Γ-CONVERGENCE FOR NONLOCAL PHASE TRANSITIONS

OVIDIU SAVIN AND ENRICO VALDINOCI

Abstract. We discuss the Γ-convergence, under the appropriate scaling, of the energy functional

$$\|u\|^2_{H^s(\Omega)} + \int_\Omega W(u)\,dx,$$

with $s \in (0,1)$, where $\|u\|_{H^s(\Omega)}$ denotes the total contribution from Ω in the H^s norm of u, and W is a double-well potential.

When $s \in [1/2, 1)$, we show that the energy Γ-converges to the classical minimal surface functional – while, when $s \in (0, 1/2)$, it is easy to see that the functional Γ-converges to the nonlocal minimal surface functional.

1. Introduction and statement of the main results

As well-known, the Γ-convergence, introduced in [11, 12], is a notion of convergence for functionals, which tends to be as compatible as possible with the minimizing features of the energy, and whose limit is capable to capture essential features of the problem. We refer to [10, 7] for a detailed presentation of several basic aspects and applications of Γ-convergence; see also [23] for applications to homogenization theory.

Making it possible to study the asymptotics of variational problems indexed by a parameter, the Γ-convergence has become a standard tool in dealing with singularly perturbed energies as the ones arising in the theory of phase transitions (see [19]), where the dislocation energy of a double well potential W is compensated by a small gradient term which avoids the formation of unnecessary interfaces, leading to a total energy which is usually written as

$$\int \varepsilon^2 |\nabla u|^2 + W(u)\,dx, \tag{1.1}$$

with $\varepsilon \to 0^+$.

The purpose of this paper is to develop a Γ-convergence theory for a nonlocal analogue of the energy above, in which the gradient term in (1.1) is replaced by a fractional, Gagliardo-type, norm of the form $\varepsilon^{2s}\|u\|^2_{H^s}$, with $s \in (0,1)$ (see below for precise definitions and statements). Notice that, formally, the gradient term in (1.1) corresponds to the case $s = 1$.

The study of such a nonlocal contribution is quite important for the applications, since the classical gradient term takes into account the interactions at small scales between the particles of the medium, but loses completely the long scale interactions. In this spirit, it is relevant to know whether or not the Γ-limit of the

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functional is local – that is, whether or not the long range interactions affect the limit interface.

From the point of view of the pure mathematics, nonlocal problems are also relevant because new techniques are usually needed to understand and estimate the contributions coming from far. We refer, in particular, to [8] for the definition and the basic features of nonlocal minimal surfaces, which are the natural analogue of the classical sets of minimal perimeter (as in [17]). In fact, we will show that the Γ-limit of our functional will be the standard minimal surface functional when $s \in [1/2, 1)$ and the nonlocal one when $s \in (0, 1/2)$.

Now, we introduce the formal setting in which we work. We consider a bounded domain Ω, with complement $\mathscr{C}\Omega$. We define

$$X := \big\{ u \in L^\infty(\mathbb{R}^n) \, : \, \|u\|_{L^\infty(\mathbb{R}^n)} \leqslant 1 \big\},$$

the space of admissible functions u. We say that a sequence $u_n \in X$ converges to u in X if u_n converges to u in $L^1_{\rm loc}(\mathbb{R}^n)$.

We define

$$\mathscr{K}(u,\Omega) := \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega\int_{\mathscr{C}\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

the Ω contribution in the H^s norm of u

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

i.e. we omit the set where $(x,y) \in \mathscr{C}\Omega \times \mathscr{C}\Omega$ since all $u \in X$ are fixed outside Ω.

The energy functional J_ε in Ω is defined as

$$J_\varepsilon(u,\Omega) := \varepsilon^{2s}\mathscr{K}(u,\Omega) + \int_\Omega W(u)\,dx.$$

Such functional may be seen as the nonlocal analogue of the classical one in (1.1).

Throughout the paper we assume that $W : [-1, 1] \to [0, \infty)$,

$$\begin{gathered} W \in C^2([-1,1]), \quad W(\pm 1) = 0, \quad W > 0 \quad \text{in } (-1,1), \\ W'(\pm 1) = 0 \quad \text{and} \quad W''(\pm 1) > 0. \end{gathered} \tag{1.2}$$

We remark that, differently from several nonlocal models considered in the literature (see e.g. [3, 5, 15] and references therein), we deal with an arbitrarily large number of space dimensions, no periodicity in space is assumed, and we consider the full interaction among all the space Ω versus $\mathbb{R}^n$ (i.e., from the physical point of view, the particles in the domain Ω interact with the ones in the whole of the space $\mathbb{R}^n$, not only with the ones in Ω).

Since Γ-convergence is expecially designed for minimizers, we recall the following notation:

Definition 1.1. We say that u is a minimizer for J_ε in an open, possibly unbounded, set $\Omega \subset \mathbb{R}^n$ if, for any open subset U compactly included in Ω, we have that

$$J_\varepsilon(u,U) < \infty,$$

and

$$J_\varepsilon(u,U) \leqslant J_\varepsilon(v,U)$$

for any v which coincides with u in $\mathscr{C}U$.

It is worth to notice that if u minimizes J_ε in Ω then it minimizes J_ε in any subdomain $\Omega' \subset \Omega$.

We deal with the functional $\mathscr{F}_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\mathscr{F}_\varepsilon(u,\Omega) := \begin{cases} \varepsilon^{-2s} J_\varepsilon(u,\Omega) & \text{if } s \in (0,\, 1/2), \\ |\varepsilon \log \varepsilon|^{-1} J_\varepsilon(u,\Omega) & \text{if } s = 1/2, \\ \varepsilon^{-1} J_\varepsilon(u,\Omega) & \text{if } s \in (1/2,\, 1). \end{cases}$$

The functional $\mathscr{F}_\varepsilon$ may be seen as the "right" scaling of J_ε, that is the one that possesses a Γ-limit.

In the case when $s \in (0, 1/2)$, the limiting functional $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\mathscr{F}(u,\Omega) := \begin{cases} \mathscr{K}(u,\Omega) & \text{if } u|_\Omega = \chi_E - \chi_{\mathscr{C}E}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

In this case, $\mathscr{F}$ agrees with the nonlocal area functional of ∂E in Ω that was studied in [8, 9, 6]. Remarkably, such nonlocal area functional is well defined exactly when $s \in (0,\, 1/2)$.

In the case when $s \in [1/2, 1)$ the limiting functional $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\mathscr{F}(u,\Omega) := \begin{cases} c_\star \text{Per}(E,\Omega) & \text{if } u|_\Omega = \chi_E - \chi_{\mathscr{C}E}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where $c_\star$ is a constant depending on n, s and W, which will be explicitly determined in the sequel, in dependence of a suitable 1D minimal profile (see Theorem 4.2 and (4.35) for details).

Here above and in the rest of the paper, we use the standard notation $\text{Per}(E, U)$ to denote the perimeter of a set E in an open set $U \subseteq \mathbb{R}^n$ (see, e.g., [17]).

Then, the results we prove here are the following:

Theorem 1.2. *Let $s \in (0,\, 1)$. Then, $\mathscr{F}_\varepsilon$ Γ-converges to $\mathscr{F}$, i.e.,*

(i) *for any u_ε converging to u in X,*

$$\mathscr{F}(u,\Omega) \leqslant \liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon,\Omega),$$

(ii) *if Ω is a Lipschitz domain, for any $u \in X$ there exists u_ε converging to u in X such that*

$$\mathscr{F}(u,\Omega) \geqslant \limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon,\Omega).$$

Theorem 1.3. *If $\mathscr{F}_\varepsilon(u_\varepsilon, \Omega)$ is uniformly bounded for a sequence of $\varepsilon \to 0^+$, then there exists a convergent subsequence*

$$u_\varepsilon \to u_* := \chi_E - \chi_{\mathscr{C}E} \quad \text{in } L^1(\Omega). \tag{1.5}$$

Moreover, let u_ε minimize $\mathscr{F}_\varepsilon$ in Ω:

(i) *if $s \in (0, 1/2)$ and u_ε converges weakly to u_o in $\mathscr{C}\Omega$, then u_* minimizes $\mathscr{F}$ in (1.3) among all the functions that agree with u_o in $\mathscr{C}\Omega$;*

(ii) *if $s \in [1/2, 1)$, then u_* minimizes $\mathscr{F}$ in (1.4). Also, for any open set $U \subset\subset \Omega$ we have*

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, U) \leqslant c_* \text{Per}\,(E, \overline{U}).$$

We recall that there are several results available in the literature concerning the approximation of the perimeter with nonlocal functionals. As far as we understand, all these results are related to our Theorems 1.2 and 1.3 (as well as to each other), but their statements are quite different from ours and the proofs are based on different techniques. In particular, we recall [4], which considered a $H^{1/2}$ norm inside a one-dimensional domain with no contribution coming from the outside. As remarked to us by [1], the extension of the results in [4] to higher dimension is implicitly contained in [5], though not explicitly mentioned. Moreover, in [14, 15] the Γ-convergence of a functional driven by a norm of type $H^{1/2}$ and a more comlicated potential on a two-dimensional square or torus, under a suitable pinning condition, was studied in detail.

Also, in [3, 2], the Γ-convergence of an interaction energy with a double integral weighted by a summable kernel is considered.

From the results in Theorem 1.2 and 1.3, it is also possible to have optimal estimates on the width of the asymptotic interface of minimizers. Indeed, in [22] we proved the following energy bound and uniform density estimate for minimizers of $\mathscr{F}_\varepsilon$.

