infty$.

Lemma 7.2. *Assume u, v are such that $J_1(u)$, $J_1(v) < \infty$ and $v - u$ is compactly supported in B_1. Then*

$$\inf_{\Omega,\bar{v}} \int_{\Omega^+} z^a (|\nabla \bar{v}|^2 - |\nabla \tilde{u}|^2) = \tilde{c}_{n,a}(J_1(v) - J_1(u)) \tag{7.3}$$

where the infimum is taken among all bounded Lipschitz domains Ω with $\Omega_0 \subset B_1$ and among all functions $\bar{v}$ such that $\bar{v} - \tilde{u}$ is compactly supported in Ω and the trace of $\bar{v}$ on $\{z = 0\}$ equals v.

Proof. If u, $v \in C_0^\infty$ then the infimum equals

$$\int z^a |\nabla \tilde{v}|^2 - \int z^a |\nabla \tilde{u}|^2 = \tilde{c}_{n,a}(\|v\|_{H^{s/2}} - \|u\|_{H^{s/2}}) = \tilde{c}_{n,a}(J_1(v) - J_1(u)).$$

In the general case, let

$$\Omega^1 \subset \Omega^2 \subset \Omega^3 ..., \qquad \bigcup \Omega^k = \mathbb{R}^{n+1} \setminus \{(x,0) | x \in \mathcal{C}B_1\}.$$

In each set Ω^k_+ let $\bar{w}_k$ be the solution to the equation (7.1) which has trace $w := v - u$ on Ω^k_0 and 0 on $\partial\Omega^k \cap \{z > 0\}$. We extend $\bar{w}_k$ to be 0 outside Ω^k. If $\Omega \subset \Omega^k$ then

$$\int z^a (|\nabla \bar{v}|^2 - |\nabla \tilde{u}|^2) \geq \int z^a (|\nabla(\tilde{u} + \bar{w}_k)|^2 - |\nabla \tilde{u}|^2).$$

$$= \int z^a |\nabla \bar{w}_k|^2 + 2 \int z^a \nabla \tilde{u} \cdot \nabla \bar{w}_k.$$

The second term is independent of k since $\tilde{u}$ solves (7.1) and $\bar{w}_{k_1} - \bar{w}_{k_2}$ is compactly supported in $\mathbb{R}^{n+1}$ and has trace 0 on $\{z = 0\}$. As we let $k \to \infty$ we find that the infimum in (7.3) equals

$$\int z^a (|\nabla \tilde{w}|^2 + \nabla \tilde{u} \cdot \nabla \bar{w}_1) = \tilde{c}_{n,a} J_1(w) + 2 \int z^a \nabla \tilde{u} \cdot \nabla \bar{w}_1,$$

and we want to show it equals $\tilde{c}_{n,a}(J_1(u + w) - J_1(u))$.

We already proved this equality when $u, w \in C_0^\infty$ thus by approximation it holds for all u, w with $J_1(u)$, $J_1(w) < \infty$.

□

As a consequence we obtain the following proposition.

Proposition 7.3. *The set E is a minimizer for $\mathcal{J}$ in B_1 if and only if the extension $\tilde{u}$ of $u = \chi_E - \chi_{\mathcal{C}E}$ satisfies*

$$\int_{\Omega_+} z^a |\nabla \bar{v}|^2 \geq \int_{\Omega_+} z^a |\nabla \tilde{u}|^2$$

for all bounded Lipschitz domains Ω with $\Omega_0 \subset\subset B_1$ and all functions $\bar{v}$ that equal $\tilde{u}$ in a neighborhood of $\partial\Omega$ and take the values ± 1 on Ω_0.

8. **Monotonicity formula**

Assume E is a minimizer for $\mathcal{J}$ in B_R. For all $r < R$ we define the functional

$$\Phi_E(r) := \frac{\int_{B_r} z^a |\nabla \tilde{u}|^2}{r^{n+a-1}}$$

where

$$u = \chi_E - \chi_{\mathcal{C}E}.$$

The functional Φ is scale invariant in the sense that the rescaled set $\lambda E = \{\lambda x, x \in E\}$ satisfies

$$\Phi_{\lambda E}(\lambda r) = \Phi_E(r).$$

From (7.2) we see that there exists a constant $C_{n,a}$ depending only on n and a such that

$$\Phi_E(r) \leq C_{n,a}$$

for all $r \leq R/2$. Moreover, if $0 \in \partial E$, the density estimates imply that there exists a small $c_{n,a} > 0$ such that

$$\Phi_E(r) \geq c_{n,a}.$$

Theorem 8.1. Monotonicity formula
The function $\Phi_E(r)$ is increasing in r.

