On Monge-Ampère Equations with Homogenous Right-Hand Sides

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Abstract

We study the regularity and behavior at the origin of solutions to the two-dimensional degenerate Monge-Ampère equation $\det D^2 u = |x|^\alpha$ with $\alpha > -2$. We show that when $\alpha > 0$, solutions admit only two possible behaviors near the origin, radial and nonradial, which in turn implies $C^{2,\delta}$-regularity. We also show that the radial behavior is unstable. For $\alpha < 0$ we prove that solutions admit only the radial behavior near the origin. ⓒ 2008 Wiley Periodicals, Inc.

1 Introduction

We consider the degenerate two-dimensional Monge-Ampère equation

$$\det D^2 u = |x|^\alpha, \quad x \in B_1, \tag{1.1}$$

on the unit disc $B_1 = \{|x| \leq 1\}$ of $\mathbb{R}^2$ and in the range of exponents $\alpha > -2$. Our goal is to investigate the behavior of solutions u near the origin, where the equation becomes degenerate.

The study of (1.1) is motivated by the Weyl problem with nonnegative curvature, posed in 1916 by Weyl [13] himself: Given a Riemannian metric g on the 2-sphere $\mathbb{S}^2$ whose Gauss curvature is everywhere positive, does there exist a global C^2 isometric embedding $X : (\mathbb{S}^2, g) \to (\mathbb{R}^3, ds^2)$, where ds^2 is the standard flat metric on $\mathbb{R}^3$?

H. Lewy [10] solved the problem under the assumption that the metric g is analytic. The solution to the Weyl problem, under the regularity assumption that g has continuous fourth-order derivatives, was given in 1953 by L. Nirenberg [12].

P. Guan and Y.Y. Li [6] considered the question: If the Gauss curvature of the metric g is nonnegative instead of strictly positive and g is smooth, is it still possible to have a smooth isometric embedding? It was shown in [6] that for any C^4 Riemannian metric g on $\mathbb{S}^2$ with nonnegative Gaussian curvature, there is always a $C^{1,1}$ global isometric embedding into $(\mathbb{R}^3, ds^2)$.

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Examples show that for some analytic metrics with positive Gauss curvature on $\mathbb{S}^2$ except at one point, there exists only a $C^{2,1}$ but not a C^3 global isometric embedding into $(\mathbb{R}^3, ds^2)$. Note that the phenomenon is global, since C. S. Lin [11] has shown that for any smooth two-dimensional Riemannian metric with nonnegative Gauss curvature there exists a smooth *local* isometric embedding into $(\mathbb{R}^3, ds^2)$.

This leads to the following question, which was posed in [6]:

> Under what conditions on a smooth metric g on $\mathbb{S}^2$ with nonnegative Gauss curvature, is there a $C^{2,\alpha}$ global isometric embedding into $(\mathbb{R}^3, ds^2)$, for some $\alpha > 0$, or even a $C^{2,1}$?

The problem can be reduced to a partial differential equation of Monge-Ampère type that becomes degenerate at the points where the Gauss curvature vanishes. It is well-known that in general one may have solutions to degenerate Monge-Ampère equations that are at most $C^{1,1}$.

One may consider a smooth Riemannian metric g on $\mathbb{S}^2$ with nonnegative Gauss curvature that has only one nondegenerate zero. In this case, if we represent the $C^{1,1}$ embedding as a graph, answering the above question amounts to studying the regularity at the origin of the degenerate Monge-Ampère equation

$$\det D^2 u = f \quad \text{on } B_1, \tag{1.2}$$

in the case where the forcing term f vanishes quadratically at $x = 0$. More precisely, it suffices to assume that $f(x) = |x|^2 g(x)$, where g is a positive Lipschitz function. This leads to equation (1.1) when $\alpha = 2$.

In addition to the results mentioned above, degenerate equations of the form (1.2) on $\mathbb{R}^2$ were previously considered by P. Guan in [5] in the case where $f \in C^\infty(B_1)$ and

$$A^{-1}(x_1^{2l} + Bx_2^{2m}) \le f(x_1, x_2) \le A(x_1^{2l} + Bx_2^{2m}) \tag{1.3}$$

for some constants $A > 0$, $B \ge 0$, and positive integers $l \le m$. The C^∞ regularity of the solution u of (1.2) was shown in [5], under the additional condition that $u_{x_2 x_2} \ge C_0 > 0$. It was conjectured in [5] that the same result must be true under the weaker condition that $\Delta u \ge C_0 > 0$. This was recently shown by P. Guan and I. Sawyer in [8].

Equation (1.1) has also an interpretation in the language of optimal transportation with quadratic cost $c(x, y) = |x - y|^2$. In this setting the problem consists in transporting the density $|x|^\alpha\, dx$ from a domain Ω_x into the uniform density dy in the domain Ω_y in such a way that we minimize the total "transport cost," namely,

$$\int_{\Omega_x} |y(x) - x|^2 |x|^\alpha \, dx.$$

Then, by a theorem of Y. Brenier [1], the optimal map $x \mapsto y(x)$ is given by the gradient of a solution of the Monge-Ampère equation (1.1). The behavior of these solutions at the origin gives information on the geometry of the optimal map near the singularity of the measure $|x|^\alpha\, dx$.

We will next state the results of this paper. We assume that u is a solution of equation (1.1). Then u is C^∞ smooth away from the origin. The following results describe the regularity of u at the origin. We begin with the case when $\alpha > 0$.

THEOREM 1.1 *If $\alpha > 0$, then $u \in C^{2,\delta}$ for a small δ depending on α.*

Theorem 1.1 is a consequence of Theorem 1.2, which shows that there are exactly two types of behavior near the origin.

THEOREM 1.2 *If $\alpha > 0$ and*

$$u(0) = 0, \quad \nabla u(0) = 0, \tag{1.4}$$

then there exist positive constants $c(\alpha)$ and $C(\alpha)$ depending on α such that either u has the radial behavior

$$c(\alpha)|x|^{2+\frac{\alpha}{2}} \le u(x) \le C(\alpha)|x|^{2+\frac{\alpha}{2}}, \tag{1.5}$$

or, in an appropriate system of coordinates, the nonradial behavior

$$u(x) = \frac{a}{(\alpha+2)(\alpha+1)}|x_1|^{2+\alpha} + \frac{1}{2a}\, x_2^2 + O\big((|x_1|^{2+\alpha} + x_2^2)^{1+\delta}\big) \tag{1.6}$$

for some $a > 0$.

The nonradial behavior (1.6) was first shown by P. Guan in [5] under the condition that $u_{x_2x_2} \ge C_0 > 0$ near the origin, and was recently generalized in [8] to assume only that $\Delta u \ge C_0 > 0$.

The next result states that the radial behavior is unstable.

THEOREM 1.3 *Suppose $\alpha > 0$, and let u_0 be the radial solution to* (1.1)*,*

$$u_0(x) = c_\alpha |x|^{2+\frac{\alpha}{2}}$$

and consider the Dirichlet problem

$$\det D^2 u = |x|^\alpha, \quad u = u_0 - \varepsilon\cos(2\theta) \ \text{ on } \partial B_1.$$

Then $u - u(0)$ has the nonradial behavior (1.6) *for small ε.*

Subsequences of blowup solutions satisfying (1.5) converge to homogeneous solutions, as shown next.

THEOREM 1.4 *Under the assumptions of Theorem* 1.2*, if u satisfies* (1.5)*, then for any sequence of $r_k \to 0$ the blowup solutions*

$$r_k^{-2-\frac{\alpha}{2}} u(r_k x)$$

have a subsequence that converges uniformly on compact sets to a homogeneous solution of (1.1).

In the case $-2 < \alpha < 0$ solutions have only the radial behavior. Actually, we prove a stronger result by showing that u converges to the radial solution u_0 in the following sense:

THEOREM 1.5 *If* $-2 < \alpha < 0$ *and* (1.4) *holds, then*

$$\lim_{x \to 0} \frac{u(x)}{u_0(x)} = 1.$$

Our results are based on the following argument: assume that a section of u, say $\{u < 1\}$, is "much longer" in the x_1-direction compared to the x_2-direction. If v is an affine rescaling of u so that $\{v < 1\}$ is comparable to a ball, then v is an approximate solution of

$$\det D^2 v(x) \approx c|x_1|^\alpha.$$

Hence, the geometry of small sections of solutions of this new equation provides information on the behavior of the small sections of u. For example, if the sections of v are "much longer" in the x_1-direction (case $\alpha > 0$), then the corresponding sections of u degenerate more and more in this direction, producing the nonradial behavior (1.6). If the sections of v are longer in the x_2-direction (case $\alpha < 0$), then the sections of u tend to become round and we end up with a radial behavior near the origin.

We close this introduction with the following remarks.

Remark 1.6. From the proofs one can see that the theorems above, with the exception of the instability result, are still valid for the equation with more general right-hand side

$$\det D^2 u = |x|^\alpha g(x)$$

with $g \in C^\delta(B_1)$, $g > 0$.

Remark 1.7.

(1) We will show in the proof of Theorem 1.1 that solutions of (1.1), with $\alpha > 0$, which satisfy the radial behavior (1.5) at the origin are of class $C^{2,\alpha/2}$.

(2) Theorems 1.1 and 1.2 and the results of Guan in [5] and Guan and Sawyer in [8] imply that solutions of (1.1), with α a positive integer, which satisfy the nonradial behavior (1.6) at the origin are C^∞ smooth.

Remark 1.8. Equations of the form

$$\det D^2 w = |\nabla w|^\beta, \quad \beta = -\alpha, \tag{1.7}$$

for which the set $\{\nabla w = 0\}$ is compactly included in the domain of definition, can be reduced to (1.1) by defining u to be the Legendre transform of w. Hence Theorem 1.5 establishes the sharp regularity of solutions w of equation (1.7) when $0 < \beta < 2$.

The paper is organized as follows. In Section 2 we introduce tools and notation to be used later in the paper. In Section 3 we prove Theorem 1.2. In Section 4 we establish the radial behavior of solutions when $-2 < \alpha < 0$, showing Theorem 1.5. In Section 5 we investigate homogeneous solutions and give the proof of Theorem 1.4. In Section 6 we prove Theorem 1.3. Finally, in Section 7 we show that Theorem 1.2 implies Theorem 1.1.

2 Preliminaries

In this section we investigate the geometry of the sections of u, namely the sets

$$S^u_{t,x_0} := \{u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + t\}.$$

We omit the indices u and x_0 whenever there is no possibility of confusion. We recall some facts about such sections.

John's lemma (cf. [9, theorem 1.8.2]) states that any bounded convex set $\Omega \subset \mathbb{R}^n$ is balanced with respect to its center of mass. That is, if Ω has center of mass at the origin, there exists an ellipsoid E (with center of mass 0) such that

$$E \subset \Omega \subset k(n)E$$

for a constant $k(n)$ depending only on the dimension n.