Theorem 1.4. *If u_ε minimizes $\mathscr{F}_\varepsilon$ in $B_{1+2\varepsilon}$ then*

$$\mathscr{F}_\varepsilon(u, B_1) \leqslant \overline{C},$$

with $\overline{C}$ depending on n, s, W.

Theorem 1.5. *If u_ε minimizes $\mathscr{F}_\varepsilon$ in B_r and $u(0) > \theta_1$ then*

$$|\{u_\varepsilon > \theta_2\} \cap B_r| \geqslant \bar{c}\, r^n$$

provided that $\varepsilon \leqslant c(\theta_1, \theta_2) r$, where $\bar{c} > 0$ depends only on n, s, W and $c(\theta_1, \theta_2) > 0$ depends also on $\theta_1, \theta_2 \in (-1, 1)$.

As a consequence of these theorems we obtained in [22] that the convergence in (1.5) is better when dealing with minimizers. More precisely, we showed that the level sets of minimizers u_ε of $\mathscr{F}_\varepsilon$ converge locally uniformly to ∂E.

For the proof of Theorems 1.4 and 1.5, see [22]. We also refer to [5, 16, 18], where other types of nonlocal models have been considered (in particular, a three-dimensional fluid with boundary and weight inhomogeneity of distance type, whose energy bounds the Gagliardo norm, see Theorem 19 in [18]).

The proof of Theorems 1.2 and 1.3 when $s \in (0, 1/2)$ is elementary and it is contained in Section 2. In Section 3 we prove the compactness needed in Theorem 1.3 in the case $s \geqslant 1/2$. In Section 4 we prove Theorem 1.2 and Theorem 1.3 (ii) when $s \in [1/2, 1)$ by interpolating the functions candidate to the minimization. For this, a careful analysis on the energy contribution across the gluing of the interpolation is needed, as well as some measure theoretic result of [22].

Several arguments in the sequel will be based on some preliminary considerations, whose detailed proofs can be found in [20].

Finally, we conclude the introduction with a notation that will be used throughout the paper. For simplicity we denote

$$u(E, F) := \int_E \int_F \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}}\, dxdy. \tag{1.6}$$

Clearly, $u(E,F)=u(F,E)$, and if E_1 and E_2 are disjoint, then

$$u(E_1\cup E_2,F)=u(E_1,F)+u(E_2,F).$$

Using this notation, the Ω contribution in the H^s norm of u can be written as

$$\mathscr{K}(u,\Omega)=\frac{1}{2}u(\Omega,\Omega)+u(\Omega,\mathscr{C}\Omega).$$

2. Proof of Theorems 1.2 and 1.3 when $s\in(0,\,1/2)$

Throughout this section we assume $s\in(0,\,1/2)$.

Proof of Theorem 1.2. Recalling (1.3), we observe that

$$\text{if } u\Big|_{\Omega}=\chi_E-\chi_{\mathscr{C}E}, \text{ then } \mathscr{F}_\varepsilon(u,\Omega)=\mathscr{F}(u,\Omega)=\mathscr{K}(u,\Omega). \tag{2.1}$$

Now, we prove (i). For this, let u_ε converging to u in X. If

$$\liminf_{\varepsilon\to0^+}\mathscr{F}_\varepsilon(u_\varepsilon,\Omega)=+\infty,$$

then (i) is obvious, so we may suppose that

$$\liminf_{\varepsilon\to0^+}\mathscr{F}_\varepsilon(u_\varepsilon,\Omega)=\ell<+\infty.$$

We take a subsequence, say u_{ε_k} attaining the above limit.

Then, we take a further subsequence, say $u_{\varepsilon_{k_j}}$, that converges to u almost everywhere. Therefore,

$$\ell=\lim_{k\to+\infty}\mathscr{F}_{\varepsilon_k}(u_{\varepsilon_k},\Omega)=\lim_{j\to+\infty}\mathscr{F}_{\varepsilon_{k_j}}(u_{\varepsilon_{k_j}},\Omega)\geqslant\lim_{j\to+\infty}\frac{1}{\varepsilon_{k_j}^{2s}}\int_\Omega W(u_{\varepsilon_{k_j}}(x))\,dx.$$

Consequently,

$$\int_\Omega W(u(x))\,dx=\lim_{j\to+\infty}\int_\Omega W(u_{\varepsilon_{k_j}}(x))\,dx=0.$$

This implies that $u(x)\in\{-1,+1\}$ for almost any $x\in\Omega$, that is, $u\Big|_\Omega=\chi_E-\chi_{\mathscr{C}E}$ for a suitable set E. And so, by Fatou Lemma and (2.1), we conclude that

$$\liminf_{\varepsilon\to0^+}\mathscr{F}_\varepsilon(u_\varepsilon,\Omega)\geqslant\liminf_{\varepsilon\to0^+}\mathscr{K}(u_\varepsilon,\Omega)\geqslant\mathscr{K}(u,\Omega)=\mathscr{F}(u,\Omega),$$

proving (i). Now, we prove (ii).

For this, we may suppose that $u\Big|_\Omega=\chi_E-\chi_{\mathscr{C}E}$ for a suitable set E, otherwise (ii) is obvious. Then, we choose $u_\varepsilon:=u$ and we use (2.1) to see that $\mathscr{F}_\varepsilon(u_\varepsilon,\Omega)=\mathscr{F}(u,\Omega)$, which obviously implies (ii). This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Since $s\in(0,\,1/2)$, the uniform bound on $\mathscr{F}_\varepsilon$ gives a uniform bound of the Gagliardo norm $\mathscr{K}(u_\varepsilon,\Omega)$, and the compactness claim in (1.5) is quite standard, see for example Lemma **??** in [20]. It remains to prove (i).

As a result of Definition 1.1, it suffices to consider the case when Ω is bounded and smooth. In this case, one has that

$$\int_{\mathscr{C}\Omega}\int_\Omega\frac{2}{|x-y|^{n+2s}}dxdy<\infty, \tag{2.2}$$

see, for instance, Lemma ??? in [20].

Let $v \in X$ be an arbitrary function with

$$v\Big|_{\Omega} = \chi_F - \chi_{\mathscr{C}F}$$

for some set F, and $v = u_o$ in $\mathscr{C}\Omega$. For any $y \in \mathscr{C}\Omega$, let

$$\psi(y) := \int_{\Omega} \frac{v(x)}{|x-y|^{n+2s}}\, dx \quad \text{ and } \quad \Psi(y) := \int_{\Omega} \frac{u_*(x)}{|x-y|^{n+2s}}\, dx\,,$$

where u_* is as in (1.5). We remark that $\psi(y)$ and $\Psi(y)$ are in $L^1(\mathscr{C}\Omega)$, since

$$\int_{\mathscr{C}\Omega} |\psi(y)| + |\Psi(y)|\, dy \leqslant \int_{\mathscr{C}\Omega}\int_{\Omega} \frac{2}{|x-y|^{n+2s}} dxdy < \infty,$$

thanks to (2.2).

By the weak convergence of u_ε, and the fact that $|u_\varepsilon|, |u_o|$ are uniformly bounded

$$\lim_{\varepsilon\to 0^+} \int_{\mathscr{C}\Omega} \big(u_\varepsilon(y) - u_o(y)\big)\phi(y)\, dy \,=\, 0\,, \tag{2.3}$$

for any $\phi \in L^1(\mathbb{R}^n)$. Moreover, by the strong convergence of u_ε in Ω, (2.2), and the Dominated Convergence Theorem, we have that

$$\lim_{\varepsilon\to 0^+} \int_{\Omega} |u_\varepsilon(x)|^2 \int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}}\, dx = \int_{\Omega} |u_*(x)|^2 \int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}}\, dx \tag{2.4}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \left| \int_{\Omega} \big(u_\varepsilon(x) - u_*(x)\big) \int_{\mathscr{C}\Omega} \frac{u_\varepsilon(y)\, dy}{|x-y|^{n+2s}}\, dx \right| \\ &\qquad \leqslant \lim_{\varepsilon\to 0^+} \int_{\Omega} \big|u_\varepsilon(x) - u_*(x)\big| \int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}}\, dx \,=\, 0. \end{aligned} \tag{2.5}$$

On the other hand, making use of the notation in (1.6), we deduce from Fatou Lemma that

$$\liminf_{\varepsilon\to 0^+} u_\varepsilon(\Omega,\Omega) \geqslant u_*(\Omega,\Omega). \tag{2.6}$$

Let also

$$v_\varepsilon(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ u_\varepsilon(x) & \text{if } x \in \mathscr{C}\Omega. \end{cases}$$

Recalling that u_ε is minimal, we obtain that

$$\begin{aligned} 0 \;&\leqslant\; \mathscr{F}_\varepsilon(v_\varepsilon,\Omega) - \mathscr{F}_\varepsilon(u_\varepsilon,\Omega) \\ &=\; \mathscr{K}(v_\varepsilon,\Omega) - \mathscr{K}(u_\varepsilon,\Omega) - \varepsilon^{-2s}\int_{\Omega} W(u_\varepsilon(x))\, dx \\ &\leqslant\; \frac{1}{2}\Big(v(\Omega,\Omega) - u_\varepsilon(\Omega,\Omega)\Big) + \int_{\Omega}\left(\int_{\mathscr{C}\Omega} \frac{|v(x)-u_\varepsilon(y)|^2 - |u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\, dy\right) dx \\ &=\; \frac{1}{2}\Big(v(\Omega,\Omega) - u_\varepsilon(\Omega,\Omega)\Big) \\ &\qquad + \int_{\Omega} |v(x)|^2 \int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}}\, dx - \int_{\Omega} |u_\varepsilon(x)|^2 \int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}} \\ &\qquad +2\int_{\mathscr{C}\Omega} u_\varepsilon(y)\Psi(y)\, dy - 2\int_{\mathscr{C}\Omega} u_\varepsilon(y)\psi(y)\, dy \\ &\qquad +2\int_{\Omega} \big(u_\varepsilon(x) - u_*(x)\big) \int_{\mathscr{C}\Omega} \frac{u_\varepsilon(y)\, dy}{|x-y|^{n+2s}}\, dx. \end{aligned}$$