Proof. We show that $\frac{d}{dr}\Phi_E(r) \geq 0$. Due to the scale invariance, it suffices to prove the inequality only for $r = 1$, that is

$$\int_{\partial B_1^+} z^a |\nabla \tilde{u}|^2 d\sigma \geq (n + a - 1) \int_{B_1^+} z^a |\nabla \tilde{u}|^2.$$

Consider the function

$$\bar{v}(x,z) := \begin{cases} \tilde{u}((1+\varepsilon)(x,z)) & |(x,z)| \leq 1/(1+\varepsilon), \\ \tilde{u}((x,z)/|(x,z)|) & 1/(1+\varepsilon) < |(x,z)| \leq 1 \end{cases}$$

and $\bar{v} = \tilde{u}$ outside B_1. The trace v of $\bar{v}$ on $\{x_{n+1} = 0\}$ is of the form $\chi_F - \chi_{\mathcal{C}F}$ for a set F which coincides with E outside B_1. The minimality of E implies

$$\int_{B_1^+} z^a |\nabla \bar{v}|^2 \geq \int_{B_1^+} z^a |\nabla \tilde{u}|^2$$

$$(1+\varepsilon)^{-n+1-a} \int_{B_1^+} z^a |\nabla \tilde{u}|^2 + \int_{B_1^+ \setminus B_{1/(1+\varepsilon)}} \frac{z^a}{|(x,z)|^2} |\nabla_\tau \tilde{u}|^2 \geq \int_{B_1^+} z^a |\nabla \tilde{u}|^2.$$

We let $\varepsilon \to 0$ and obtain

$$\int_{\partial B_1^+} z^a |\nabla_\tau \tilde{u}|^2 d\sigma \geq (n + a - 1) \int_{B_1^+} z^a |\nabla \tilde{u}|^2,$$

where ∇_τ represents the tangential component of the gradient. Hence

$$\int_{\partial B_1^+} z^a |\nabla \tilde{u}|^2 d\sigma \geq (n + a - 1) \int_{B_1^+} z^a |\nabla \tilde{u}|^2 + \int_{\partial B_1^+} z^a |\tilde{u}_\nu|^2 d\sigma. \tag{8.1}$$

□

From (8.1) we see that $\frac{d}{dr}\Phi_E(r) = 0$ only if $\tilde{u}_\nu = 0$ on ∂B_r^+. We obtain the following corollary.

Corollary 8.2. *The function $\Phi_E(r)$ is constant if and only if $\tilde{u}$ is homogenous of degree 0.*

9. **Minimal cones**

Proposition 9.1. *Assume E_k are minimizers for $\mathcal{J}$ in B_k and*

$$E_k \to E \quad in\ L^1_{loc}.$$

Then the corresponding extensions $\tilde{u}_k$, respectively $\tilde{u}$ satisfy

$$\tilde{u}_k \to \tilde{u} \quad \text{uniformly on compact sets of } \mathbb{R}^{n+1}_+,$$

$$\nabla\tilde{u}_k \to \nabla\tilde{u} \quad in\ L^2_{loc}(z^a\ dxdz).$$

In particular $\Phi_{E_k}(r) \to \Phi_E(r)$.

Proof. The functions $\tilde{u}_k$ are uniformly Lipschitz continuous on each compact set of $\{z > 0\}$. Consider a subsequence $\tilde{u}_{k_i}$ that converges uniformly on compact sets to a function $\tilde{v}$. We will show that $\tilde{v} = \tilde{u}$. Since both $\tilde{u}$, $\tilde{v}$ are bounded and satisfy the equation (7.1) it suffices to prove that their traces on $\{z = 0\}$ are equal.