Sections S^u_{t,x_0} of solutions to Monge-Ampère equations with doubling measure μ on the right-hand side also satisfy a balanced property with respect to x_0. We recall the following definition:

DEFINITION 2.1 (Doubling Measure) The measure μ is *doubling* with respect to ellipsoids in Ω if there exists a constant $c > 0$ such that for any point $x_0 \in \Omega$ and any ellipsoid $x_0 + E \subset \Omega$

$$\mu(x_0 + E) \geq c\mu((x_0 + 2E) \cap \Omega). \tag{2.1}$$

The following theorem, due to L. Caffarelli [2], holds.

THEOREM 2.2 (Caffarelli) *Let $u : \Omega \to \mathbb{R}$ be a (Alexandrov) solution of*

$$\det D^2 u = \mu$$

with μ a doubling measure. Then for each $S_{t,x_0} \subset \Omega$ there exists a unimodular matrix A_t such that

$$k_0^{-1} A_t B_r \subset S_{t,x_0} - x_0 \subset k_0 A_t B_r \tag{2.2}$$

with

$$r = t\,(\mu(S_{t,x_0}))^{-\frac{1}{n}}, \quad \det A_t = 1,$$

for a constant $k_0(c, n) > 0$.

The ellipsoid $E = A_t B_r$ remains invariant if we replace A_t with $A_t O$ with O orthogonal; thus we may assume that A is triangular. If (2.2) is satisfied we write

$$S_t \sim A_t$$

and say that the eccentricity of S_t is proportional to $|A_t|$.

The measure that appears in (1.1), namely

$$\mu := |x|^\alpha \, dx$$

is clearly doubling with respect to ellipsoids for $\alpha > 0$. We will see in Section 4 that this property is still true for $-1 < \alpha < 0$ but fails for $-2 < \alpha \leq -1$.

Next we discuss the case when the right-hand side in the Monge-Ampère equation depends only on one variable, i.e.,

$$\det D^2 u = h(x_1). \tag{2.3}$$

We will show in Section 3 that such equations are satisfied by blowup limits of solutions to $\det D^2 u = |x|^\alpha$ at the origin, when $\alpha > 0$. These equations remain invariant under affine transformations. Also, by taking derivatives along the x_2-direction, one obtains the Pogorelov-type estimate

$$u_{22} \leq C$$

in the interior of the sections of u.

Assume that u satisfies equation (2.3) in $B_1 \subset \mathbb{R}^n$, in any dimension $n \geq 2$, and perform the following partial Legendre transformation:

$$y_1 = x_1, \quad y_i = u_i(x), \ \ i \geq 2, \quad u^*(y) = x' \cdot \nabla_{x'} u - u(x), \tag{2.4}$$

with $x' = (x_2, \ldots, x_n)$. The function u^* is obtained by taking the Legendre transform of u on each slice $x_1 = \text{const}$. We claim that u^* (which is convex in y' and concave in y_1) satisfies

$$u^*_{11} + h(y_1) \det D^2_{y'} u^* = 0. \tag{2.5}$$

To see this, we first notice that by the change of variable

$$v(x_1, x') \to u(x_1, x' + x_1 \xi')$$

v satisfies the same equation as u and

$$v^*(y) = u^*(y) - y_1 \, \xi' \cdot y'.$$

Thus we may assume that $D^2 u$ is diagonal at x. Now it is easy to check that

$$u^*_1 = -u_1, \quad \nabla_{y'} u^* = x',$$

and

$$u^*_{11} = -u_{11}, \quad D^2_{y'} u^* = [D^2_{x'} u]^{-1}.$$

Hence u^* satisfies (2.5).

Remark 2.3. The following hold:

(1) The partial Legendre transform of u^* is u, i.e., $(u^*)^* = u$.

(2) The inequality $|u - v| \le \varepsilon$ implies that $|u^* - v^*| \le \varepsilon$ on their common domain of definition.

(3) In dimension $n = 2$, the partial Legendre transform of the function

$$p(x_1, x_2) = a|x_1|^{2+\alpha} + bx_1x_2 + dx_2^2$$

is given by

$$\begin{aligned} p^*(y_1, y_2) &= (a|x_1|^{2+\alpha} + bx_1x_2 + dx_1^2)^* \\ &= -ay_1^{2+\alpha} + \frac{1}{4d}(y_2 - by_1)^2. \end{aligned} \tag{2.6}$$

Notice that p is a solution of the equation $\det D^2u = c|x_1|^\alpha$ for an appropriate constant c, and p^* is a solution of the equation $w_{11} + c|y_1|^\alpha w_{22} = 0$.

From now on we will restrict our discussion to dimension $n = 2$ and the special case where $h(x_1) = |x_1|^\alpha$.

LEMMA 2.4 *Assume that for some $\alpha > 0$, w solves the equation*

$$Lw := w_{11} + |y_1|^\alpha w_{22} = 0 \quad \textit{in } B_1 \subset \mathbb{R}^2$$

with $|w| \le 1$. Then in $B_{1/2}$, w satisfies

$$\begin{aligned} w(y) = {} & a_0 + a_1 \cdot y + a_2y_1y_2 \\ & + a_3\left(\frac{1}{2}y_2^2 - \frac{1}{(\alpha+2)(\alpha+1)}|y_1|^{2+\alpha}\right) + O((y_2^2 + |y_1|^{2+\alpha})^{1+\delta}) \end{aligned}$$

with $|a_i|$ and $O(\cdot)$ bounded by a universal constant and $\delta = \delta(\alpha) > 0$.

PROOF: First we prove that w_2 is bounded in the interior. Since $Lw_2 = 0$, the same argument applied inductively would imply that the derivatives of w with respect to y_2 of any order are bounded in the interior.

To establish the bound on w_2, we show that

$$L(Cw^2 + \varphi^2w_2^2) \ge 0 \tag{2.7}$$

for a smooth cutoff function φ, to be made precise later. Indeed, a direct computation shows that

$$L(w^2) = 2\,(w_1^2 + |y_1|^\alpha w_2^2)$$

and

$$\begin{aligned} L(\varphi^2w_2^2) &= L(\varphi^2)w_2^2 + \varphi^2L(w_2^2) + 2(\varphi^2)_1(w_2^2)_1 + 2|y_1|^\alpha(\varphi^2)_2(w_2^2)_2 \\ &= L(\varphi^2)w_2^2 + 2\varphi^2(w_{21}^2 + |y_1|^\alpha w_{22}^2) + 8(\varphi_1w_2)(\varphi w_{21}) \\ &\quad + 8|y_1|^\alpha(\varphi_2w_2)(\varphi w_{22}); \end{aligned}$$

hence

$$\begin{aligned} L(Cw^2 + \varphi^2w_2^2) \ge {} & 2C|y_1|^\alpha w_2^2 + 2\varphi^2(w_{21}^2 + |y_1|^\alpha w_{22}^2) \\ & + L(\varphi^2)w_2^2 + 8(\varphi_1w_2)(\varphi w_{21}) + 8|y_1|^\alpha(\varphi_2w_2)(\varphi w_{22}). \end{aligned}$$

By choosing the cutoff function φ such that $\varphi_1 = 0$ for $|y_1| \le \frac{1}{4}$, then

$$L(\varphi^2) \ge -C_1|y_1|^\alpha, \quad |\varphi_1 w_2| \le C_1|y_1|^{\frac{\alpha}{2}}|w_2|,$$

and we obtain (2.7) if C is large. Therefore w_2 is bounded in the interior by the maximum principle.

The equation $w_{11} + |y_1|^\alpha\, w_{22} = 0$ and the bound $|w_{22}| \le C$ imply the bound

$$|w_{11}| \le C\, |y_1|^\alpha.$$

Thus w_1 is bounded. The same estimates as above show that w_{12} and w_{122} are bounded as well. By Taylor's formula, namely,

$$f(t) = f(0) + f'(0)t + \int_0^t (t-s) f''(s) ds,$$

and the equation $Lw = 0$, we conclude that

$$w(y_1, 0) = w(0) + w_1(0) y_1 - \frac{w_{22}(0)}{(\alpha+2)(\alpha+1)} y_1^{2+\alpha} + O(|y_1|^{3+\alpha}),$$

$$w(y_1, y_2) = w(y_1, 0) + w_2(y_1, 0)\, y_2 + \frac{w_{22}(0)}{2}\, y_2^2 + O(|y_2|^3 + |y_1 y_2^2|),$$

and

$$w_2(y_1, 0) = w_2(0) + w_{12}(0) y_1 + O(|y_1|^{2+\alpha}),$$

from which the lemma follows. $\square$

Notation. By universal constants we understand positive constants that may also depend on the exponent α. Also, when there is no possibility of confusion, we use the letters c and C for various universal constants that change from line to line.

3 Proof of Theorem 1.2

Throughout this section we assume that $\alpha > 0$ and that u satisfies

$$u(0) = 0, \quad \nabla u(0) = 0.$$

We simply write S_t for the section $S^u_{t,0}$.

Let

$$\Gamma := \{|x_1|^{2+\alpha} + x_2^2 < 1\}$$

be the 1-section of $|x_1|^{2+\alpha} + x_2^2$ at 0. If a set Ω satisfies

$$(1-\theta)\Gamma \subset \Omega \subset (1+\theta)\Gamma$$

we write

$$\Omega \in \Gamma \pm \theta.$$

The following approximation lemma constitutes the basic step in the proof of Theorem 1.2.

LEMMA 3.1 *Assume that u in the section S_1 satisfies*

$$\det D^2 u = c f(x), \quad |f(x) - |x_1|^\alpha| \le \varepsilon, \tag{3.1}$$

and

$$S_1 \in \Gamma \pm \theta \tag{3.2}$$

with $\varepsilon \le \varepsilon_0$ and $\varepsilon^{1/8} \le \theta$, $\theta < 1$ small. Then, for some small universal t_0, we have

$$S_{t_0} \in A D_{t_0}(\Gamma \pm \theta\, t_0^\delta)$$

where

$$A := \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}, \quad D_{t_0} := \begin{pmatrix} t_0^{1/(2+\alpha)} & 0 \\ 0 & t_0^{1/2} \end{pmatrix},$$

and

$$|A - I| \le C\theta, \quad C \text{ universal.}$$

Moreover, the constant c in (3.1) *satisfies*

$$|c - 2(1+\alpha)(2+\alpha)| \le C\theta. \tag{3.3}$$

PROOF: We consider the solution

$$v := \frac{c^{1/2}}{[2(1+\alpha)(2+\alpha)]^{1/2}} (|x_1|^{2+\alpha} + x_2^2) \tag{3.4}$$

of the equation

$$\det D^2 v = c|x_1|^\alpha$$

and compute that

$$\det D^2(v + \sqrt{c\varepsilon}\,|x|^2) > c(|x_1|^\alpha + \varepsilon) \ge \det D^2 u \tag{3.5}$$

and

$$\det D^2(u + \sqrt{c\varepsilon}\,|x|^2) > c(f(x) + \varepsilon) \ge \det D^2 v \tag{3.6}$$

because $|f(x) - |x_1|^\alpha| \le \varepsilon$ by assumption.