Consequently, recalling that $v(y) = u_*(y) = u_o(y)$ for any $y \in \mathscr{C}\Omega$ and using (2.3), (2.4), (2.5) and (2.6), we obtain

$$\begin{aligned}
0 &\leqslant \frac{1}{2}\Big(v(\Omega,\Omega) - u_*(\Omega,\Omega)\Big) \\
&\quad + \int_\Omega |v(x)|^2 \int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}}\,dx - \int_\Omega |u_*(x)|^2 \int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}} \\
&\quad +2\int_{\mathscr{C}\Omega} u_o(y)\Psi(y)\,dy - 2\int_{\mathscr{C}\Omega} u_o(y)\psi(y)\,dy \\
&= \frac{1}{2}\Big(v(\Omega,\Omega) - u_*(\Omega,\Omega)\Big) + \int_\Omega \left(\int_{\mathscr{C}\Omega} \frac{|v(x)-v(y)|^2 - |u_*(x)-u_*(y)|^2}{|x-y|^{n+2s}}\,dy \right) dx \\
&= \mathscr{F}(v,\Omega) - \mathscr{F}(u_*,\Omega).
\end{aligned}$$

This proves claim (i) of Theorem 1.3, and it ends the proof of Theorem 1.3.

3. Compactness for $s \geqslant 1/2$

Here, we prove the compactness claimed in Theorem 1.3 when $s \in [1/2,\,1)$ (and this range of s will be assumed throughout this section). An important tool for our estimate is Proposition 4.3 of [22], which provides a lower bound for the double integral

$$L(A,D) := \int_A \int_D \frac{1}{|x-y|^{n+2s}}\,dxdy.$$

For the convenience of the reader we state it below.

Proposition 3.1. *Let $s \in [1/2,1)$. Let A, D be disjoint subsets of a cube $Q \subset \mathbb{R}^n$ with*

$$\min\{|A|,|D|\} \geqslant \sigma|Q|, \tag{3.1}$$

for some $\sigma > 0$. Let $B = Q \setminus (A \cup D)$. Then,

$$L(A,D) \geqslant \begin{cases} \delta |Q|^{\frac{n-1}{n}} \log(|Q|/|B|) & \text{if } s = 1/2, \\ \delta |Q|^{\frac{n-2s}{n}} (|Q|/|B|)^{2s-1} & \text{if } s \in (1/2,1). \end{cases}$$

with $\delta > 0$ depending on σ, n and s.

Also, it is convenient to define

$$I_\varepsilon(u,\Omega) = \begin{cases} \dfrac{1}{2|\log\varepsilon|} u(\Omega,\Omega) + \dfrac{1}{\varepsilon|\log\varepsilon|}\displaystyle\int_\Omega W(u)\,dx & \text{if } s = 1/2, \\[2ex] \dfrac{\varepsilon^{2s-1}}{2} u(\Omega,\Omega) + \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u)\,dx & \text{if } s \in (1/2,\,1). \end{cases} \tag{3.2}$$

Notice that $I_\varepsilon(u_\varepsilon,\Omega)$ depends only on the values of u in Ω. We list some useful properties of $\mathscr{F}_\varepsilon$ and I_ε that follow immediately from their definition:

a) I_ε is bounded by $\mathscr{F}_\varepsilon$, i.e.

$$\mathscr{F}_\varepsilon(u,\Omega) \geqslant I_\varepsilon(u,\Omega),$$

b) $\mathscr{F}_\varepsilon$ is subadditive, i.e. if E and F are disjoint sets then

$$\mathscr{F}_\varepsilon(u,E\cup F) \leqslant \mathscr{F}_\varepsilon(u,E) + \mathscr{F}_\varepsilon(u,F),$$

c) I_ε is superadditive, i.e. if E and F are disjoint sets then

$$I_\varepsilon(u, E \cup F) \geqslant I_\varepsilon(u, E) + I_\varepsilon(u, F).$$

As a consequence of (3.2) and Proposition 3.1, we obtain

Lemma 3.2. *Let $\sigma \in (0, 1/4)$. Let Q be a cube in $\mathbb{R}^n$. If*

$$|\{u \geqslant 1 - \sigma\} \cap Q| \geqslant \sigma|Q| \quad \textit{and} \quad |\{u \leqslant -1 + \sigma\} \cap Q| \geqslant \sigma|Q| \tag{3.3}$$

then, for all small ε

$$I_\varepsilon(u, Q) \geqslant c(\sigma)|Q|^{\frac{n-1}{n}}. \tag{3.4}$$

where $c(\sigma) > 0$ depends on σ and on n, s, W.

Proof. Define

$$A := \{u \geqslant 1 - \sigma\} \cap Q, \quad D := \{u \leqslant -1 + \sigma\} \cap Q,$$
$$B := \{|u| \leqslant 1 - \sigma\} \cap Q = Q \setminus (A \cup D).$$

If

$$|B| \geqslant \begin{cases} \varepsilon|\log \varepsilon||Q|^{\frac{n-1}{n}}, & \text{and } s = 1/2 \\ \\ \varepsilon|Q|^{\frac{n-1}{n}}, & \text{and } s > 1/2, \end{cases}$$

then the potential energy in $I_\varepsilon(u, Q)$ satisfies (3.4) for some small $c(\sigma)$, and there is nothing to prove. Otherwise we apply Proposition 3.1, noticing that (3.1) is satisfied because of (3.3): we obtain

$$u(Q, Q) \geqslant u(A, D) \geqslant L(A, D) \geqslant \begin{cases} \delta(\sigma) \log\left(\dfrac{|Q|^{\frac{1}{n}}}{\varepsilon \log \varepsilon}\right) |Q|^{\frac{n-1}{n}}, & \text{and } s = 1/2 \\ \\ \delta(\sigma)\varepsilon^{1-2s}|Q|^{\frac{n-1}{n}}, & \text{and } s > 1/2. \end{cases}$$

This shows that the kinetic energy in $I_\varepsilon(u, Q)$ satisfies (3.4) provided that, in the case $s = 1/2$, $\varepsilon \leqslant \varepsilon_0(|Q|)$. □

Here is the compactness needed for Theorem 1.3:

Proposition 3.3. *Let Ω be an open, bounded subset of $\mathbb{R}^n$. If*

$$\liminf_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, \Omega) < +\infty,$$

then u_ε has a subsequence converging in $L^1(\Omega)$ to $\chi_E - \chi_{\mathscr{C}E}$, for a suitable $E \subseteq \mathbb{R}^n$. Moreover,

$$\mathrm{Per}(E, \Omega) < \infty.$$

Proof. We prove that the set u_ε is totally bounded in $L^1(\Omega)$, i.e. for any $\delta > 0$ there exists a finite set $\mathscr{S} \subset L^1(\Omega)$ such that for any small ε there exists $\psi_\varepsilon \in \mathscr{S}$ with

$$\|u_\varepsilon - \psi_\varepsilon\|_{L^1(\Omega)} \leqslant \delta. \tag{3.5}$$

By passing if necessary to a subsequence we assume

$$C_0 \geqslant \mathscr{F}_\varepsilon(u_\varepsilon, \Omega), \tag{3.6}$$

for some constant C_0. Fix $\sigma > 0$ small. We decompose the space in cubes Q_i of size ρ with $\rho > 0$ small, depending on σ and δ, to be made precise later. Let

$$K := \bigcup_{Q_i \subset \Omega} Q_i$$

denote the collection of these cubes which are included in Ω. We decompose K in three sets K_+, K_-, K_0 as follows

$$\begin{aligned}
K_+ &:= \bigcup_{Q_i \subset F_+} Q_i, \qquad F_+ = \Big\{ Q_i \in K \text{ s.t. } |\{u_\varepsilon < -1+\sigma\} \cap Q_i| < \sigma |Q_i| \Big\}, \\
K_- &:= \bigcup_{Q_i \in F_-} Q_i, \qquad F_- := \Big\{ Q_i \in K \setminus K_+ \text{ s.t. } |\{u_\varepsilon > 1-\sigma\} \cap Q_i| < \sigma |Q_i| \Big\}, \\
K_0 &:= K \setminus (K_+ \cup K_-).
\end{aligned}$$

We define ψ_ε to be 1 in K_+, and -1 otherwise. If ρ is sufficiently small then

$$|\Omega \setminus K| \leqslant \delta/8. \tag{3.7}$$

We have

$$C_0 \geqslant \mathscr{F}_\varepsilon(u_\varepsilon, \Omega) \geqslant I_\varepsilon(u_\varepsilon, K_0) \geqslant \sum_{Q_i \subset K_0} I_\varepsilon(u_\varepsilon, Q_i) \geqslant \frac{|K_0|}{\rho^n} c(\sigma) \rho^{n-1},$$

where in the last inequality we used Lemma 3.2. Hence

$$|K_0| \leqslant C(\sigma, C_0)\rho \leqslant \delta/8, \tag{3.8}$$

provided that ρ is small enough.