Clearly

$$\int_{B_r^+} z^a|\nabla\tilde{v}|^2 \le \liminf \int_{B_r^+} z^a|\nabla\tilde{u}_{k_i}|^2 \le r^{n+a-1}C_{n,a}.$$

Using Holder inequality we obtain

$$\int_{B_r\cap\{0<z<\delta\}} |\nabla(\tilde{u}_{k_i} - \tilde{v})| \le C(r)\delta^{\frac{1-a}{2}} \left(\int_{B_r^+} z^a|\nabla(\tilde{u}_{k_i} - \tilde{v})|^2\right)^{1/2}$$

$$\le C'(r)\delta^{\frac{1-a}{2}}.$$

Since $\nabla\tilde{u}_{k_i}$ converges uniformly on compact sets to ∇v we find $\tilde{u}_{k_i} \to \tilde{v}$ in $W^{1,1}(B_r^+)$ which implies the convergence of the traces $u_{k_i} \to v$ in L^1. Thus $v = u$ and the first part of the theorem is proved.

For the second part we use (7.2) and find

$$\limsup \int_{B_1^+} z^a|\nabla(\tilde{u}_k - \tilde{u})|^2 \le C \limsup J_2(u_k - u).$$

We will prove that the right hand side equals 0. Since $u_k \to u$ in L^1_{loc}, any sequence of the u_k contains a subsequence u_{k_i} that converges pointwise to u.

Define

$$f_k(x,y) = \frac{u_k(x) - u_k(y)}{|x-y|^{\frac{n+s}{2}}}\chi_{B_2}(x)(\chi_{B_2}(y) + \sqrt{2}\chi_{\mathbb{R}^n\setminus B_2}(y)),$$

and notice that

$$\|f_k\|_{L^2} = J_2(u_k).$$

According to Theorem 3.3

$$\lim J_2(u_k) = J_2(u).$$

Now we use the following standard lemma.

If $f_k \in L^2$ converge pointwise to f and $\|f_k\|_{L^2} \to \|f\|_{L^2}$ then $f_k \to f$ in L^2.

The lemma implies that any sequence of the u_k's contains a subsequence such that $J_2(u_{k_i} - u) \to 0$ thus

$$J_2(u_k - u) \to 0 \quad \text{as } k \to \infty$$

which concludes the proof.

We finish with a short proof of the lemma above. Indeed, from the pointwise convergence we find that f_k converges weakly to f in L^2 hence

$$\int |f_k - f|^2 = \int f_k^2 + \int f^2 - 2\int f_k f \to 0.$$

□

Theorem 9.2. Blow-up limit

Assume E is minimal in B_1 and $0 \in \partial E$. Let $\lambda_k \to \infty$ be a sequence such that

$$\lambda_k E \to C \quad \text{in } L^1_{loc}. \tag{9.1}$$

Then C is a minimal cone, i.e $tC = C$ for all $t > 0$.

Proof. The fact that C is minimal is proved in Theorem 3.3.

From Proposition 9.1 $\Phi_{\lambda_k E}(r) = \Phi_E(r/\lambda_k)$ converges to $\Phi_C(r)$, thus

$$\Phi_C(r) = \lim_{s \to 0} \Phi_E(s).$$

Since Φ_C is constant we conclude that the extension $\tilde{u}_C$ (and its trace) are homogenous of degree 0.

□

Definition 9.3. We say that a cone C as in Theorem 9.2 is a tangent cone for E at the origin.

Corollary 4.4 implies the following: for any $\varepsilon > 0$ all but a finite number of the sets $\lambda_k \partial E \cap B_1$ lie in a ε neighborhood of ∂C. As a consequence of the improvement of flatness Theorem 6.1 we obtain the following result.

Theorem 9.4. *If C is a half-space then ∂E is a $C^{1,\alpha}$ surface in a neighborhood of the origin.*

Definition 9.5. A point $x_0 \in \partial E \cap \Omega$ that has a half-space as a tangent cone is called a regular point. The points in $\partial E \cap \Omega$ which are not regular are called singular points.

For a minimal cone C we denote by Φ_C its "energy" i.e. the value of the constant function $\Phi_C(r)$. Let $\Pi := \{x_1 > 0\}$ be a half-space.