We first notice that the assumption (3.2) implies that the constant c in equation (3.1) is bounded from above by a universal constant if ε_0 is small. This can be easily seen from equation (3.6), which, with the aid of the maximum principle, implies that $u + \sqrt{c\varepsilon}\,|x|^2 \ge v$ on $\{u = 1\}$ (notice that both v and $w = u + \sqrt{c\varepsilon}\,|x|^2$ satisfy $v(0) = w(0) = 0$ and $\nabla w(0) = \nabla v(0) = 0$). Since $\{u = 1\} \in \Gamma \pm \theta$, this readily gives a bound on c if we assume that θ is small.

We will next show that

$$\{v < 1\} \in \Gamma \pm 2\theta, \tag{3.7}$$

which implies the bound (3.3). Indeed, if

$$\{v < 1\} \subset (1 - 2\theta)\Gamma,$$

then $v > u + \tilde{c}\theta|x|^2$ on $\{u = 1\}$ for a universal $\tilde{c}$; thus

$$v > u + \sqrt{c\varepsilon}\,|x|^2 \quad \text{on } \{u = 1\}$$

since, by the assumptions of the lemma, $\sqrt{\varepsilon} < \varepsilon^{1/8} \leq \theta$ and $\varepsilon \leq \varepsilon_0$ with ε_0 sufficiently small. We conclude from the maximum principle (see (3.5)) that $v > u + \sqrt{c\varepsilon}\,|x|^2$ in S_1. This is a contradiction, since $u(0) = v(0) = 0$. If

$$(1 + 2\theta)S_1 \subset \{v < 1\},$$

then similarly we obtain $v + \sqrt{c\varepsilon}\,|x|^2 < u$ in S_1, a contradiction.

Let w be the solution of the problem

$$\det D^2 w = c x_1^2 \ \text{ in } S_1, \quad w = u \ \text{ on } \partial S_1.$$

By the maximum principle

$$w + \sqrt{c\varepsilon}(|x|^2 - 2) \leq u \leq w - \sqrt{c\varepsilon}(|x|^2 - 2);$$

thus

$$|w - u| \leq C\sqrt{\varepsilon}.$$

Also from (3.7) we obtain

$$|w - v| \leq C\theta.$$

Hence, by Remark 2.3, the corresponding partial Legendre transforms defined in Section 2 satisfy in $B_{1/2}$

$$|w^* - v^*| \leq C\theta, \tag{3.8}$$

$$|w^* - u^*| \leq C\sqrt{\varepsilon}, \quad u^*(0) = 0, \ \ \nabla u^*(0) = 0, \tag{3.9}$$

and w^* and v^* solve the same linear equation

$$w^*_{11} + c|y_1|^\alpha w^*_{22} = 0.$$

Using Lemma 2.4 for the difference $w^* - v^*$ together with (2.6), (3.4), (3.3), and (3.8) yields to

$$\begin{aligned} w^* = {} & -|y_1|^{2+\alpha} + \frac{1}{4}\,y_2^2 + a + b_1 y_1 + b_2 y_2 \\ & + \theta\big(c y_1 y_2 + d_1|y_1|^{2+\alpha} + d_2 y_2^2 + O((|y_1|^{2+\alpha} + y_2^2)^{1+\delta})\big) \end{aligned} \tag{3.10}$$

with the coefficients a, b_i, c, and d_i bounded by a universal constant.

From (3.9) we find that

$$w^*(0, y_2) \geq -C\sqrt{\varepsilon} \quad \text{and} \quad w^*(y_1, 0) \leq C\sqrt{\varepsilon}$$

since, from the convexity in y_2 and concavity in y_1 of u^*,

$$u^*(0, y_2) \geq 0 \quad \text{and} \quad u^*(y_1, 0) \leq 0.$$

This and (3.10) imply the bounds

$$|a| \leq C\varepsilon^{\frac{1}{2}}, \quad |b_1| \leq C\varepsilon^{\frac{1}{4}}, \quad |b_2| \leq C\varepsilon^{\frac{1}{4}}.$$

Thus, if $|y_1|^{2+\alpha} + y_2^2 \leq 10\,t_0$, then

$$w^* = -(1 - d_1\theta)\,|y_1|^{2+\alpha} + \left(\frac{1}{4} + d_2\theta\right)y_2^2 + c\theta y_1 y_2 + O(\varepsilon^{\frac{1}{4}} + \theta t_0^{1+\delta}).$$

Hence, by performing the partial Legendre transform on w^* (using that $(w^*)^* = w$ and (2.6)), we obtain

$$\left|w - [e_1|x_1|^{2+\alpha} + e_2\,(x_2 + e_3\,x_1)^2]\right| \le C(\varepsilon^{\frac{1}{4}} + \theta t_0^{1+\delta}) \tag{3.11}$$

for

$$|x_1|^{2+\alpha} + 4e_2^2\,(x_2 + e_3 x_1)^2 \le 10 t_0$$

with $|e_1 - 1|$, $|e_2 - 1|$, and $|e_3|$ bounded by $C\theta$.

We next observe that if $p(x) = e_1|x_1|^{2+\alpha} + e_2\,(x_2 + e_3\,x_1)^2$, then the function

$$\tilde{p}(y) := \frac{1}{t_0}\,p(Fy)$$

with F given by

$$F^{-1} := \begin{pmatrix} t_0^{-1/(2+\alpha)} & 0 \\ 0 & t_0^{-1/2} \end{pmatrix} \begin{pmatrix} e_1^{1/(2+\alpha)} & 0 \\ e_3 e_2^{1/2} & e_2^{1/2} \end{pmatrix} = D_{t_0}^{-1} A^{-1}$$

satisfies

$$\tilde{p}(y) = |y_1|^{2+\alpha} + y_2^2.$$

Hence, defining

$$\tilde{w}(y) = \frac{1}{t_0}\,w(Fy),$$

we conclude from (3.11) that

$$|\tilde{w}(y) - (|y_1|^{2+\alpha} + y_2^2)| \le C(\varepsilon^{\frac{1}{4}} t_0^{-1} + \theta t_0^{\delta}) \quad \text{for } |y_1|^{2+\alpha} + y_2^2 \le 2.$$

Since $|\tilde{w} - \tilde{u}| \le C\varepsilon^{1/2} t_0^{-1}$ (because $|w - u| \le C\varepsilon^{1/2}$), we find for $\varepsilon < \min(\theta^8, \varepsilon_0)$, with ε_0 small, that

$$\{\tilde{u} < 1\} \in \Gamma \pm \gamma$$

with

$$\gamma = C(\varepsilon^{\frac{1}{4}}\,t_0^{-1} + \theta\,t_0^{\delta}) \le \theta\,t_0^{\delta'}.$$

The proof is now complete since $S_{t_0} = F\{\tilde{u} < 1\} = A D_{t_0}\{\tilde{u} < 1\}$. □

The proof given above also proves the following lemma.

LEMMA 3.2 *Assume that u satisfies*

$$\det D^2 u = c f(x) \;\text{ on } S_1 \quad \text{and} \quad B_{1/k_0} \subset S_1 \subset B_{k_0}.$$

Then, given θ_0, there exist $\varepsilon_1(\theta_0, k_0)$ and $t_1(\theta_0, k_0)$ small such that if

$$\left|f(x) - |x_1|^{\alpha}\right| \le \varepsilon_1, \quad \text{then} \quad S_{t_1} \in A_0 D_{t_1}(\Gamma \pm \theta_0)$$

with

$$A_0 := \begin{pmatrix} a_{0,11} & 0 \\ a_{0,21} & a_{0,22} \end{pmatrix} \tag{3.12}$$

and

$$c(k_0) \le a_{0,ii} \le C(k_0), \quad |a_{0,12}| \le C(k_0),$$

for some universal constants $c(k_0)$ and $C(k_0)$.

The next proposition shows that if the section S_t has large eccentricity for some t, then u enjoys the nonradial behavior (1.6) at the origin.

PROPOSITION 3.3 *Assume that u solves the equation*

$$\det D^2 u = |x|^\alpha \quad \text{on } S_1$$

and that S_1 has large eccentricity, i.e.,

$$FB_{1/k_0} \subset S_1 \subset FB_{k_0}, \quad F := c\begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix}$$

with $b \geq C_0$. Then there exists a z-system of coordinates such that

$$u(z) = \frac{a}{(\alpha+2)(\alpha+1)}|z_1|^{2+\alpha} + \frac{1}{2a}z_2^2 + O\big((|z_1|^{2+\alpha} + z_2^2)^{1+\delta}\big) \tag{3.13}$$

for some $a > 0$.

Theorem 1.2 readily follows from Proposition 3.3. Indeed, since $|x|^\alpha\, dx$ is a doubling measure, we know from Theorem 2.2 that there exists k_0 depending on α such that each section $S_t \subset B_1$ satisfies

$$F_t B_{1/k_0} \subset S_t \subset F_t B_{k_0}$$

for some symmetric matrix F_t. If for any $t > 0$ the quotient between the largest and the smallest eigenvalue of F_t is greater than C_0^2, then we satisfy the hypothesis of the proposition above for a rescaling of u, and therefore obtain the nonradial behavior. In the case when the quotient is bounded from above by C_0^2 for all small $t > 0$, then we clearly obtain the radial behavior.

PROOF OF PROPOSITION 3.3: The proof will be based on an inductive argument, where at each step we use Lemma 3.1.

Define

$$v_1(x) := u(Fx),$$

and compute v_1 such that it satisfies the equation

$$\det D^2 v_1(x) = (\det F)^2 |Fx|^\alpha = c^{4+\alpha} b^\alpha |(x_1, b^{-2}x_2)|^\alpha.$$

Also,

$$\{v_1 < 1\} = F^{-1} S_1.$$

If b is large, then v_1 satisfies the hypothesis of Lemma 3.2. Hence, for some fixed θ_0 we obtain

$$S_{t_1} = F\{v_1 < t_1\} \in FA_0 D_{t_1}(\Gamma \pm \theta_0)$$

with A_0 satisfying (3.12).