From (3.6) we also see that for all small ε

$$|\{|u| \leqslant 1-\sigma\} \cap \Omega| \leqslant C(\sigma) \int_\Omega W(u_\varepsilon)\, dx \leqslant C(\sigma, C_0)\varepsilon^{1/2} \leqslant \delta/8.$$

Therefore

$$\int_{\{|u| \leqslant 1-\sigma\} \cap \Omega} |u_\varepsilon - \psi_\varepsilon|\, dx \leqslant 2|\{|u| \leqslant 1-\sigma\} \cap \Omega| \leqslant \delta/4. \tag{3.9}$$

Moreover,

$$\begin{aligned}
|K_+ \cap \{u_\varepsilon < -1+\sigma\}| &= \sum_{Q_i \subset K_+} |Q_i \cap \{u_\varepsilon < -1+\sigma\}| \\
&< \sigma \sum_{Q_i \subset K_+} |Q_i| = \sigma |K_+|
\end{aligned}$$

and so

$$\int_{K_+ \cap \{u_\varepsilon < -1+\sigma\}} |u_\varepsilon - \psi_\varepsilon|\, dx \leqslant 2|K_+ \cap \{u_\varepsilon < -1+\sigma\}| \leqslant 2\sigma |K_+|. \tag{3.10}$$

In the same way, we obtain

$$\int_{K_- \cap \{u_\varepsilon > 1-\sigma\}} |u_\varepsilon - \psi_\varepsilon|\, dx \leqslant 2\sigma |K_-|. \tag{3.11}$$

On the other hand,

$$\begin{aligned} &\int_{K_-\cap\{u_\varepsilon<-1+\sigma\}} |u_\varepsilon-\psi_\varepsilon|\,dx+\int_{K_+\cap\{u_\varepsilon>1-\sigma\}} |u_\varepsilon-\psi_\varepsilon|\,dx\\ &=\int_{K_-\cap\{u_\varepsilon<-1+\sigma\}} |u_\varepsilon+1|\,dx+\int_{K_+\cap\{u_\varepsilon>1-\sigma\}} |u_\varepsilon-1|\,dx\\ &\leqslant \sigma|K_-\cap\{u_\varepsilon<-1+\sigma\}|+\sigma|K_+\cap\{u_\varepsilon>1-\sigma\}|\\ &\leqslant \sigma|K_+\cup K_-|. \end{aligned} \tag{3.12}$$

From (3.10), (3.11) and (3.12), we conclude that

$$\int_{(K_-\cup K_+)\cap\{|u_\varepsilon|>1-\sigma\}} |u_\varepsilon-\psi_\varepsilon|\,dx\leqslant 3\sigma|K_+\cup K_-|.$$

This and (3.9) yield that

$$\int_{K_-\cup K_+} |u_\varepsilon-\psi_\varepsilon|\,dx\leqslant \delta/2$$

as long as σ is small enough.

From the latter inequality and the ones in (3.7), and (3.8) we obtain

$$\int_\Omega |u_\varepsilon-\psi_\varepsilon|\,dx\leqslant 2|\Omega\setminus K|+2|K_0|+\int_{K_+\cup K_-} |u_\varepsilon-\psi_\varepsilon|\,dx\leqslant\delta.$$

The set S of all ψ_ε is clearly finite and our claim is proved. Since $|\psi_\varepsilon|\equiv 1$ we can easily conclude that there exists a convergent subsequence of u_ε's in $L^1(\Omega)$ to a function of the form $\chi_E-\chi_{\mathscr{C}E}$ for some set E. It remains to show that if u_ε converges to $\chi_E-\chi_{\mathscr{C}E}$ then E has finite perimeter in Ω. As above, we decompose $\mathbb{R}^n$ into cubes Q_i of size ρ and define

$$\phi_\rho=\begin{cases} 1 & \text{in } Q_i \text{ if } |E\cap Q_i|\geqslant 1/2|Q_i|\\ -1 & \text{otherwise.}\end{cases}$$

We also define

$$\tilde\phi_\rho:=\phi_\rho * g_\rho$$

where g_ρ is a mollifier defined in B_ρ, and we remark that

$$|\nabla\tilde\phi_\rho|\leqslant C/\rho.$$

From Lebesgue Theorem, ψ_ρ and $\tilde\psi_\rho$ converge to $\chi_E-\chi_{\mathscr{C}E}$ as $\rho\to 0^+$. Now we estimate the BV norm of $\tilde\psi_\rho$ by counting the number of cubes Q_i in Ω at distance greater than $\sqrt{n}\rho$ from $\partial\Omega$, i.e. $Q_i\in\Omega_{\sqrt{n}\rho}$, for which $\tilde\psi_\rho$ is not constant (1 or -1) in Q_i. Denote the set of such cubes by F. If $Q_i\in F$, then the cube $3Q_i$ of size 3ρ which contains Q_i in the interior, satisfies

$$|3Q_i\cap E|\geqslant c_0|Q_i|,\quad |3Q_i\cap\mathscr{C}E|\geqslant c_0|Q_i|,$$

for some explicit constant $c_0>0$. This implies that for all small ε,

$$|\{u_\varepsilon>1-\sigma\}\cap 3Q_i|\geqslant\sigma|3Q_i|,\quad |\{u_\varepsilon<-1+\sigma\}\cap 3Q_i|\geqslant\sigma|3Q_i|,$$

for some small, fixed $\sigma>0$. By Lemma 3.2 we obtain

$$I_\varepsilon(u_\varepsilon,3Q_i)\geqslant c\rho^{n-1}\quad\text{if } Q_i\in F.$$

We write

$$\bigcup_{Q_i \in F} 3Q_i = \bigcup_{k=1}^{N} \bigcup_{Q_i \in F_k} 3Q_i$$

with N depending only on n so that for each F_k, all cubes $3Q_i$ with $Q_i \in F_k$ are disjoint. We obtain

$$\sum_{Q_i \in F} I_\varepsilon(u_\varepsilon, 3Q_i) \leqslant N I_\varepsilon(u_\varepsilon, \Omega) \leqslant N C_0,$$

hence the number of cubes Q_i in F is bounded by $C\rho^{1-n}$. In conclusion

$$\int_{\Omega_{\sqrt{n}\rho}} |\nabla \tilde{\phi}_\rho| \, dx \leqslant C,$$

with C depending on n, s and W. Since $\tilde{\phi}_\rho \to \chi_E - \chi_{\mathscr{C}E}$ as $\rho \to 0^+$, the desired result follows from the lower-semicontinuity of the BV norm. □

4. Γ-convergence when $s \in [1/2, 1)$

In this section we prove Theorem 1.2 and Theorem 1.3 (ii) when $s \in [1/2, 1)$. In the classical case $s = 1$, the Γ-convergence is obtained by relating the energy $\mathscr{F}_\varepsilon(u, \Omega)$ with the area of the level sets of u using the coarea formula:

$$\begin{aligned}\int_\Omega \frac{1}{2\varepsilon} |\nabla u|^2 + \varepsilon W(u) \, dx &\geqslant \int_\Omega |\nabla u| \sqrt{2W(u)} \, dx \\ &= \int_{-1}^{1} \sqrt{2W(s)} \mathscr{H}^{n-1}\big(\{u = s\}\big) \, ds.\end{aligned}$$

Such formula is not available when $s < 1$, so we need a careful analysis of the local and nonlocal contributions in the energy functional $\mathscr{F}_\varepsilon$. We will see that in the case when $s \geqslant 1/2$ the contribution $u(\Omega, \mathscr{C}\Omega)$ in the kinetic term of $\mathscr{F}_\varepsilon(u, \Omega)$ for a minimizer u becomes negligible as $\varepsilon \to 0^+$.

Let $D \subseteq \Omega$ be a non-empty open bounded subset of Ω with smooth boundary. For all small $t > 0$ define

$$D_t = \{x \in D : \ d_{\partial D}(x) > t\},$$

where $d_{\partial D}(x)$ represents the distance from the point x to ∂D.

Next result gives an energy bound for the interpolation of two functions u_k, w_k across ∂D: for this a fine analysis on the integrals is needed.

Proposition 4.1. *Let $\varepsilon_k \to 0^+$, and let u_k, w_k be two sequences respectively in $L^1(D)$ and in $L^1(\mathbb{R}^n)$ such that*

$$u_k - w_k \to 0 \quad \text{in } L^1(D \setminus D_\delta).$$

Then, there exists a sequence v_k with the following properties:

1)

$$v_k(x) = \begin{cases} u_k(x) & \text{if } x \in D_\delta \\ w_k(x) & \text{if } x \in \Omega \setminus D \end{cases}$$

2)

$$\limsup_{k \to +\infty} \mathscr{F}_{\varepsilon_k}(v_k, \Omega) \leqslant \limsup_{k \to +\infty} \Big(\mathscr{F}_{\varepsilon_k}(w_k, \Omega) - \mathscr{F}_\varepsilon(w_k, D_\delta) + I_{\varepsilon_k}(u_k, D) \Big).$$

Proof. Assume that there exists $C_0 > 0$ such that

$$\mathscr{F}_{\varepsilon_k}(w_k, \Omega) - \mathscr{F}_{\varepsilon}(w_k, D_\delta) + I_{\varepsilon_k}(u_k, D) \leqslant C_0, \tag{4.1}$$

otherwise there is nothing to prove.