Theorem 9.6. Energy gap

Let C be a minimal cone. Then

$$\Phi_C \geq \Phi_\Pi. \tag{9.2}$$

Moreover, if C is not a half-space then

$$\Phi_C \geq \Phi_\Pi + \delta_0$$

where δ_0 is a constant depending only on n, s.

Proof. Consider a small ball included in C which is tangent to ∂C at a point x_0. Clearly, ∂C is $C^{1,\alpha}$ in a neighborhood of x_0 hence the tangent cone of C at x_0 is a half-space which implies

$$\lim_{r \to 0} \Phi_{C-x_0}(r) = \Phi_\Pi.$$

On the other hand, since $\frac{1}{k}(C - x_0) = C - \frac{1}{k}x_0$ we obtain

$$\frac{1}{k}(C - x_0) \to C \quad \text{in } L^1_{loc}$$

hence

$$\Phi_{C-x_0}(k) \to \Phi_C \quad \text{as } k \to \infty.$$

The monotonicity of Φ_{C-x_0} gives (9.2). We have equality only when $C - x_0$ is a cone, thus $C - x_0$ is a half-space which in turn implies C is a half-space.

The second part of the proof is by compactness. Assume by contradiction that there exist minimal cones C_k with $\Phi_{C_k} \le \Phi_\Pi + 1/k$ that are not half-spaces. Then, we can find a convergent subsequence C_{k_i} in L^1_{loc} to C_0. Then $\Phi_{C_0} = \Phi_\Pi$ hence C_0 is a half-space. Once again from Corollary 4.4, the sets $\partial C_{k_i} \cap B_1$ lie in any neighborhood of a hyperplane for all large k_i. From Theorem 6.1 we obtain that ∂C_{k_i} are $C^{1,\alpha}$ surfaces around 0, thus C_{k_i} are half-spaces for all large k_i and we reached a contradiction.

□

10. **Dimension reduction**

Finally, in this section we briefly discuss how the classical dimension reduction argument from Federer [F] applies to our case. Since our starting point is that in two dimensions minimal cones consist of a finite number of rays, we prove that the singular set has $\mathcal{H}^{n-2}$ Hausdorff dimension in $\mathbb{R}^n$.

Theorem 10.1. *The set E is a local minimizer for J in $\mathbb{R}^n$ if and only if $E \times \mathbb{R}$ is a local minimizer for J in $\mathbb{R}^{n+1}$.*

Proof. Let $\tilde{u}(x, z)$ be the extension in $\mathbb{R}^{n+1}$ for $\chi_E - \chi_{CE}$. Clearly by making $\tilde{u}$ to be constant in the x_{n+1} variable we obtain the extension in $\mathbb{R}^{n+2}$ corresponding to $E \times \mathbb{R}$.

($\Rightarrow$) Assume E is a local minimizer.

Let $\bar{v}(x, x_{n+1}, z)$ be such that the set where $\bar{v} \neq \tilde{u}$ is compactly supported in a cube Q in $\mathbb{R}^{n+2}$, and the trace of $\bar{v}$ on $\{z = 0\}$ takes only the values ± 1.

We have

$$\int_Q z^a |\nabla \bar{v}|^2 \ge \int \left(\int_{Q_t} z^a |\nabla_{x,z} \bar{v}|^2 dx dz \right) dt,$$

where $Q_t = Q \cap \{x_{n+1} = t\}$. From the minimality of E we find that for a.e t

$$\int_{Q_t} z^a |\nabla_{x,z} \bar{v}|^2 dx dz \ge \int_{Q_t} z^a |\nabla \tilde{u}|^2 dx dz$$

which implies

$$\int_Q z^a|\nabla \bar v|^2 \geq \int_Q z^a|\nabla \tilde u|^2.$$

($\Leftarrow$) Assume $E\times\mathbb{R}$ is a local minimizer.

Let $\bar v(x,z)$ be such that the set where $\bar v \neq \tilde u$ is compactly supported in $B_R \subset \mathbb{R}^{n+1}$, and the trace of $\bar v$ on $\{z=0\}$ takes only the values ± 1. We need to show that

$$\int_{B_R^+} z^a|\nabla \bar v|^2 \geq \int_{B_R^+} z^a|\nabla \tilde u|^2. \tag{10.1}$$

We can assume the first integral is finite otherwise there is nothing to prove. Notice that local minimality of $E\times\mathbb{R}$ gives

$$\int_{-1}^1 \left(\int_{B_R^+} z^a|\nabla \tilde u|^2\right) dx_{n+1} < \infty$$

thus the integral of $\tilde u$ in (10.1) is also finite.