We assume by induction that for $t = t_1 t_0^k$ we have

$$S_t \in FA_k D_t(\Gamma \pm \theta_0 t_0^{(k-1)\delta})$$

with

$$A_k := \begin{pmatrix} a_{k,11} & 0 \\ a_{k,21} & a_{k,22} \end{pmatrix}$$

and

$$\frac{c}{2} \le a_{k,ii} \le 2C, \quad |a_{k,21}| \le 2C. \tag{3.14}$$

We will show that

$$S_{t_0 t} \in FA_{k+1} D_{t_0 t}(\Gamma \pm \theta_0 t_0^{k\delta})$$

where

$$A_{k+1} = A_k E_k \quad \text{and} \quad E_k := \begin{pmatrix} e_{k,11} & 0 \\ e_{k,21} & e_{k,22} \end{pmatrix}$$

with

$$|e_{k,ii} - 1| \le C\theta_0 t_0^{(k-1)\delta}, \quad |e_{k,21}| t^{-\frac{\alpha}{2(2+\alpha)}} \le C\theta_0 t_0^{(k-1)\delta}. \tag{3.15}$$

Notice that condition (3.15) implies the bound

$$|A_{k+1} - A_k| \le C\theta_0 t_0^{(k-1)\delta}. \tag{3.16}$$

To prove this inductive step, we observe that the function

$$v_t(x) := t^{-1} u(FA_k D_t x)$$

satisfies in $\{v_t < 1\}$ the equation

$$\det D^2 v_t = c_t |\tilde{x}|^\alpha$$

with

$$|\tilde{x}|^\alpha = \left|\left(a_{k,11} t^{\frac{1}{2+\alpha}} x_1, b^{-2}(a_{k,21} t^{\frac{1}{2+\alpha}} x_1 + a_{k,22} t^{\frac{1}{2}} x_2)\right)\right|^\alpha = c_t' f_t(x)$$

and

$$\left| f_t - |x_1|^\alpha \right| \le b^{-2} t^{\frac{\alpha}{2(2+\alpha)}}.$$

Also,

$$\{v_t < 1\} \in \Gamma \pm \theta_0 t_0^{(k-1)\delta}$$

since $S_t \in FA_k D_t(\Gamma \pm \theta_0 t_0^{(k-1)\delta})$ by the inductive assumption. Hence, if $\delta = \delta(\alpha)$ is chosen small, then v_t satisfies the assumptions of Lemma 3.1, leading to

$$\{v_t < t_0\} \in \tilde{A} D_{t_0}(\Gamma \pm \theta_0 t_0^{k\delta})$$

with

$$|\tilde{A} - I| \le C\theta_0 t_0^{(k-1)\delta}. \tag{3.17}$$

Thus

$$S_{t_0 t} \in FA_k D_t \tilde{A} D_{t_0}(\Gamma \pm \theta_0 t_0^{k\delta}).$$

Defining E_k such that $D_t \tilde{A} = E_k D_t$, we see from (3.17) that E_k satisfies (3.15). We conclude the proof of the induction step by first choosing θ_0 small so that (3.12) and (3.16) imply that (3.14) is always satisfied.

Define

$$A^* := \lim_{k\to\infty} A_k.$$

We will next prove that

$$S_t \in FA^* D_t(\Gamma \pm C' t^\delta). \tag{3.18}$$

As before, let $t = t_1 t_0^k$. Notice that

$$A^* = A_k E_k^*, \quad E_k^* := \Pi_{i=k}^\infty E_i,$$

and it is straightforward to check from (3.15) that

$$|e^*_{k,ii} - 1| \le C_1 t^\delta, \quad |e^*_{k,12}| t^{-\frac{\alpha}{2(2+\alpha)}} \le C_1 t^\delta. \tag{3.19}$$

We have

$$A_k D_t = A^* (E_k^*)^{-1} D_t = A^* D_t \tilde{E}$$

with

$$|\tilde{e}_{k,ii} - 1| \le C_2 t^\delta, \quad |\tilde{e}_{k,12}| \le C_2 t^\delta.$$

Now (3.18) follows since

$$\tilde{E}(\Gamma \pm C_2 t^\delta) \subset \Gamma \pm C' t^\delta.$$

Finally, from (3.18) we see that

$$u(FA^* x) = |x_1|^{2+\alpha} + x_2^2 + O\big((|x_1|^{2+\alpha} + x_2^2)^{1+\delta}\big),$$

which implies that in a z-system of coordinates

$$u(z) = \beta_1 |z_1|^{2+\alpha} + \beta_2 z_2^2 + O\big((|z_1|^{2+\alpha} + z_2^2)^{1+\delta}\big).$$

The rescaled functions

$$r^{-1} u\big(r^{\frac{1}{2+\alpha}} z_1, r^{\frac{1}{2}} z_2\big)$$

converge, as $r \to 0$, to

$$\tilde{u}(z) := \beta_1 |z_1|^{2+\alpha} + \beta_2 z_2^2.$$

Moreover, this function solves the limiting equation

$$\det D^2 \tilde{u} = |z_1|^\alpha.$$

Hence

$$2(2+\alpha)(1+\alpha)\beta_1\beta_2 = 1,$$

which implies (3.13). $\square$

4 Negative Powers

In this section we consider the equation

$$\det D^2 u = |x|^\alpha \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{4.1}$$

in the negative range of exponents $-2 < \alpha < 0$. We will assume, throughout the section, that $0 \in \Omega$ and

$$u(0) = 0, \quad \nabla u(0) = 0.$$

Our goal is to prove the following proposition, which shows that solutions of equation (4.1) admit radial behavior only near the origin. This is in contrast with the case $0 < \alpha < \infty$, where both the radial behavior and the nonradial behavior (3.13) occur (see Proposition 3.3).

PROPOSITION 4.1 *There exist positive constants c and C (depending on u) such that*

$$c\,|x|^{2+\frac{\alpha}{2}} \le u(x) \le C|x|^{2+\frac{\alpha}{2}}$$

near the origin.

We distinguish two cases depending on whether the measure $|x|^\alpha\,dx$ is doubling with respect to all ellipsoids (see the discussion in Section 2).

Case 1: $-1 < \alpha < 0$. In this case the measure

$$\mu := |x|^\alpha\,dx$$

is doubling with respect to ellipsoids. Indeed, it suffices to show that there exists $c > 0$ such that for any ellipsoid E, we have

$$\mu(x_0 + E) \ge c\mu(x_0 + 2E). \tag{4.2}$$

Since $-1 < \alpha$, the density

$$(x_1^2 + x_2^2)^{\frac{\alpha}{2}}$$

is doubling on each line $x_2 = \text{const}$ with the doubling constant independent of x_2. This implies that the density $\mu = |x|^\alpha\,dx$ is doubling with respect to any line in the plane. From this and the fact that $x_0 + 2E$ can be covered with translates of $x_0 + \frac{E}{2}$ over a finite number of directions, we obtain (4.2).

From Theorem 2.2, there exists a matrix A_t such that $S_t \sim A_t$, i.e.,

$$k_0^{-1} A_t B_r \subset S_t \subset k_0 A_t B_r, \tag{4.3}$$

with

$$r = t(\mu(S_t))^{-\frac{1}{2}}, \quad \det A_t = 1.$$

In this case Proposition 4.1 follows from the lemma below.

LEMMA 4.2 *There exist universal constants $C > 0$ large and $\delta > 0$ such that if $S_t \sim A_t$ with $|A_t| > C$, then*

$$S_{\delta t} \sim A_{\delta t} \quad \textit{with } |A_{\delta t}| \le \frac{|A_t|}{2}. \tag{4.4}$$

In particular, $|A_t| \le C|A_{t_0}|$ if $t \le t_0$.

PROOF: We will use a compactness argument. Assume, by contradiction, that the conclusion of the lemma is not true. Then we can find a sequence of solutions u_k of (4.1) with sections $S^{u_k}_{t_k}$ at 0 such that $S^{u_k}_{t_k} \sim A^{u_k}_{t_k}$ with $|A^{u_k}_{t_k}| \to \infty$ and (4.4) does not hold for any $\delta > 0$.

Without loss of generality, we may assume that

$$A^{u_k}_{t_k} := \begin{pmatrix} a_k & 0 \\ 0 & a_k^{-1} \end{pmatrix}, \quad a_k \to \infty. \tag{4.5}$$

We renormalize the functions u_k as

$$v_k(x) := \frac{1}{t_k}\, u_k(r_k A^{u_k}_{t_k} x) \tag{4.6}$$

so that

$$\det D^2 v_k = c_k |A^{u_k}_{t_k} x|^\alpha = c_k' |x_1^2 + a_k^{-4} x_2^2|^{\frac{\alpha}{2}}$$

and

$$k_0^{-1} B_1 \subset S^{v_k}_1 \subset k_0 B_1.$$

Since $S^{u_k}_{t_k} \sim A^{u_k}_{t_k}$, i.e., $r_k = t_k(\mu(S^{u_k}_{t_k}))^{-1/2}$, the Monge-Ampère measure $\det D^2 v_k\, dx$ satisfies

$$\det D^2 v_k(S^{v_k}_1) = r_k^2 t_k^{-2} \mu(S^{u_k}_{t_k}) = 1.$$

Hence, as $k \to \infty$ we can find a subsequence of the v_k's that converge uniformly to a function v that satisfies

$$\det D^2 v = c|x_1|^\alpha\, dx \tag{4.7}$$

and

$$k_0^{-1} B_1 \subset S^v_1 \subset k_0 B_1, \quad \det D^2 v_k(S^v_1) = 1.$$

Obviously the constant c in (4.7) is bounded from above and below by universal constants. Since the right-hand side of (4.7) does not depend on x_2 and v is constant on ∂S^v_1, Pogorelov's interior estimate holds and we obtain the bound

$$v_{22} < C_1 \quad \text{in } (2k_0)^{-1} B_1.$$

This implies that the section S^v_δ contains a segment of size $\delta^{1/2}$ in the x_2-direction, namely,

$$\{x_1 = 0,\ |x_2| \le (\delta/C_1)^{\frac{1}{2}}\} \subset S^v_\delta. \tag{4.8}$$

From Theorem 2.2 there exists

$$A_\delta = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}, \quad 0 < a < C(\delta), \quad |b| \le C(\delta), \tag{4.9}$$

with

$$k_0^{-1} A_\delta B_r \subset S^v_\delta \subset k_0 A_\delta B_r \tag{4.10}$$

and

$$r = \delta[\det D^2 v(S^v_\delta)]^{-\frac{1}{2}}. \tag{4.11}$$

From (4.8) and (4.10) we have

$$\frac{r}{a} \geq c_1 \delta^{\frac{1}{2}}$$

while from (4.9), (4.10), and (4.11) we get

$$\delta^2 = r^2 \det D^2 v(S_\delta^v) \geq c_2 r^2 \frac{r}{a}(ar)^{1+\alpha}.$$

From the last two inequalities we obtain

$$a \leq C_2 \delta^{\frac{-\alpha}{4(2+\alpha)}} \leq \frac{1}{4} \quad \text{for } \delta \text{ small.} \tag{4.12}$$

Since the v_k's converge uniformly to v, their δ-sections also converge uniformly; thus

$$S_\delta^{v_k} \sim A_\delta \quad \text{for } k \text{ large}$$

and hence

$$S_{\delta t_k}^{u_k} \sim A_{t_k}^{u_k} A_\delta.$$

From (4.5), (4.9), and (4.12) we conclude

$$|A_{t_k}^{u_k} A_\delta| \leq \frac{|A_{t_k}^{u_k}|}{3} \quad \text{for } k \text{ large,}$$

which implies that the function u_k satisfies (4.4), a contradiction. □

Case 2: $-2 < \alpha \leq -1$. In this case the measure μ is not doubling with respect to any convex set but it is still doubling with respect to convex sets that have the origin as the center of mass.

We proceed as in the first case but replace the sections S_t with the sections T_t that have 0 as the center of mass. The existence of these sections follows from the following lemma, due to L. Caffarelli [3, lemma 2].

LEMMA 4.3 (Centered Sections) *Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a globally defined convex function (we set $u = \infty$ outside Ω). Also, assume u is bounded in a neighborhood of 0 and the graph of u does not contain an entire line. Then for each $t > 0$, there exists a "t-section" T_t centered at 0; that is, there exists p_t such that the convex set*

$$T_t := \{u(x) < u(0) + p_t \cdot x + t\}$$

is bounded and has 0 as a center of mass.