For simplicity of notation we drop the subindex k.

Since

$$\begin{aligned} \mathscr{K}(w, \Omega) - \mathscr{K}(w, D_\delta) &= \frac{1}{2} w(\Omega \setminus D_\delta, \Omega \setminus D_\delta) + w(\Omega \setminus D_\delta, \mathscr{C}\Omega) \\ &\geqslant \frac{1}{2} w(\Omega \setminus D_\delta, \mathscr{C}D_\delta), \end{aligned} \tag{4.2}$$

from (4.1) we obtain for $s > 1/2$ that

$$w(\Omega \setminus D_\delta, \mathscr{C}D_\delta) + u(D \setminus D_\delta, D) \leqslant 2C_0 \varepsilon^{1-2s},$$

and for $s = 1/2$ that

$$w(\Omega \setminus D_\delta, \mathscr{C}D_\delta) + u(D \setminus D_\delta, D) \leqslant 2C_0 |\log \varepsilon|.$$

Fix $\sigma > 0$ small. Let

$$\tilde{\delta} := \frac{\delta}{M}$$

for some large M depending on σ, and we partition $D \setminus D_\delta$ into M sets (i.e., "shells")

$$D \setminus D_{\tilde{\delta}}, \quad D_{\tilde{\delta}} \setminus D_{2\tilde{\delta}}, \cdots, D_{(M-1)\tilde{\delta}} \setminus D_{M\tilde{\delta}}.$$

If $s > 1/2$,

$$\begin{aligned} 2C_0 \varepsilon^{1-2s} &\geqslant w(D \setminus D_\delta, \mathscr{C}D_\delta) + u(D \setminus D_\delta, D) \\ &= \sum_{j=0}^{M-1} \Big(w(D_{j\tilde{\delta}} \setminus D_{(j+1)\tilde{\delta}}, \mathscr{C}D_\delta) + u(D_{j\tilde{\delta}} \setminus D_{(j+1)\tilde{\delta}}, D) \Big), \end{aligned}$$

thus there exists $j \leqslant M - 1$ such that

$$w(D_{j\tilde{\delta}} \setminus D_{(j+1)\tilde{\delta}}, \mathscr{C}D_\delta) + u(D_{j\tilde{\delta}} \setminus D_{(j+1)\tilde{\delta}}, D) \leqslant \sigma \varepsilon^{1-2s},$$

provided that we choose M sufficiently large. We denote

$$\tilde{D} := D_{j\tilde{\delta}}, \tag{4.3}$$

hence, if $s > 1/2$, we see that

$$w(\tilde{D} \setminus \tilde{D}_{\tilde{\delta}}, \mathscr{C}D_\delta) + u(\tilde{D} \setminus \tilde{D}_{\tilde{\delta}}, \tilde{D}) \leqslant \sigma \varepsilon^{1-2s}. \tag{4.4}$$

Similarly, if $s = 1/2$, then

$$w(\tilde{D} \setminus \tilde{D}_{\tilde{\delta}}, \mathscr{C}D_\delta) + u(\tilde{D} \setminus \tilde{D}_{\tilde{\delta}}, \tilde{D}) \leqslant \sigma |\log \varepsilon|. \tag{4.5}$$

We remark that, since $j \leqslant M - 1$ in (4.3), we have that $j\tilde{\delta} + \tilde{\delta} \leqslant \delta$, and so

$$\tilde{D}_{\tilde{\delta}} \supseteq D_\delta. \tag{4.6}$$

Next we consider N shells of width $\varepsilon \ll \tilde{\delta}$ of $\tilde{D}$, namely

$$A_i := \Big\{ x \in \tilde{D} \,:\, i\varepsilon < d_{\partial \tilde{D}}(x) \leqslant (i+1)\varepsilon \Big\}$$

for $0 \leqslant i \leqslant N - 1$, with N equal the integer part of $\tilde{\delta}/(2\varepsilon)$.

We note that

$$A_i \subseteq \tilde{D} \setminus \tilde{D}_{\tilde{\delta}}. \tag{4.7}$$

Also, denote by

$$d_i(x) := d_{\partial \tilde{D}_{i\varepsilon}}(x). \tag{4.8}$$

Notice that

$$\begin{gathered}\text{for any } x \in A_i, \text{ we have } d_i(x) \leqslant \varepsilon, \text{ i.e.} \\ 1 = \min\{1, (\varepsilon/d_i(x))^{2s}\},\end{gathered} \tag{4.9}$$

while

$$\begin{gathered}\text{for any } x \in \tilde{D}_{(i+1)\varepsilon} \setminus \tilde{D}_{\tilde{\delta}}, \text{ we have } d_i(x) \geqslant \varepsilon, \text{ i.e.} \\ (\varepsilon/d_i(x))^{2s} = \min\{1, (\varepsilon/d_i(x))^{2s}\}.\end{gathered} \tag{4.10}$$

Now, we claim that there exists $0 \leqslant i \leqslant N-1$ such that if $s > 1/2$

$$\int_{A_i} |u-w|\,dx + \varepsilon^{2s} \int_{\tilde{D}_{(i+1)\varepsilon} \setminus \tilde{D}_{\tilde{\delta}}} |u-w| d_i(x)^{-2s}\,dx \leqslant \sigma\varepsilon, \tag{4.11}$$

or if $s = 1/2$

$$\int_{A_i} |u-w|\,dx + \varepsilon^{2s} \int_{\tilde{D}_{(i+1)\varepsilon} \setminus \tilde{D}_{\tilde{\delta}}} |u-w| d_i(x)^{-2s}\,dx \leqslant \sigma\varepsilon|\log\varepsilon|. \tag{4.12}$$

Indeed, by (4.7), (4.9) and (4.10), we have that the sum of all N left hand sides for $i = 0, .., N-1$ is bounded by

$$2 \int_{\tilde{D} \setminus \tilde{D}_{\tilde{\delta}}} |u-w| \left(\sum_{i=0}^{N-1} \min\{1, (\varepsilon/d_i(x))^{2s}\} \right) dx. \tag{4.13}$$

Now, fix $x \in \tilde{D} \setminus \tilde{D}_{\tilde{\delta}}$ and consider the shell containing x, that is let $i_x \in \mathbb{N} \cap [0, 4N]$ such that $x \in A_{i_x}$. Then, if $d_i(x) \leqslant \varepsilon$, we have that $|i - i_x| \leqslant 1$, since the other shells are more than ε far apart from x. Also, if $|i - i_x| \geqslant 2$, then $d_i(x) \geqslant (\varepsilon/2)|i - i_x|$. From these considerations, we see that the sum inside the integral is bounded by a universal constant if $s > 1/2$ or by a constant times $\log N$ if $s = 1/2$.

Thus the integral in (4.13) is bounded by

$$\int_{D \setminus D_\delta} |u-w|\,dx, \quad \text{if } s > 1/2, \tag{4.14}$$

or

$$\log N \int_{D \setminus D_\delta} |u-w|\,dx, \quad \text{if } s = 1/2, \tag{4.15}$$

up to multiplicative constants.

By hypothesis (for all k large enough), the quantity

$$\int_{D \setminus D_\delta} |u-w|\,dx$$

can be made arbitrarily small, and so the claims in (4.11) and (4.12) follow easily from (4.14) and (4.15).

Now, fix a shell A_i for which (4.11) or (4.12) holds. Then we partition $\mathbb{R}^n$ into five regions P, Q, R, S, T where

$$\begin{gathered} P := \tilde{D}_{\tilde{\delta}}, \qquad Q := \tilde{D}_{(i+1)\varepsilon} \setminus \tilde{D}_{\tilde{\delta}}, \\ R := A_i, \qquad S := \Omega \setminus \tilde{D}_{i\varepsilon}, \qquad T = \mathscr{C}\Omega. \end{gathered}$$

Notice that

$$Q \cup R = \tilde{D}_{i\varepsilon} \setminus \tilde{D}_{\tilde{\delta}} \subseteq \tilde{D} \setminus \tilde{D}_{\tilde{\delta}}$$

and, by (4.6),

$$R \cup S \cup T = \mathscr{C}\tilde{D}_{(i+1)\varepsilon} \subseteq \mathscr{C}\tilde{D}_{\tilde{\delta}} \subseteq \mathscr{C}D_{\delta}.$$

Therefore, (4.4) gives that

$$w(Q \cup R, R \cup S \cup T) \leqslant \sigma \varepsilon^{1-2s}. \tag{4.16}$$

We choose

$$v = \phi u + (1 - \phi) w$$

where ϕ is a smooth cutoff function with $\phi = 1$ on $P \cup Q$, $\phi = 0$ on $S \cup T$, and

$$\|\nabla \phi\|_{L^\infty} \leqslant 3/\varepsilon.$$

Next we use (4.4) and (4.11) and we bound

$$\mathscr{K}(v, \Omega) = \frac{1}{2} v(\Omega, \Omega) + v(\Omega, \mathscr{C}\Omega),$$

in terms of double integrals of u and w. We consider only the case $s > 1/2$ since the only difference when $s = 1/2$ is, as in (4.12), the presence of an extra $|\log \varepsilon|$ on the right hand side.