We consider the function $\bar v_*(x,x_{n+1},z)$ defined in $D := B_R^+\times[-(a+1),a+1]$

$$\bar v_* = \begin{cases} \bar v(x,z), & \text{if } |x_{n+1}| \leq a-1 \\ \bar v(x,z) + \bar w_*(x,|x_{n+1}|-a,z), & \text{if } -1<|x_{n+1}|-a\leq 1 \end{cases}$$

where $\bar w_*$ is chosen such that $\bar v_* = \tilde u$ in a neighborhood of $\partial D\cap\{z>0\}$, the trace of $\bar w_*$ on $\{z=0\}$ takes only the values ±1 and

$$\int_{B_R^+\times[-1,1]} z^a|\nabla \bar w_*|^2 < \infty.$$

The existence of such a function is given in Lemma 10.2 below by taking $\bar w = \tilde u - \bar v$.

The minimality of $E\times\mathbb{R}$ implies

$$\int_D z^a|\nabla \bar v_*|^2 \geq \int_D z^a|\nabla \tilde u|^2$$

hence

$$2(a-1)\int_{B_R^+} z^a|\nabla \bar v|^2 + 2\int_{B_R^+\times[-1,1]} z^a|\nabla \bar w_*|^2 \geq 2(a+1)\int_{B_R^+} z^a|\nabla \tilde u|^2.$$

We obtain the result by letting $a\to\infty$. □

Lemma 10.2. *Assume $\bar w(x,z)$ is a bounded function in $B_1^+\subset\mathbb{R}^{n+1}$, $\bar w = 0$ in a neighborhood of ∂B_1^+ and*

$$\int_{B_1^+} z^a|\nabla \bar w| < \infty.$$

There exists a function $\bar w_*(x,x_{n+1},z)$ defined in $B_1^+\times[-1,1]$ such that

$$\bar w_* = 0 \quad \textit{if } x_{n+1} < -1/2, \qquad \bar w_* = \bar w \quad \textit{if } x_{n+1} > 1/2,$$

$$\bar w_* = 0 \quad \textit{near } \partial B_1^+\times[-1,1] \tag{10.2}$$

and

$$\int z^a |\nabla \bar{w}_*|^2 < \infty, \qquad w_* = \begin{cases} 0 & \text{if } x_{n+1} < 0, \\ w & \text{if } x_{n+1} > 0. \end{cases} \tag{10.3}$$

Proof. First we assume that $0 \le \bar{w} \le 1$ and we think $\bar{w}$ is defined in $\mathbb{R}^{n+2}$ and it is constant in the x_{n+1} variable. Let π be the extension in $\mathbb{R}^{n+2}$ corresponding to $\chi_{\{x_{n+1}>0\}}$. The function

$$\bar{w}_1 := \min\{w, \pi\}$$

satisfies (10.2), (10.3). Now we modify $\bar{w}_1$ so that the other condition also holds.

For this let ϕ_1 be a smooth cutoff function on $\mathbb{R}$ with $\phi_1 = 0$ outside $[-1/2, 1/2]$ and $\phi_1 = 1$ on $[-1/4, 1/4]$. Define $\phi_2 = 1 - \phi_1$ on $[0, \infty)$ and $\phi_2 = 0$ on $(-\infty, 0)$. Then

$$\bar{w}_* := \phi_1(x_{n+1})\bar{w}_1 + \phi_2(x_{n+1})\bar{w}$$

has all the required properties.

The general case follows by applying the construction above to $\bar{w}^+$ and $\bar{w}^-$ and then subtracting the functions. $\square$

Theorem 10.3. Dimension reduction

Let C be a minimal cone in $\mathbb{R}^n$ and $x_0 = e_n \in \partial C$. Any sequence converging to ∞ has a subsequence $\lambda_k \to \infty$ such that

$$\lambda_k(C - x_0) \to A \times \mathbb{R} \quad \textit{in } L^1_{loc}$$

where A is a minimal cone in $\mathbb{R}^{n-1}$.