Using the lemma above, one can obtain Theorem 2.2 (similarly as in [2]), with S_t is replaced by T_t: *for every $T_t \subset \Omega$ as above, there exists a unitary matrix A_t such that*

$$k_0^{-1} A_t B_r \subset T_t \subset k_0 A_t B_r \tag{4.13}$$

with $r = t(\mu(T_t))^{-1/2}$. If (4.13) is satisfied, we write $T_t \sim A_t$.

Next we will show the analogue of Lemma 4.2 for this case.

LEMMA 4.4 *There exist universal constants $C > 0$ large and $\delta > 0$ such that if $T_t \sim A_t$ with $|A_t| > C$, then $T_{\delta t} \subset T_t$ and*

$$T_{\delta t} \sim A_{\delta t} \quad \text{with } |A_{\delta t}| \leq \frac{|A_t|}{2}. \tag{4.14}$$

PROOF: We argue similarly as in the proof of Lemma 4.2. We assume by contradiction that the conclusion does not hold for a sequence of functions u_k. Proceeding as in the proof of Lemma 4.2, we work with the renormalizations v_k of u_k defined by (4.6), which satisfy

$$\det D^2 v_k = c_k' |x_1^2 + a_k^{-4} x_2^2|^{\alpha/2} =: \mu_k$$

and

$$k_0^{-1} B_1 \subset T_1^{v_k} \subset k_0 B_1, \quad \det D^2 v_k(T_1^{v_k}) = 1.$$

As $k \to \infty$, we can find a subsequence of the v_k's that converge uniformly to a function v. Since $a_k \to \infty$ and $-2 < \alpha \leq -1$, the corresponding measures μ_k, when restricted to a line $x_2 = \text{const}$, converge weakly to the measure $c|x_2|^{1+\alpha}\delta_{\{x_1=0\}}$. This implies that the measures μ_k converge weakly to $c|x_2|^{1+\alpha}\, d\mathcal{H}^1_{\{x_1=0\}}$, where $d\mathcal{H}^1$ is the one-dimensional Hausdorff measure. The limit function v therefore satisfies

$$\begin{gathered} \det D^2 v = c|x_2|^{1+\alpha}\, d\mathcal{H}^1_{\{x_1=0\}}, \\ k_0^{-1} B_1 \subset T_1^v \subset k_0 B_1, \quad \det D^2 v_k(T_1^v) = 1. \end{gathered} \tag{4.15}$$

Clearly c is bounded from above and below by universal constants.

We notice that the measure $d\mathcal{H}^1_{\{x_1=0\}}$ is doubling with respect to any convex set with the center of mass on the line $\{x_1 = 0\}$. Using the same methods as in the case of the classical Monge-Ampère equation, one can show that the graph of v contains no line segments when restricted to $\{x_1 = 0\}$ (see Lemma 4.5 below). From this and the fact that v is the convex envelope of its restriction on ∂T_1^v and $\{x_1 = 0\}$ (see (4.15)), we conclude that there exist two supporting planes with slopes $\beta e_2 \pm \gamma e_1$ to the graph of v at 0. Moreover, it follows from the compactness of the equation (4.15) that γ can be chosen to be universal, and the sections T_δ^v satisfy

$$T_\delta^v \subset (2k_0)^{-1} B_1$$

when $\delta \leq \delta_0$, a universal constant. We have

$$T_\delta^v \subset \{|x_1| \leq c(\gamma)\delta\}. \tag{4.16}$$

Let A_δ be of the form (4.9) with

$$k_0^{-1} A_\delta B_r \subset T_\delta^v \subset k_0 A_\delta B_r$$

and

$$r = \delta[\det D^2 v(T_\delta^v)]^{-\frac{1}{2}} \sim \delta\left(\frac{r}{a}\right)^{1+\frac{\alpha}{2}}. \tag{4.17}$$

On the other hand, (4.16) implies

$$a\, r \le c\delta,$$

which together with (4.17) yields

$$a \le c\delta^{\frac{2+\alpha}{6+\alpha}} \le \frac{1}{4}$$

for δ small enough. Now the contradiction follows as in Lemma 4.2. □

LEMMA 4.5 *If v satisfies* (4.15)*, then*

$$v_0(t) := v(0,t)$$

is strictly convex.

PROOF: Assume that the conclusion does not hold. Then, after subtracting a linear function, we can assume that

$$v \ge 0 \quad \text{in } T_1^v$$

and

$$v_0(t) = 0 \text{ for } t \le 0, \quad v_0(t) > 0 \text{ for } t > 0.$$

Let

$$l_\varepsilon := \varepsilon t + a_\varepsilon$$

be such that

$$0 \in \{v_0 < l_\varepsilon\} = (b_\varepsilon, c_\varepsilon) \to 0, \quad \frac{c_\varepsilon}{|b_\varepsilon|} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.18}$$

We consider the linear function p_ε in $\mathbb{R}^2$ such that $\{u < p_\varepsilon\}$ has center of mass on $\{x_1 = 0\}$ and $p_\varepsilon = l_\varepsilon$ on $\{x_1 = 0\}$. We claim that for ε small, $\{u < p_\varepsilon\}$ is compactly included in T_1^v. Otherwise, the graph of v would contain a segment passing through 0; hence $v = 0$ is an open set that intersects the line $\{x_1 = 0\}$ and we contradict (4.15). Since $d\mathcal{H}^1_{\{x_1=0\}}$ is doubling with respect to the center of mass of $\{u < p_\varepsilon\}$, we conclude that this set is also balanced around 0, which contradicts (4.18). □

We are now in a position to exhibit the final steps of the proof of Proposition 4.1 in the case $-2 < \alpha \le -1$.

PROOF OF PROPOSITION 4.1: We choose t_0 small such that $T_{t_0} \subset \Omega$. The existence of t_0 follows from the fact that the graph of u cannot contain any line segments.

From Lemma 4.4 we conclude that there exists a large constant $K > 0$ depending on the eccentricity of T_{t_0} such that

$$T_t \sim A_t \quad \text{with } |A_t| \leq K \text{ for all } t \leq \delta t_0.$$

Claim. There exists γ depending on K such that $S_{\gamma t} \subset T_t$.

To show this, first observe that by rescaling we can assume that $t = 1$. We use the compactness of the problem for fixed K. If there exist a sequence $\gamma_k \to 0$ and functions u_k for which the conclusion does not hold, then the graph of the limiting function u_∞ (of a subsequence of $\{u_k\}$) contains a line segment. This is a contradiction since u_∞ solves the Monge-Ampère equation (4.1), which proves the claim.

If $t = 1$, then from simple geometrical considerations and the claim above we obtain

$$\gamma k_0^{-1} K^{-1} B_1 \subset S_\gamma \subset k_0 K B_1.$$

By rescaling, we find that S_t has bounded eccentricity for t small, and the proposition is proved. □

5 Homogenous Solutions and Blowup Limits

We will consider in this section homogeneous solutions of the equation

$$\det D^2 w(x) = |x|^\alpha \quad \text{in } \mathbb{R}^2$$

for $\alpha > -2$, namely, solutions of the form

$$w(x) = r^{2+\frac{\alpha}{2}} g(\theta) := r^\beta g, \quad \beta = 2 + \frac{\alpha}{2}.$$

In the polar system of coordinates

$$D^2 w(x) = r^{\beta-2} \begin{pmatrix} \beta(\beta-1)g & (\beta-1)g' \\ (\beta-1)g' & g'' + \beta g \end{pmatrix}.$$

Thus, the function g satisfies the following ODE:

$$\beta g(g'' + \beta g) - (\beta - 1)(g')^2 = \frac{1}{\beta - 1}. \tag{5.1}$$

We consider g as the new variable in a maximal interval $[a, b]$ where g is increasing, and define h on $[g(a), g(b)]$ as $g' = \sqrt{2h(g)}$. We have $g'' = h'(g)$; thus h satisfies

$$\beta t(h'(t) + \beta t) - 2(\beta - 1)h(t) = \frac{1}{\beta - 1}.$$

Solving for h we obtain

$$2h_c(t) = ct^{2(1-\frac{1}{\beta})} - \beta^2 t^2 - \frac{1}{(\beta-1)^2} \tag{5.2}$$

for some c positive.

The function g on $[a,b]$ is the inverse of

$$a+\int_{g(a)}^{\xi}\frac{1}{\sqrt{2h_c(t)}}\,dt,$$

and the length of the interval $[a,b]$ is given by

$$b-a=\int_{\{h_c>0\}}\frac{1}{\sqrt{2h_c(t)}}\,dt:=I_c. \tag{5.3}$$

Solutions of (5.1) are periodic, of period $2(b-a)$; thus a global solution g on the circle exists if and only if I_c equals $\frac{\pi}{k}$ for some integer k.

Next we investigate the existence of such solutions. First we notice that for any quadratic polynomial $f(s)=-l^2s^2+d_1s+d_2$ of opening $-2l^2$, we have

$$\int_{\{f>0\}}\frac{1}{\sqrt{f(s)}}\,ds=\frac{\pi}{l}. \tag{5.4}$$

Therefore if $\phi(s)$ denotes any convex function that intersects the parabola l^2s^2 at two points, and we set $f(s)=-l^2s^2+d_1s+d_2$, with d_1s+d_2 denoting the line through the intersection points between $\phi(s)$ and l^2s^2, then

$$\int_{\{\phi(s)-l^2s^2>0\}}\frac{1}{\sqrt{\phi(s)-l^2s^2}}\,ds\geq\int_{\{f>0\}}\frac{1}{\sqrt{f(s)}}\,ds=\frac{\pi}{l}.$$

If $\phi(s)$ is concave, we obtain the opposite inequality.

Applying the above to $h_c(s)$, we find that depending on the convexity of the first term in (5.2), we obtain that the integral I_c in (5.3) is less (or greater) than $\frac{\pi}{\beta}$ for $\beta<2$ (or $\beta>2$), i.e.,

$$I_c<\frac{\pi}{\beta}\ \text{ if }\beta<2\quad\text{and}\quad I_c>\frac{\pi}{\beta}\ \text{ if }\beta>2. \tag{5.5}$$

On the other hand, by performing the change of variable $t=s^{\beta/2}$ in the integral (5.3), we obtain the integral (5.4) with

$$f(s):=c_1s-4s^2-c_2s^{2-\beta}$$

for some positive constants c_1 and c_2 depending on c. Hence, depending on the convexity of the last term of f, the integral I_c is greater (or less) than $\frac{\pi}{2}$ for $\beta<2$ (or $\beta>2$), i.e.,

$$I_c>\frac{\pi}{2}\ \text{ if }\beta<2\quad\text{and}\quad I_c<\frac{\pi}{2}\ \text{ if }\beta>2. \tag{5.6}$$

Let $-2<\alpha<0$, or equivalently $1<\beta<2$. It follows from (5.5) and (5.6) that $\frac{\pi}{2}<I_c<\frac{\pi}{\beta}$; hence $I_c=\frac{\pi}{k}$ for an integral k only when $k=1$. This readily implies that the only homogeneous solution in this case is the radial one.