First we notice that

$$\int_{\mathscr{C}B_\alpha(x)} \frac{dy}{|x-y|^{n+2s}} \leqslant C \int_\alpha^{+\infty} r^{-1-2s}\, dr \leqslant C\alpha^{-2s} \tag{4.17}$$

for any $\alpha > 0$, and that

$$\begin{gathered} v(S,S) = w(S,S), \quad v(S,T) = w(S,T), \\ v(P \cup Q, P \cup Q) = u(P \cup Q, P \cup Q). \end{gathered} \tag{4.18}$$

If $x \in P$ and $y \in R \cup S \cup T$ then

$$|x - y| \geqslant \tilde{\delta}/2 \quad \text{and} \quad |v(x) - v(y)|^2 \leqslant 4.$$

So, we use (4.17) and we integrate the inequality

$$\int_{R \cup S \cup T} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dy \leqslant \int_{\mathscr{C}B_{\tilde{\delta}/2}(x)} \frac{4}{|x-y|^{n+2s}} dy \leqslant C\tilde{\delta}^{-2s},$$

over $x \in P$ and obtain

$$v(P, R \cup S \cup T) \leqslant C\tilde{\delta}^{-2s}, \tag{4.19}$$

where $C > 0$ may also depend on $|\Omega|$.

On the other hand, recalling (4.8), we see that if $x \in Q$ and $y \in S \cup T$ then

$$|x - y| \geqslant d_i(x), \tag{4.20}$$

and

$$|v(x) - v(y)|^2 \leqslant 2|u(x) - w(x)|^2 + 2|w(x) - w(y)|^2. \tag{4.21}$$

Thus, using (4.17) again, we deduce from (4.20) that

$$\int_{S \cup T} \frac{1}{|x-y|^{n+2s}} dy \leqslant C d_i(x)^{-2s},$$

for any $x \in Q$, and so we obtain, by (4.11), (4.16) and (4.21) that

$$\begin{aligned} v(Q, S \cup T) &\leqslant 2w(Q, S \cup T) + C \int_Q |u-w|^2 d_i(x)^{-2s}\, dx \\ &\leqslant C\sigma\varepsilon^{1-2s}. \end{aligned} \tag{4.22}$$

Moreover, if $y \in Q$ and $x \in R$ then

$$|1-\phi(x)| \leqslant \frac{3}{\varepsilon} d_{i+1}(x), \tag{4.23}$$

and

$$|v(x)-v(y)|^2 \leqslant 2|u(x)-u(y)|^2 + 2|1-\phi(x)|^2|u(x)-w(x)|^2. \tag{4.24}$$

Since, by (4.17), we know that

$$\int_Q \frac{1}{|x-y|^{n+2s}} dy \leqslant C d_{i+1}(x)^{-2s}$$

for any $x \in R$, we obtain, by (4.4), (4.6), (4.11), (4.23) and (4.24), that

$$\begin{aligned} v(Q, R) &\leqslant 2u(Q, R) + \frac{C}{\varepsilon^2} \int_R |u-w|^2 d_{i+1}(x)^{2-2s}\, dx \\ &\leqslant 2u(Q, R) + C\varepsilon^{-2s} \int_R |u-w|^2\, dx \\ &\leqslant 2u(\tilde{D} \setminus \tilde{D}_{\tilde{\delta}}, \tilde{D} \setminus \tilde{D}_{\tilde{\delta}}) + C\varepsilon^{-2s} \int_R |u-w|\, dx \\ &\leqslant C\sigma\varepsilon^{1-2s}. \end{aligned} \tag{4.25}$$

Similarly we find

$$v(S \cup T, R) \leqslant C\sigma\varepsilon^{1-2s}. \tag{4.26}$$

Furthermore, if $x \in R$ and $y \in R$ then

$$\begin{aligned} |v(x)-v(y)| &\leqslant |w(x)-w(y)| + |\phi(x)(u-w)(x) - \phi(y)(u-w)(y)| \\ &\leqslant |w(x)-w(y)| + |(u-w)(x)||\phi(x)-\phi(y)| \\ &\qquad + \phi(y)|(u-w)(x)-(u-w)(y)| \\ &\leqslant 2|w(x)-w(y)| + |u(x)-u(y)| + |(u-w)(x)||\phi(x)-\phi(y)|, \end{aligned}$$

hence

$$\begin{aligned} |v(x)-v(y)|^2 \leqslant C(&|u(x)-u(y)|^2 + |w(x)-w(y)|^2 \\ &+ |(u-w)(x)|^2|\phi(x)-\phi(y)|^2). \end{aligned} \tag{4.27}$$

Also, since

$$|\phi(x)-\phi(y)| \leqslant \min\{1, \frac{3}{\varepsilon}|x-y|\},$$

we find

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{n+2s}} dy &\leqslant \frac{C}{\varepsilon^2} \int_0^\varepsilon r^{1-2s} dr + C \int_\varepsilon^\infty r^{-1-2s} dr \\ &\leqslant C\varepsilon^{-2s}. \end{aligned} \tag{4.28}$$

Therefore, using (4.4), (4.6), (4.11), (4.16), (4.27) and (4.28), we can conclude that

$$
\begin{aligned}
v(R,R) &\leqslant C\left(u(R,R)+w(R,R)+\varepsilon^{-2s}\int_R |u-w|^2 dx\right)\\
&\leqslant C\sigma\varepsilon^{1-2s}.
\end{aligned}
\tag{4.29}
$$

Also, noticing that $S\subseteq\Omega\setminus D_\delta$, we obtain

$$
\begin{aligned}
&\frac{1}{2}w(S,S)+w(S,T)+\mathscr{K}(w,D_\delta)\\
=\;&\frac{1}{2}w(S,S)+w(S,\mathscr{C}\Omega)+\frac{1}{2}w(D_\delta,D_\delta)+\frac{1}{2}w(D_\delta,\Omega\setminus D_\delta)\\
&\quad+\frac{1}{2}w(\Omega\setminus D_\delta,D_\delta)+w(D_\delta,\mathscr{C}\Omega)\\
=\;&\frac{1}{2}w(S,S)+\frac{1}{2}w(\Omega\setminus D_\delta,D_\delta)+\frac{1}{2}w(D_\delta,\Omega)+w(S\cup D_\delta,\mathscr{C}\Omega)\\
\leqslant\;&\frac{1}{2}w(\Omega\setminus D_\delta,S)+\frac{1}{2}w(\Omega\setminus D_\delta,D_\delta)+\frac{1}{2}w(D_\delta,\Omega)+w(\Omega,\mathscr{C}\Omega)\\
\leqslant\;&\frac{1}{2}w(\Omega\setminus D_\delta,\Omega)+\frac{1}{2}w(D_\delta,\Omega)+w(\Omega,\mathscr{C}\Omega)\\
=\;&\frac{1}{2}w(\Omega,\Omega)+w(\Omega,\mathscr{C}\Omega)\\
=\;&\mathscr{K}(w,\Omega).
\end{aligned}
$$

As a consequence, observing that $P\cup Q\subseteq D$, and making use of (4.2), (4.18), (4.19), (4.22), (4.25), (4.26) and (4.29), we find

$$
\begin{aligned}
\mathscr{K}(v,\Omega)\leqslant&\frac{1}{2}w(S,S)+w(S,T)+\frac{1}{2}u(P\cup Q,P\cup Q)\\
&+C\left(\sigma\varepsilon^{1-2s}+\tilde\delta^{-2s}\right)\\
\leqslant&\mathscr{K}(w,\Omega)-\mathscr{K}(w,D_\delta)+\frac{1}{2}u(D,D)+C\left(\sigma\varepsilon^{1-2s}+\tilde\delta^{-2s}\right).
\end{aligned}
\tag{4.30}
$$

Also, if $x\in R$ then

$$
W(v)\leqslant W(w)+C|v-w|\leqslant W(w)+C|u-w|,
$$

hence (4.11) gives that

$$
\begin{aligned}
\int_\Omega W(v)&\leqslant\int_{P\cup Q}W(u)+\int_{R\cup S}W(w)+\int_R|u-w|\,dx\\
&\leqslant\int_D W(u)+\int_{\Omega\setminus D_\delta}W(w)+\sigma\varepsilon.
\end{aligned}
\tag{4.31}
$$

From (4.30) and (4.31) we obtain (for all $\varepsilon=\varepsilon_k$ small)

$$
\mathscr{F}_\varepsilon(v,\Omega)\leqslant\mathscr{F}_\varepsilon(w,\Omega)-\mathscr{F}_\varepsilon(w,D_\delta)+I_\varepsilon(u,D)+C\left(\sigma+\varepsilon^{2s-1}\tilde\delta^{-2s}\right),
$$

where C depends only on $|\Omega|$, n and s. We remark that when $s=1/2$, the last term becomes $C(\sigma+\tilde\delta^{-1}/|\log\varepsilon|)$. Since σ is arbitrary the proof is complete. □

We recall the following result about the one-dimensional minimizer which is proved in [20] (see, in particular, ????? there).