Moreover, if x_0 is a singular point for ∂C then 0 is a singular point for ∂A.

Proof. In view of Theorems 9.2 and 10.1, the only thing that remains to be proved is that the limiting set D is constant in the x_n direction.

Let x be an interior point of D, i.e. $B_\varepsilon(x) \subset D$. Then by uniform density,

$$B_{\varepsilon/2}(x) \subset C_k := \lambda_k(C - x_0) \quad \text{for all large } k.$$

Since the cones generated by $-\lambda_k x_0$ and $B_{\varepsilon/2}(x)$ are in C_k and converge in L^1_{loc} to the set

$$\bigcup_{t\in\mathbb{R}} \{B_{\varepsilon/2}(x) + te_n\},$$

we conclude that this set is included in D. Hence the line $x + te_n$ is included in the interior of D and the theorem is proved. $\square$

Theorem 10.4. Dimension of the singular set

The singular set $\Sigma_E \subset \partial E \cap \Omega$ has Hausdorff dimension $n-2$, i.e.

$$\mathcal{H}^s(\Sigma_E) = 0 \quad \textit{for } s > n-2.$$

Proof. The proof is the same as the one for the classical minimal surfaces. We present a sketch of the proof for completeness.

Step 1: Assume $\mathcal{H}^s(\Sigma_C) = 0$ for all minimal cones C. Then $\mathcal{H}^s(\Sigma_E) = 0$.

First we notice that Σ_E satisfies the following property: for every $x \in \Sigma_E$ there exists $\delta(x) > 0$ such that for any $\delta \leq \delta(x)$ and any set $D \subset \Sigma_E \cap B_\delta(x)$ there exists a cover of D with balls $B_{r_i}(x_i)$ with $x_i \in D$ and

$$\sum r_i^s \leq \frac{1}{2}\delta^s.$$

This follows from compactness and the fact that the statement is true for minimal cones in B_1 since we assumed $\mathcal{H}^s(\Sigma_C) = 0$.

Next we show that $\mathcal{H}^s(D_k) = 0$ where

$$D_k := \{x \in \Sigma_E | \quad \delta(x) \geq 1/k\}.$$

We cover D_k with a countable family of balls of radius $\delta = 1/k$ and centers in D_k. In each such ball B_δ we cover $D_k \cap B_\delta$ with balls of smaller radius that satisfy the property above. For each smaller ball we apply again the property above, and so on. After m steps we find that we can cover $D_k \cap B_\delta$ with balls of radii $B_{r_i}(x_i)$, $x_i \in D_k$ so that

$$\sum r_i^s \leq \frac{1}{2^m}\delta^s.$$

In conclusion $\mathcal{H}^s(D_k \cap B_\delta) = 0$, or $\mathcal{H}^s(D_k) = 0$, thus $\mathcal{H}^s(\Sigma_E) = \mathcal{H}^s(\cup D_k) = 0$.

Step 2: If $\mathcal{H}^s(\Sigma_C) = 0$ for all minimal cones $C \subset \mathbb{R}^n$, then $\mathcal{H}^{s+1}(\Sigma_{\tilde{C}}) = 0$ for all minimal cones $\tilde{C} \subset \mathbb{R}^{n+1}$.

It suffices to show that $\mathcal{H}^s(\Sigma_{\tilde{C}} \cap \partial B_1) = 0$. Using the induction hypothesis and Theorem 10.3 one can deduce by compactness that, when restricted to ∂B_1, $\Sigma_{\tilde{C}}$ satisfies the same property as Σ_E above. From here we obtain again the desired conclusion as in Step 1.

Now the result follows since the singular set of a cone in $\mathbb{R}^2$ can be only the origin. Otherwise, from Theorem 10.3 we will find a singular cone at the origin in $\mathbb{R}^1$ and reach a contradiction.

□

As a consequence of the theorem above and the fact that ∂E is a $C^{1,\alpha}$ surface in a neighborhood of a regular point we obtain the following corollary.

Corollary 10.5. *Let E be a minimizer for $\mathcal{J}$ in Ω. Then ∂E has Hausdorff dimension $n-1$, i.e.*

$$\mathcal{H}^s(\partial E \cap \Omega) = 0 \quad \textit{for } s > n-1.$$

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