Assume next that $\alpha > 0$. We will now show that in this case, depending on the value of β, more homogeneous solutions may exist. To this end, denote by $c_0 = c_0(\alpha)$ the value of c for which the two functions

$$f_1(t) = ct^{2(1-\frac{1}{\beta})} \quad \text{and} \quad f_2(t) = \beta^2 t^2 + \frac{1}{(\beta-1)^2}$$

become tangent. When $c < c_0$, then the set where $h_c(t) > 0$ is empty. As $c \to c_0^+$ the set $\{t : h_c(t) > 0\}$ approaches the point t_0 at which the two functions $f_1(t)$ and $f_2(t)$ become tangent when $c = c_0$. Since $f_1'(t_0) = f_2'(t_0)$ when $c = c_0$, the point t_0 satisfies

$$2c\left(1 - \frac{1}{\beta}\right)t_0^{1-2/\beta} = 2\beta^2 t_0,$$

which implies that

$$c\left(1 - \frac{1}{\beta}\right)t_0^{-2/\beta} = \beta^2.$$

As $c \to c_0^+$, $f_1(t)$ behaves as its Taylor quadratic polynomial, namely,

$$f_1(t) \approx f(t_0) + f'(t_0)\,t^2 + \frac{f''(t_0)}{2}\,t^2$$

and

$$\frac{f''(t_0)}{2} = c\left(1 - \frac{1}{\beta}\right)\left(1 - \frac{2}{\beta}\right)t_0^{-2/\beta} = \beta^2\left(1 - \frac{2}{\beta}\right).$$

We conclude that, as $c \to c_0^+$, $(h_c)^+$ behaves as a quadratic polynomial of opening -4β, and thus I_c converges to $\pi/\sqrt{2\beta}$. Hence

$$\left(\frac{\pi}{\sqrt{2\beta}}, \frac{\pi}{2}\right) \subset \{I_c : c > c_0\} \quad \text{and also} \quad \{I_c : c > c_0\} \subset \left(\frac{\pi}{\beta}, \frac{\pi}{2}\right)$$

by (5.5) and (5.6).

Summarizing the discussion above yields the following:

PROPOSITION 5.1 *Homogenous solutions to* (1.1) *are periodic on the unit circle.*

(i) *If $-2 < \alpha < 0$, then the only homogeneous solution is the radial one.*
(ii) *If $\alpha > 0$, then there exists a homogeneous solution of principal period $\frac{2\pi}{k}$ if and only if*

$$\frac{\pi}{k} \in \{I_c,\ c > c_0(\alpha)\}.$$

In addition,

$$\left(\frac{\pi}{\sqrt{2\beta}}, \frac{\pi}{2}\right) \subset \{I_c,\ c > c_0\} \subset \left(\frac{\pi}{\beta}, \frac{\pi}{2}\right) \quad \textit{with } \beta = 2 + \frac{\alpha}{2}.$$

Using the proposition above, we will now prove Theorem 1.4. We begin with two useful remarks.

Remark 5.2. From (5.2) we see that any point in the positive quadrant can be written as $(t, \sqrt{2h_c})$ for a suitable c. Hence, given any point $x_0 \in \partial B_1$ and any positive symmetric unimodular matrix A, there exists a homogeneous solution w in a neighborhood of x_0 such that $D^2 w(x_0) = A$.

Remark 5.3. Equation (5.2) gives

$$[(h' + \beta t) + \beta(\beta - 1)t]\, t^{\frac{2}{\beta}-1} = c\left(1 - \frac{1}{\beta}\right);$$

hence $\Delta w[r^2 w_{rr}]^{2/\beta-1}$ is constant for any local homogeneous solution w. This quantity will play a crucial role in the proof of Theorem 1.4.

DEFINITION 5.4 For any solution u of equation (1.1), we define

$$J_u(x) := (\Delta u)(r^2 u_{rr})^\gamma, \quad \gamma := \frac{2}{\beta} - 1.$$

Remark 5.5. The quantity $J_u(x)$ remains invariant under the homogeneous scaling

$$v(x) = r^{-\beta} u(rx), \quad J_v(x) = J_u(rx).$$

We denote by J_0 the constant obtained when we evaluate J on the radial solution u_0 of (1.1).

PROPOSITION 5.6 *The function $|J_u - J_0|$ cannot have an interior maximum in $\Omega \setminus \{0\}$ unless it is constant.*

PROOF: We compute the linearized operator $u^{ij} M_{ij}$ for

$$M = \log J_u = \log(\Delta u) + \gamma \log(x_i x_j u_{ij})$$

at a point $x \in \Omega \setminus \{0\}$ where $J_u(x) \neq J_0$.

By choosing an appropriate system of coordinates and by rescaling, we can assume that $|x| = 1$ and $D^2 u$ is diagonal. By differentiating the equation (1.1) twice, we obtain

$$u^{ii} u_{kii} = \alpha x_k \tag{5.7}$$

and

$$u^{ii} u_{klii} = u^{ii} u^{jj} u_{kij} u_{lij} + \alpha(\delta_k^l - 2x_k x_l).$$

Since the linearized equation of each second derivative of u depends on $D^3 u$, $D^2 u$, and x, we see that

$$u^{ij} M_{ij} = H(D^3 u, D^2 u, x) \tag{5.8}$$

where H is a quadratic polynomial in $D^3 u$ for fixed $D^2 u > 0$ and x.

Let w denote the (local) homogeneous solution for which $D^2 u(x) = D^2 w(x)$. Since $M_w = \log J_w$ is constant, we have

$$H(D^3 w, D^2 w, \cdot) = 0$$

in a neighborhood of x.

Claim. We have

$$\|D^3u(x) - D^3w(x)\| \le C|\nabla M|$$

with the constant C depending on D^2u and x.

To prove this claim, we obtain from (5.7) and the equalities

$$M_k = \frac{u_{iik}}{\Delta u} + \gamma\,\frac{x_i x_j u_{ijk} + 2x_i u_{ik}}{x_i x_j u_{ij}}$$

the following system for the third derivatives of u:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ b_1 & d_1 & b_2 & 0 \\ 0 & b_1 & d_2 & b_2 \end{pmatrix} \begin{pmatrix} \frac{u_{111}}{u_{11}} \\ \frac{u_{112}}{u_{11}} \\ \frac{u_{221}}{u_{22}} \\ \frac{u_{222}}{u_{22}} \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ M_1 - 2\gamma\frac{x_1u_{11}}{u_{rr}} \\ M_2 - 2\gamma\frac{x_2u_{22}}{u_{rr}} \end{pmatrix}$$

and

$$b_i = \frac{u_{ii}}{\Delta u} + \gamma\,\frac{x_i^2 u_{ii}}{u_{rr}}, \qquad d_i = 2\gamma u_{ii}\,\frac{x_1x_2}{u_{rr}}.$$

The third-order derivatives of w solve the same system but with no dependence on M in the right-hand side vector (since the corresponding M for w is constant).

It is enough to show that the determinant of the third-order derivatives coefficient matrix above is positive. This determinant is equal to

$$d_1d_2 + (b_1 - b_2)^2 = 4\gamma^2\left(\frac{x_1x_2}{u_{rr}}\right)^2 + (b_1 - b_2)^2$$

and can vanish only if one of the coordinates, say, $x_2 = 0$, and $b_1 = b_2$, i.e.,

$$u_{11}^2 = \frac{1-\gamma}{1+\gamma} = \beta - 1.$$

This implies that $J(x) = J_0$, which is a contradiction. Thus, the determinant is positive and the claim is proved.

Since H depends quadratically on D^3u and $D^2u = D^2w$ at x, the claim above implies that

$$\begin{aligned} |H(D^3u, D^2u, x)| &= |H(D^3u, D^2u, x) - H(D^3w, D^2u, x)| \\ &\le C(x, D^2u)(|\nabla M| + |\nabla M|^2). \end{aligned}$$

Hence (5.8) implies that on the set where $J(x) \ne J_0$, there exists a smooth function $C(x)$ depending on u such that

$$|u^{ij}M_{ij}| \le C(x)(|\nabla M| + |\nabla M|^2).$$

From the strong maximum principle, we conclude that M cannot have a local maximum or minimum in this set unless it is constant. With this the proposition is proved. □

Theorem 1.4 will follow from the proposition below.

PROPOSITION 5.7 *Suppose that u is a solution u of* (1.1)*, with $\alpha > -2$, which satisfies*

$$c|x|^{\beta} \le u(x) \le C|x|^{\beta}, \quad \beta = 2 + \frac{\alpha}{2}. \tag{5.9}$$

Then the limit

$$J_u(0) := \lim_{x \to 0} J_u(x)$$

exists. Moreover, if for a sequence of $r_k \to 0$ the blowup solutions

$$v_{r_k} := r_k^{-\beta} u(r_k x)$$

converge uniformly on compact sets to the solution w, then w is homogeneous of degree β with $J_w = J_u(0)$.

PROOF: From (5.9) we find that as $x \to 0$, $J_u(x)$ is bounded away from 0 and ∞ by constants depending on c and C. We will first show that $\lim_{x \to 0} J_u(x) = J(0)$ exists.

We may assume, without loss of generality, that

$$\limsup_{x \to 0} J_u(x) := k > J_0.$$

Let x_i be a sequence of points for which lim sup is achieved. The blowup solutions v_{r_i}, $r_i = |x_i|$, have a subsequence that converges uniformly on compact sets of $\mathbb{R}^2$ to a solution v. Moreover, there exists a point y on the unit circle for which

$$J_v(y) = k \ge \limsup_{x \to 0} J_v;$$

hence, by Proposition 5.6, J_v is constant.

This argument also shows that if

$$J_u(z) \le k - \varepsilon \quad \text{then} \quad J_u(x) \le k - \delta(\varepsilon) \text{ on the circle } |x| = |z|.$$

Thus, if there exists a sequence of points $y_j \to 0$ with

$$\lim_{y_j \to 0} J_u(y_j) < k,$$

then J_u would have an interior maximum in the annulus $\{x : |y_j| \le |x| \le |y_{j'}|\}$ that contains one of the points x_i given above, a contradiction. This shows that $\lim_{x \to 0} J_u(x)$ exists.

It remains to prove that if J_v is constant, then v is homogeneous. It suffices to show that $D^2 v$ is homogeneous of degree $\beta - 2$, or more precisely that for each second derivative v_{ij}, we have

$$x \cdot \nabla v_{ij} = (\beta - 2) v_{ij}. \tag{5.10}$$

To this end, for a fixed point x, we consider the homogeneous solution w with $D^2 w(x) = D^2 v(x)$. Since

$$\nabla J_v(x) = \nabla J_w(x) = 0,$$

the third derivatives of v and w solve the same system. We have seen in the proof of Proposition 5.6 that this system is solvable provided $J_v \neq J_0$. Thus $D^3 v(x) = D^3 w(x)$ if $J_v \neq J_0$. Since (5.10) is obviously true for w, this implies that the equality holds for u as well.