Theorem 4.2. *There exists a unique (up to translations and rotations) nontrivial global minimizer u_0 of the energy*

$$\mathscr{E}(u,\Omega) := \mathscr{K}(u,\Omega) + \int_\Omega W(u)\,dx,$$

which depends only on one variable. If the function u_0 depends only on x_n, then $u_0 \in C^{1,s}$ is increasing in x_n and

$$1 - |u_0(x_n)| \leqslant C|x_n|^{-2s}, \qquad |u_0'(x_n)| \leqslant C|x_n|^{-1-2s}. \tag{4.32}$$

There exists a constant $b_\star > 0$ depending only on s, n and W such that as $R \to \infty$

a) if $s < 1/2$ then

$$\frac{\mathscr{E}(u_0,B_R)}{R^{n-2s}} \to b_\star \quad \text{and} \quad \frac{u_0(B_R,\mathscr{C}B_R)}{R^{n-2s}} \to d_\star > 0;$$

b) if $s = 1/2$ then

$$\frac{\mathscr{E}(u_0,B_R)}{R^{n-1}\log R} \to b_\star, \quad \text{and} \quad \frac{u_0(B_R,\mathscr{C}B_R)}{R^{n-1}\log R} \to 0;$$

c) if $s \in (1/2,1)$ then

$$\frac{\mathscr{E}(u_0,B_R)}{R^{n-1}} \to b_\star, \quad \text{and} \quad \frac{u_0(B_R,\mathscr{C}B_R)}{R^{n-1}} \to 0.$$

Theorem 4.2 says that, as R gets larger and larger, the contribution in $\mathscr{K}(u_0,B_R)$ from $\mathscr{C}B_r$ becomes negligible if $s \geqslant 1/2$, however when $s < 1/2$ this does not happen.

The energy $\mathscr{F}_\varepsilon$ is a rescaling of the energy $\mathscr{E}$ in the sense that if u is defined in $\mathbb{R}^n$ and $u_\varepsilon(x) := u(x/\varepsilon)$, then

$$\mathscr{F}_\varepsilon(u_\varepsilon,B_\rho) = \begin{cases} \varepsilon^{n-2s}\,\mathscr{E}(u,B_{\rho/\varepsilon}) & \text{if } s<1/2 \\ \dfrac{\varepsilon^{n-1}}{|\log\varepsilon|}\,\mathscr{E}(u,B_{\rho/\varepsilon}) & \text{if } s=1/2 \\ \varepsilon^{n-1}\,\mathscr{E}(u,B_{\rho/\varepsilon}) & \text{if } s>1/2. \end{cases}$$

Hence if

$$w_\varepsilon(x) := u_0(x/\varepsilon), \tag{4.33}$$

denotes the rescaling of the one-dimensional solution u_0, then w_ε is a global minimizer of $\mathscr{F}_\varepsilon$. Moreover, Theorem 4.2 can be stated in terms of w_ε and $\mathscr{F}_\varepsilon$ as

$$\lim_{\varepsilon\to0^+} \mathscr{F}_\varepsilon(w_\varepsilon,B_\rho) = b_\star\rho^{n-2s} > \lim_{\varepsilon\to0^+} I_\varepsilon(w_\varepsilon,B_\rho) \quad \text{if } s<1/2,$$

and

$$\lim_{\varepsilon\to0^+} \mathscr{F}_\varepsilon(w_\varepsilon,B_\rho) = \lim_{\varepsilon\to0^+} I_\varepsilon(w_\varepsilon,B_\rho) = c_\star\omega_{n-1}\rho^{n-1}, \quad \text{if } s\geqslant 1/2, \tag{4.34}$$

where

$$c_\star := \frac{b_\star}{\omega_{n-1}}. \tag{4.35}$$

As a consequence of Proposition 4.1 and Theorem 4.2 we obtain the following

Proposition 4.3. *Let $\alpha > 0$. If u_ε is a sequence of functions that satisfies*

$$\lim_{\varepsilon\to 0^+} \int_{B_\rho} |u_\varepsilon(x) - \mathrm{sign}(x_n)|\,dx \leqslant \alpha\rho^n, \tag{4.36}$$

for some $\rho > 0$, then

$$\liminf_{\varepsilon\to 0^+} I_\varepsilon(u_\varepsilon, B_\rho) \geqslant \omega_{n-1}\rho^{n-1}(c_\star - \eta(\alpha)).$$

with $\eta(\alpha)$ depending on α (and n, s and W) and

$$\lim_{\alpha\to 0^+} \eta(\alpha) = 0. \tag{4.37}$$

Proof. First we prove the statement in the particular case $\rho = 1$.

Assume by contradiction that the statement fails. Then we can find a sequence of functions u_ε such that

$$\lim_{\varepsilon\to 0^+} \int_{B_1} |u_\varepsilon(x) - \mathrm{sign}(x_n)|\,dx = 0,$$

and

$$\limsup_{\varepsilon\to 0^+} I_\varepsilon(u_\varepsilon, B_1) \leqslant \omega_{n-1}c_\star - \mu, \tag{4.38}$$

for some small $\mu > 0$.

Let w_ε be defined by (4.33). Then w_ε is a global minimizer for $\mathscr{F}_\varepsilon$ i.e.

$$\mathscr{F}_\varepsilon(w_\varepsilon, B_1) \leqslant \mathscr{F}_\varepsilon(v_\varepsilon, B_1) \tag{4.39}$$

for any v_ε that coincides with w_ε outside B_1. Since

$$\int_{B_1} |w_\varepsilon(x) - \mathrm{sign}(x_n)|\,dx \to 0 \quad \text{as } \varepsilon \to 0^+,$$

we can apply Proposition 4.1 for u_ε and w_ε with $D = \Omega = B_1$ and obtain

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(v_\varepsilon, B_1) \leqslant \limsup_{\varepsilon\to 0^+} \left(\mathscr{F}_\varepsilon(w_\varepsilon, B_1) - \mathscr{F}_\varepsilon(w_\varepsilon, B_{1-\delta}) + I_\varepsilon(u_\varepsilon, B_1)\right). \tag{4.40}$$

On the other hand, by (4.34)

$$\lim_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(w_\varepsilon, B_1) = \omega_{n-1}c_\star, \quad \lim_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(w_\varepsilon, B_{1-\delta}) = (1-\delta)^{n-1}\omega_{n-1}c_\star,$$

hence, by (4.39) and (4.40)

$$(1-\delta)^{n-1}\omega_{n-1}c_\star \leqslant \limsup_{\varepsilon\to 0^+} I_\varepsilon(u_\varepsilon, B_1),$$

and we reach a contradiction with (4.38) by choosing δ sufficiently small.

For the general case we define $\tilde u_{\tilde\varepsilon}$ in B_1 as

$$\tilde u_{\tilde\varepsilon}(x) := u_\varepsilon(\rho x).$$

Then $\tilde u_{\tilde\varepsilon}$ satisfies the hypothesis above in B_1 with $\tilde\varepsilon := \varepsilon/\rho$ and the result follows by scaling since

$$I_\varepsilon(u_\varepsilon, B_\rho) = \rho^{n-1} I_{\tilde\varepsilon}(\tilde u_{\tilde\varepsilon}, B_1) \qquad \text{if } s > 1/2,$$

and

$$I_\varepsilon(u_\varepsilon, B_\rho) = \rho^{n-1}\frac{|\log(\tilde\varepsilon)|}{|\log\varepsilon|}\, I_{\tilde\varepsilon}(\tilde u_{\tilde\varepsilon}, B_1) \qquad \text{if } s = 1/2.$$

∎

4.1. **Reduced boundary analysis.** The idea now is to consider any u_ε approaching $\chi_E - \chi_{\mathscr{C}E}$, with E of finite perimeter. Then (4.36) holds, suitably scaled, near the reduced boundary of E, that will be denoted, as usual, by $\partial^* E$. We refer to [17] for the basics of the theory of sets with finite perimeter and the definition of the reduced boundary.

Precisely, let $\nu(p)$ denote the measure theoretic unit inner normal at any $p \in \partial^* E$ (see Definitions 3.3 and 3.6 of [17]). Then, (4.36) holds true in small balls:

Corollary 4.4. *Let E be a set of finite perimeter, with $0 \in \partial^* E$ and*

$$\nu(0) = e_n. \tag{4.41}$$

Suppose that, as $\varepsilon \to 0^+$, u_ε converges to $\chi_E - \chi_{\mathscr{C}E}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Then, for any $\alpha > 0$ there exists $\rho(\alpha) > 0$ (depending also on n, s and E) such that if $\rho \in \big(0, \rho(\alpha)\big]$, we have that

$$\lim_{\varepsilon \to 0^+} \int_{B_\rho} |u_\varepsilon(x) - \operatorname{sign}(x_n)|\, dx \leqslant \alpha \rho^n.$$

Corollary 4.4 is a consequence of the following known property of $\partial^* E$:

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho} |\chi_E - \chi_{\mathscr{C}E} - \operatorname{sign}(x_n)| = 0.$$

4.2. **Bounding the energy from below.** We are now in the position of obtaining a lower bound for the energy with respect to the perimeter of the asymptotic interface for $s \in [1/2, 1)$, and thus proving Theorem 1.2 (i).

Proposition 4.5. *Suppose that, as $\varepsilon \to 0^+$, u_ε converges to $\chi_E - \chi_{\mathscr{C}E}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then,*

$$\liminf_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, \Omega) \geqslant c_\star \operatorname{Per}(E, \Omega).$$

Proof. From Proposition 3.3 (see Section 3), we may assume that E has finite perimeter in Ω. Then by Theorem 4.4 of [17], we have

$$\operatorname{Per}(E, \Omega) = \mathscr{H}^{n-1}(\partial^* E \cap \Omega).$$

Consequently, by fixing $\alpha > 0$, we can find a collection of balls $\{B_j\}_{j \in \mathbb{N}}$ centered at points of $\partial^* E$ and of radius $\rho_j > 0$, conveniently small in dependence of α, such that

$$\operatorname{Per}(E, \Omega) \leqslant \alpha + \omega_{n-1} \sum_{j=0}^{+\infty} \rho_j^{n-1}. \tag{4.42}$$

In fact, we can take the above balls disjoint, because of the Vitali's Covering Theorem (see, e.g., [13]), thus

$$\mathscr{F}_\varepsilon(u_\varepsilon, \Omega) \geqslant I_\varepsilon(u_\varepsilon, \Omega) \geqslant \sum_{j=0}^{+\infty} I_\varepsilon(u_\varepsilon, B_j). \tag{4.43}$$

Also, Corollary 4.4 makes (4.36) hold, and so we can use Proposition 4.3 in any of these balls B_j. Hence, we obtain

$$\liminf_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, \Omega) \geqslant \omega_{n-1}(c_\star - \eta(\alpha)) \sum_{j=0}^{+\infty} \rho_j^{n-1} \geqslant (c_\star - \eta(\alpha))(\operatorname{Per}(E, \Omega) - \alpha),$$

and the desired result follows by letting $\alpha \to 0^+$. □

4.3. **Bounding the energy from above.** Now we prove part (ii) of Theorem 1.2.

Proposition 4.6. *Let Ω be a bounded domain with Lipschitz boundary. Given a set E, there exists a sequence u_ε converging in $L^1(\Omega)$ to $\chi_E - \chi_{\mathscr{C}E}$ such that*

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon,\Omega) \leqslant c_\star \,\mathrm{Per}\,(E,\Omega).$$

Proof. It was proved in [19] that there exist open sets with smooth boundaries which approximate E in Ω. Precisely, given any $\sigma > 0$, there exists A open with ∂A smooth, such that

$$\|\chi_{A\cap\Omega} - \chi_E\|_{L^1(\Omega)} \leqslant \sigma, \qquad P(A,\Omega) \leqslant P(E,\Omega) + \sigma,$$
$$\mathscr{H}^{n-1}(\partial A \cap \partial\Omega) = 0.$$

This shows that it suffices to prove the theorem with A instead of E. Fix $\alpha > 0$ small.

Let $d(x)$ be the signed distance of x to ∂A with the convention that $d(x) \geqslant 0$ if $x \in A$ and $d(x) \leqslant 0$ if $x \in \mathscr{C}A$.

We define

$$u_\varepsilon(x) := u_0\left(\frac{d(x)}{\varepsilon}\right),$$

where $u_0 : \mathbb{R} \to [-1,1]$ is the profile of the one-dimensional minimizer of $\mathscr{E}$ (see Theorem 4.2).

Let us take a finite overlapping family of balls $\{B_{\rho_j}(x_j)\}_{j\in\mathbb{N}}$ centered at $x_j \in \partial A$, with $\sup_{j\in\mathbb{N}} \rho_j \leqslant \alpha$, such that

$$\partial A \cap \overline{\Omega} \subseteq \bigcup_{j=0}^{+\infty} B_{\rho_j}(x_j)$$

and

$$\mathrm{Per}(A,\overline{\Omega}) + \alpha \geqslant \omega_{n-1} \sum_{j=0}^{+\infty} \rho_j^{n-1}.$$

By compactness, we may suppose that

$$\partial A \cap \overline{\Omega} \subseteq V := \bigcup_{j=0}^{N} B_{\rho_j}(x_j),$$

for a suitable $N \in \mathbb{N}$. Notice that

$$\delta := \inf_{x\in\Omega\setminus V} |d(x)| > 0.$$

Recalling (4.32), we have that

$$\begin{aligned} \int_{\Omega\setminus V} W(u_\varepsilon(x))\,dx &\leqslant C\int_{\Omega\setminus V} |u_\varepsilon(x) - 1|\,dx \\ &\leqslant C\int_{\Omega\setminus V} \left|\frac{d(x)}{\varepsilon}\right|^{-2s} dx \leqslant C(\delta)\varepsilon^{2s}, \end{aligned} \tag{4.44}$$

thus, the contribution in $\mathscr{F}_\varepsilon(u_\varepsilon)$ from the potential energy in $\Omega\setminus V$ tends to 0 as $\varepsilon \to 0^+$. Moreover if $|d(x)| \geqslant \delta/2$ then we use (4.32) and obtain

$$|\nabla u_\varepsilon(x)| = \left|u_0'\left(\frac{d(x)}{\varepsilon}\right)\right| \frac{1}{\varepsilon} \leqslant C(\delta).$$

If $x \in \Omega \setminus V$ then

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leqslant C(\delta)|x-y|, \quad \text{if } |x-y| \leqslant \delta/2,$$

thus

$$\int_{\mathbb{R}^n} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\, dy \leqslant C(\delta) \left(\int_0^{\delta/2} r^{1-2s} + \int_{\delta/2}^{\infty} r^{-1-2s} dr \right) \leqslant C(\delta).$$

We find

$$u_\varepsilon(\Omega \setminus V, \mathbb{R}^n) \leqslant C(\delta),$$

which together with (4.44) gives,

$$\limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, \Omega) \leqslant \limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, V) \leqslant \limsup_{\varepsilon \to 0^+} \sum_{j=0}^{+\infty} \mathscr{F}_\varepsilon(u_\varepsilon, B_{\rho_j}).$$

Now we estimate each term $\mathscr{F}_\varepsilon(u_\varepsilon, B_{\rho_j})$. We will denote by $\eta_i(\alpha)$ suitable functions depending only on α, n, s and A satisfying

$$\lim_{\alpha \to 0^+} \eta_i(\alpha) = 0.$$

If α is small enough, then for any $B_{\rho_j}(x_j)$ there exists a diffeomorphism

$$x \in B_{\rho_j}(x_j) \longrightarrow z(x) \in U_j \quad \text{with} \quad z_n = d(x),$$

$$|D_x z - I| \leqslant \eta_0(\alpha), \quad U_j \subset B_{1+\eta_0(\alpha)}.$$

Changing coordinates from x to z we find

$$\begin{aligned} \mathscr{F}_\varepsilon(u_\varepsilon, B_{\rho_j}(x_j)) &\leqslant (1+\eta_1(\alpha))\mathscr{F}_\varepsilon(w_\varepsilon, U_j) \\ &\leqslant (1+\eta_1(\alpha))\mathscr{F}_\varepsilon(w_\varepsilon, B_{1+\eta_0(\alpha)}), \end{aligned}$$

where $w_\varepsilon(z) = u_0(z_n/\varepsilon)$. From Theorem 4.2,

$$\limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, B_{\rho_j}(x_j)) \leqslant (1+\eta_2(\alpha)) c_\star \omega_{n-1} \rho_j^{n-1},$$

and the desired result follows by letting $\alpha \to 0^+$. □

We denote

$$\operatorname{Per}(E, \overline{U}) := \lim_{\delta \to 0^+} \operatorname{Per}(E, U^\delta), \tag{4.45}$$

where

$$U^\delta := \{x \in \mathbb{R}^n \text{ s.t. } \operatorname{dist}(x, U) \leqslant \delta\}.$$

Notice that the limit in (4.45) exists by Monotone Convergence Theorem.

Next we prove part (ii) of Theorem 1.3.

Proposition 4.7. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose that u_ε minimizes $\mathscr{F}_\varepsilon$ in Ω and that, as $\varepsilon \to 0^+$, u_ε converges to $\chi_E - \chi_{\mathscr{C}E}$ in $L^1(\Omega)$, for some measurable $E \subseteq \Omega$.*

Then E has minimal perimeter in Ω and for any open set $U \subset\subset \Omega$, we have that

$$\limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, U) \leqslant c_\star \operatorname{Per}(E, \overline{U}). \tag{4.46}$$

Proof. Let $U \subset\subset \Omega$ have smooth boundary and δ be small so that $U^\delta \subset \Omega$.

Let F be a measurable set in Ω such that F and E coincide outside U. By Propositions 4.6 and 4.5, there exists a sequence $w_\varepsilon \in L^1(U^\delta)$ which converges to $\chi_F - \chi_{\mathscr{C}F}$ such that

$$\lim_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(w_\varepsilon, U^\delta) = c_\star \mathrm{Per}(F, U^\delta).$$

From Proposition 4.1 we construct a sequence v_ε which coincides with w_ε in U and with u_ε in $\mathscr{C}U^\delta$ such that such that

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(v_\varepsilon, \Omega) \leqslant \limsup_{\varepsilon\to 0^+} \left(\mathscr{F}_\varepsilon(u_\varepsilon, \Omega) - \mathscr{F}_\varepsilon(u_\varepsilon, U) + \mathscr{F}_\varepsilon(w_\varepsilon, U^\delta)\right).$$

Since u_ε is a minimizer,

$$\mathscr{F}_\varepsilon(u_\varepsilon, \Omega) \leqslant \mathscr{F}_\varepsilon(v_\varepsilon, \Omega),$$

hence

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, U) \leqslant c_\star \mathrm{Per}(F, U^\delta).$$

We let $\delta \to 0^+$ and use Proposition 4.5 to find

$$c_\star \mathrm{Per}(E, U) \leqslant \limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, U) \leqslant c_\star \mathrm{Per}(F, \overline{U}). \tag{4.47}$$

Since this inequalities are valid if we replace U with U^δ for all small δ, we can conclude that

$$\mathrm{Per}(E, \Omega) \leqslant \mathrm{Per}(F, \Omega),$$

i.e. E has minimal perimeter in Ω. Also, by taking $F = E$ in (4.47) we obtain (4.46) for smooth subsets U. Now the general case follows easily by approximating U with smooth domains from the exterior. □

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