If $J_v = J_0$, we denote by Γ the set where $D^2 u(x)$ does not coincide with the Hessian of the radial solution. From the proof of Proposition 5.6, we still obtain $D^3 v(x) = D^3 w(x)$ if $x \in \Gamma$, and by continuity (5.10) holds for $x \in \bar{\Gamma}$. If x is in the open set $\bar{\Gamma}^c$, then $D^2 v$ coincides with $D^2 u_0$ and (5.10) is again satisfied. This finishes the proof of the proposition. □

PROOF OF THEOREM 1.4: The proof of the theorem readily follows from Propositions 4.1, 5.1, and 5.7. □

6 Proof of Theorem 1.3

We consider the Dirichlet problem

$$\begin{cases} \det D^2 u = |x|^\alpha & \text{in } B_1 \\ u = u_0 - \varepsilon \cos(2\theta) & \text{on } \partial B_1 \end{cases} \tag{6.1}$$

in the range of exponents $\alpha > 0$. Here

$$u_0(x) = c_\alpha \, |x|^\beta, \quad \beta = 2 + \frac{\alpha}{2},$$

denotes the radial solution of the equation, i.e., $\det D^2 u_0 = |x|^\alpha$. We write the solution as

$$u = u_0 - \varepsilon v. \tag{6.2}$$

Heuristically, if ε is small, v satisfies the linearized equation at u_0, namely,

$$(D^2 u_0)^{-1} : D^2 v = 0,$$

where we use the notation $A : B = \sum_{ij} a_{ij} b_{ij}$ for the Frobenius inner product between two $n \times n$ matrices A and B.

At any point $x_0 \in B_1$, we denote by ν and τ the unit normal (radial) and unit tangential direction, respectively, to the circle $|x| = |x_0|$ at x_0. In (ν, τ) coordinates,

$$D^2 u_0 = c_\alpha r^{\beta - 2} \begin{pmatrix} \beta(\beta - 1) & 0 \\ 0 & \beta \end{pmatrix};$$

hence v satisfies the equation

$$v_{\nu\nu} + (\beta - 1) v_{\tau\tau} = 0.$$

Solving this equation with boundary data $v = \cos(2\theta)$, we obtain the solution

$$v = r^\rho \cos(2\theta)$$

with

$$\rho(\rho - 1) + (\beta - 1)(\rho - 4) = 0.$$

Solving the quadratic equation with respect to ρ gives

$$\rho = \frac{2-\beta \pm \sqrt{\beta^2+12\beta-12}}{2}.$$

Since $\beta := 2+\frac{\alpha}{2} > 2$, the only acceptable solution is

$$\rho = \frac{2-\beta + \sqrt{\beta^2+12\beta-12}}{2},$$

and it satisfies

$$2 < \rho < \beta, \tag{6.3}$$

which suggests that close to the origin the perturbation term εv dominates u_0.

We wish to show that the solution u of the Dirichlet problem (6.1) admits at the origin the nonradial behavior (1.6) if $\varepsilon \le \varepsilon_0$ with ε_0 sufficiently small. We will argue by contradiction.

Assume that u has the radial behavior

$$c_0\,|x|^\beta \le u(x) \le C_0|x|^\beta$$

with c_0 and C_0 universal constants. By rescaling, we deduce that

$$cI \le |x|^{2-\beta} D^2u(x) \le CI$$

with I denoting the identity matrix.

The function v that is defined by (6.2) satisfies

$$|v| \le 1, \quad v = \cos(2\theta) \text{ on } \partial B_1,$$

and solves the equation $a^{ij}v_{ij} = 0$ with

$$\begin{aligned} A = (a^{ij}) &= \int_0^1 (tD^2u_0 + (1-t)D^2u)^{-1}\,dt \\ &= \int_0^1 (D^2u_0 + \varepsilon(t-1)D^2v)^{-1}\,dt. \end{aligned}$$

Hence

$$cI \le r^{\beta-2}A \le CI. \tag{6.4}$$

The solution u has bounded third-order derivatives in $B_1 \setminus B_{1/2}$; thus

$$|D^2v(x)| \le C\|v\|_{L^\infty} \le C \quad \text{in } B_1 \setminus B_{1/2}.$$

By rescaling we obtain the bound

$$|D^2v(x)| \le C|x|^{-2}.$$

From this we find that

$$r^{\beta-2}|A - D^2u_0^{-1}| \le C\varepsilon r^{-\beta};$$

hence v satisfies the Dirichlet problem

$$\begin{cases} f^{ij} v_{ij} = 0 & \text{in } B_1 \\ v = \cos(2\theta) & \text{on } \partial B_1 \end{cases} \tag{6.5}$$

with

$$F := c r^{\beta-2} A.$$

Hence by (6.4),

$$cI \le F \le CI \quad \text{and} \quad |F - F_0| \le C\varepsilon r^{-\beta}$$

with

$$F_0 := \nu \otimes \nu + (\beta - 1)\, \tau \otimes \tau.$$

(As before, we denote by ν and τ the unit normal (radial) and unit tangential directions to the circle $|x| = |x_0|$ at each point $x_0 \in B_1$.)

Also,

$$|v| \le 1 \quad \text{on } B_1.$$

From the definitions of A and F we also obtain

$$\|\nabla(F - F_0)\| \le C(r_0)\varepsilon \quad \text{for } |x| \ge r_0. \tag{6.6}$$

Set

$$w := r^\rho \cos(2\theta).$$

Then w satisfies the equation

$$F_0 : D^2 w = 0;$$

thus we have

$$|f^{ij}\, w_{ij}| \le C r^{\rho-2} \min\{\varepsilon r^{-\beta}, 1\}.$$

Applying the Aleksandrov maximum principle on $v - w$ (see [4, theorem 9.1]), we find that

$$|v - w| \le C\varepsilon^\delta$$

and therefore (see (6.6))

$$|D^2 v - D^2 w| \le C'(r_0)\varepsilon^\delta \quad \text{for } |x| \ge r_0. \tag{6.7}$$

We next compute

$$M_u(x) := \log(\Delta u) + \gamma \log(r^2 u_{rr}), \quad \gamma := \frac{2}{\beta} - 1,$$

in terms of M_{u_0} for $|x| \ge r_0$ with r_0 small and fixed. We recall that M_{u_0} is constant in x. Since $u = u_0 - \varepsilon v$, we find that

$$\begin{aligned} M_u(x) = M_{u_0} &- \varepsilon\left(\frac{\Delta v}{\Delta u_0} + \gamma\, \frac{v_{rr}}{u_{0,rr}}\right) \\ &- \frac{\varepsilon^2}{2}\left(\left(\frac{\Delta v}{\Delta u_0}\right)^2 + \gamma\left(\frac{v_{rr}}{u_{0,rr}}\right)^2\right) + O(\varepsilon^3). \end{aligned}$$

Because

$$\det D^2 u = \det D^2 u_0,$$

the function v satisfies the equation

$$-u_{0,rr}\, v_{\tau\tau} - u_{0,\tau\tau}\, v_{rr} + \varepsilon \det D^2 v = 0$$

or, equivalently (since $u_0(r) = c_\alpha\, r^\beta$)

$$v_{rr} + (\beta - 1) v_{\tau\tau} = \varepsilon \, \frac{r^{2-\beta}}{c_\alpha \beta} \det D^2 v.$$

The last equality implies that

$$\frac{\Delta v}{\Delta u_0} + \gamma \, \frac{v_{rr}}{u_{0,rr}} = \varepsilon \, \frac{r^{2(2-\beta)}}{c_\alpha^2 \beta^3 (\beta - 1)} \det D^2 v,$$

and also that

$$\begin{aligned}\left(\frac{\Delta v}{\Delta u_0}\right)^2 + \gamma \left(\frac{v_{rr}}{u_{0,rr}}\right)^2 &= \left(1 + \frac{1}{\gamma}\right)\left(\frac{\Delta v}{\Delta u_0}\right)^2 + O(\varepsilon) \\ &= -\frac{2 r^{2(2-\beta)}}{c_\alpha^2 \beta^2 (\beta - 2)} (\Delta v)^2 + O(\varepsilon).\end{aligned}$$

From (6.7) and the above we conclude that

$$M_u(x) = M_{u_0} + \varepsilon^2 r^{2(2-\beta)} [a_1(-\det D^2 w) + a_2 (\Delta w)^2] + O(\varepsilon^{2+\delta})$$

for $|x| \geq r_0$, with $O(\varepsilon^{2+\delta})$ depending on r_0. The constants a_1 and a_2 are given by

$$a_1 = \frac{1}{c_\alpha^2 \beta^3 (\beta - 1)} \quad \text{and} \quad a_2 = \frac{2}{c_\alpha^2 \beta^2 (\beta - 2)}.$$

We recall that $w(r, \theta) = r^\rho \cos(2\theta)$. Then, a direct computation shows that each term in the square brackets above is positive. Thus the ε^2-term is positive and homogeneous of degree $2(\rho - \beta)$ with $\rho < \beta$ (as shown in (6.3)). We conclude from Proposition 5.6 that

$$\lim_{x \to 0} M_u(x) > M_{u_0}.$$

Hence, from Proposition 5.7, the blowup limit of u at the origin cannot be u_0. On the other hand, from the symmetry of the boundary data for u, we conclude that the function $v - v(0)$ has exactly two disconnected components where it is positive (or negative). Thus the blowup limit at the origin for u has period π on the unit circle, which contradicts Proposition 5.1. $\square$

7 Proof of Theorem 1.1

In this final section we will present the last steps of the proof of Theorem 1.1. We distinguish the two different cases of behavior at the origin, (1.5) and (1.6).

Case 1: *Radial Behavior*. We will show that solutions of (1.1) with the radial behavior (1.5) are $C^{2,\alpha/2}$.

We begin by observing that solutions of (1.1) satisfy, in $B_1 \setminus B_{1/2}$, the estimate

$$\|D^2u\|_{C^{0,1}(B_1\setminus B_{1/2})} \le C(\alpha) \tag{7.1}$$

provided that

$$c(\alpha)|x|^{2+\frac{\alpha}{2}} \le u(x) \le C(\alpha)|x|^{2+\frac{\alpha}{2}}. \tag{7.2}$$

For any $r > 0$, the rescaled functions

$$u^r(x) := r^{-2-\frac{\alpha}{2}}\, u(rx) \tag{7.3}$$

solve the equation (1.1). Since u has the radial behavior (1.5) at the origin, each function u^r satisfies (7.2). Hence, applying (7.1) to u^r, we obtain for $x, y \in B_1 \setminus B_{1/2}$ the estimates

$$|D^2u(rx) - D^2u(ry)| \le r^{\frac{\alpha}{2}}|x-y|, \quad |D^2u(rx)| \le Cr^{\frac{\alpha}{2}}.$$

The above estimates readily imply that $u \in C^{2,\alpha/2}$.

Case 2: *Nonradial Behavior*. In the rest of the section we will show that solutions of (1.1) that satisfy the nonradial behavior (1.6) are also of class $C^{2,\delta}$ for some $\delta > 0$. The idea is simple: we approximate u with quadratic polynomials in the x_2-direction. However, the proof is quite technical.

In order to simplify the constants, we assume that u solves the equation

$$\det D^2u = 2(2+\alpha)(1+\alpha)|x|^\alpha \tag{7.4}$$

instead of (1.1) and (after rescaling) that

$$u(x) = |x_1|^{2+\alpha} + x_2^2 + O\big((|x_1|^{2+\alpha} + x_2^2)^{1+\delta}\big) \quad \text{as } |x| \to 0. \tag{7.5}$$

From now on, we will denote points in $\mathbb{R}^2$ with capital letters, $X = (x_1, x_2)$.

The Hölder continuity of the second-order derivatives of u follows easily from the next proposition.

PROPOSITION 7.1 *Let* $\lambda > 0$ *be small and*

$$Y \in \Omega_\lambda := \{\lambda \le |x_1|^{2+\alpha} + x_2^2 \le 2\lambda\}.$$

Then there exist universal constants C *and* μ *such that in* $B := B(Y, \lambda^{1+\alpha})$*, we have*

$$\|D^2u\|_{C^\mu(B)} \le C \quad \textit{and} \quad \|D^2u - D^2u(0)\|_{L^\infty(B)} \le \lambda^\mu.$$

We will show that in the sections

$$S_{X_0,t} := \{X : u(X) < u(X_0) + \nabla u(X_0) \cdot (X - X_0) + t\}$$

of u at the point

$$X_0 = (0, x_0), \quad |x_0| \le 2\lambda^{\frac{1}{2}},$$

we can approximate u by quadratic polynomials of opening 2 on vertical segments. We begin by making the following definition:

DEFINITION 7.2 We say that $u \in Q(e, \varepsilon, \Omega)$ if for any vertical segment $l \subset \Omega$ of length less than e, there exists a quadratic polynomial $P_{x_1,l}(x_2)$ of opening 2, namely

$$P_{x_1,l}(x_2) = x_2^2 + p(x_1, l)x_2 + r(x_1, l)$$

such that

$$|u(x_1, x_2) - P_{x_1,l}(x_2)| \le \varepsilon e^2 \quad \text{on } l.$$

Notice that for $c < 1$ we have

$$Q(e, \varepsilon, \Omega) \subset Q(ce, c^{-2}\varepsilon, \Omega).$$

The plan of the proof of Theorem 1.1 is as follows: We prove Proposition 7.1 for points $Y \in S_{X_0,t}$ with $t \le \lambda$. We first show that u belongs to some appropriate Q classes and distinguish two cases one when $t \ge \lambda^{\alpha/2+1-\delta_1}$ for some fixed $\delta_1 > 0$, and the other when $t = \lambda^{\alpha/2+1-\delta_1}$. In the first case we use the same method as in Lemma 3.1 and approximate the right-hand side $|f(X)|^{\alpha/2}$ of the rescaled Monge-Ampère equation with $|x_1|^\alpha$ (see Lemma 7.3). In the second case we approximate $f(X)$ with a more general polynomial $x_1^2 + px_1 + q$ and obtain a better approximation (Q class) for u (Lemma 7.4).

The Hölder estimates for points $Y \in S_{X_0,t}$, $|x_0| \ge \lambda^{1/2}$, are obtained in appropriate sections $S_{Y,\sigma}$ in which all the values of $|x|$ are comparable. In these sections the Monge-Ampère equation is nondegenerate and the classical estimates apply. To obtain the appropriate section $S_{Y,\sigma}$, we distinguish two cases, depending on the distance from Y to the x_2-axis. If $|y_1| \ge \lambda^{1/2}$, then we take σ so that $S_{Y,\sigma}$ is at a distance greater than $|y_1|/2$ from the x_2-axis (Lemma 7.5). If $|y_1| \le \lambda^{1/2}$, then we take $\sigma = \lambda^{(2+\alpha)/2}$ and $S_{Y,\sigma}$ is close enough to the x_2-axis so that all its points are at a distance comparable to $\lambda^{1/2}$ from the origin (Lemma 7.6).

In what follows we will denote by A_t and D_t the matrices

$$A_t = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}, \quad D_t = \begin{pmatrix} t^{1/(2+\alpha)} & 0 \\ 0 & t^{1/2} \end{pmatrix}.$$

LEMMA 7.3 *Let* $X_0 = (0, x_0)$ *with* $|x_0| \le 2\lambda^{1/2}$, $0 < \lambda < 1$. *Then, for any* $\delta_1 > 0$ *and*

$$\lambda^{\frac{\alpha}{2}+1-\delta_1} \le t \le \lambda, \tag{7.6}$$

there exists a small $\delta_2 > 0$, depending on δ_1, such that

$$S_{X_0,t} - X_0 \in A_t D_t (\Gamma \pm t^{\delta_2}) \tag{7.7}$$

with

$$|A_t - I| \le t^{\delta_2}. \tag{7.8}$$

Moreover,

$$u \in Q(t^{\frac{1}{2}}, \lambda^{\delta_2}, S_{X_0,t}).$$

PROOF: We begin by observing that if $t = \lambda$, then the conclusion of the lemma follows from the expansion (7.5) with matrix $A_t = I$. We will show by induction, using at each step the approximation Lemma 3.1, that (7.7) and (7.8) hold for every $t = \lambda\, t_0^k$, $k \in \mathbb{N}$, which satisfies (7.6).

Assume that (7.7) and (7.8) hold for some $t = \lambda\, t_0^k$ satisfying (7.6), with A_t bounded and $a_{t,11}$ bounded from below. Consider the rescaling

$$v(X) := \frac{1}{t}\big(u(X_0 + A_t D_t X) - u(X_0) - \nabla u(X_0)(A_t D_t X)\big). \tag{7.9}$$

Since u satisfies (7.4), the function v satisfies the equation

$$\det D^2 v = 2(2+\alpha)(1+\alpha) a_{11}^2 a_{22}^2 t^{-\frac{\alpha}{2+\alpha}} |X_0 + A_t D_t X|^\alpha. \tag{7.10}$$

Since

$$|X_0 + A_t D_t X|^2 = (t^{\frac{1}{2+\alpha}} a_{11} x_1)^2 + (t^{\frac{1}{2+\alpha}} a_{12} x_1 + t^{\frac{1}{2}} a_{22} x_2 + x_0)^2 \tag{7.11}$$

and $|x_0| \le 2\,\lambda^{1/2}$, we conclude from the above that v satisfies

$$\det D^2 v = c|f(X)|^{\frac{\alpha}{2}}, \quad S_{0,1}^v \in \Gamma \pm t^{\delta_2}, \tag{7.12}$$

with

$$|f(X) - x_1^2| \le C\big(\lambda^{\frac{1}{2}} t^{-\frac{1}{2+\alpha}} + t^{\frac{\alpha}{2(2+\alpha)}}\big) \le t^{\frac{\delta_1}{2(2+\alpha)}}.$$

Notice that the last inequality holds if (7.6) is satisfied.

Lemma 3.1 with $\varepsilon = t^{\delta'}$ and $\delta'(\delta_1, \alpha) > 0$ small yields

$$S^u_{X_0, t_0 t} - X_0 \in A_{t_0 t} D_{t_0 t} (\Gamma \pm (t_0 t)^{\delta_2})$$

with

$$A_{t_0 t} = A_t E_t, \quad |E_t - I| \le C t^{\delta_2}.$$

Thus, (7.7) and (7.8) hold for $t' = t t_0$. If $t' \le \lambda^{\alpha/2 + 1 - \delta_1}$, we stop; otherwise we continue the induction.

From (7.12) we find that

$$\big|v - (|x_1|^{2+\alpha} + x_2^2)\big| \le C t^{\delta_2} \quad \text{in } S^v_{0,1} \tag{7.13}$$

which, together with (7.9) and (7.8), yields

$$u \in Q(t^{\frac{1}{2}}, C\lambda^{\delta_2}, S^u_{X_0,t}).$$

The lemma is proved by replacing δ_2 with $\delta_2/2$. □

We next examine the borderline case $t = \lambda^{\alpha/2+1-\delta_1}$ and show the better approximation (7.15) of u by quadratic polynomials in the x_2-variable.

We begin by observing that the conclusion of the previous lemma implies that

$$S_{X_0,t} - X_0 \in A_t D_t(\Gamma \pm \lambda^{\delta_2}), \quad |A_t - I| \le \lambda^{\delta_2},$$

for all $\lambda^{\alpha/2+1-\delta_1} \le t \le \lambda$.

LEMMA 7.4 *Assume that, for $t = \lambda^{\alpha/2+1-\delta_1}$ and $\delta_2 \ll \delta_1$, we have*

$$S_{X_0,t} - X_0 \in A_t D_t(\Gamma \pm \lambda^{\delta_2}), \quad |A_t - I| \le \lambda^{\delta_2}. \tag{7.14}$$

Then if δ_1 is small and universal, we have

$$u \in Q(e, C\lambda^{\delta_2}, S_{X_0,\frac{t}{2}}) \quad \text{for all } e \text{ with } \lambda^{\frac{2+\alpha}{4}} \le e \le t^{\frac{1}{2}}. \tag{7.15}$$

PROOF: Let v be the rescaling defined in (7.9). It follows from (7.10), (7.11), and (7.14) that v satisfies

$$\det D^2 v = cf(X)^{\frac{\alpha}{2}}, \quad S^v_{0,1} \in \Gamma \pm \lambda^{\delta_2},$$

with

$$|f(X) - x_1^2 - px_1 - q| \le t^{\frac{\alpha}{2(2+\alpha)}}, \quad |p|, |q| \le \lambda^{\frac{\delta_1}{2+\alpha}};$$

thus

$$\left| f(X)^{\frac{\alpha}{2}} - (x_1^2 + px_1 + q)^{\frac{\alpha}{2}} \right| \le \varepsilon := t^{\delta_0(\alpha)}, \quad \delta_0(\alpha) = \frac{\alpha \min\{\alpha, 2\}}{4(2+\alpha)}.$$

Similarly, as in the proof of Lemma 3.1, we define the function w as the solution to

$$\det D^2 w = c(x_1^2 + px_1 + q)^{\frac{\alpha}{2}}, \quad w = 1 \text{ on } \partial S^v_{0,1},$$

and obtain (see (7.13)) that

$$|v - w| \le C\varepsilon^{\frac{1}{2}} = Ct^{\frac{\delta_0(\alpha)}{2}} \quad \text{and} \quad \left| w - (|x_1|^{2+\alpha} + x_2^2) \right| \le C\lambda^{\delta_2}.$$

By considering the partial Legendre transform w^*, one can deduce from the last inequality, the bounds on $|p|$ and $|q|$, and Lemma 2.4 that

$$|w_{22} - 2| \le C\lambda^{\delta_2} \quad \text{in } S^v_{0,1/2}.$$

This implies that

$$w \in Q(e, C\lambda^{\delta_2}, S^v_{0,1/2}) \quad \text{for any } e;$$

hence

$$v \in Q(e, C\lambda^{\delta_2}, S^v_{0,1/2}) \quad \text{for } e \ge t^{\frac{\delta_0(\alpha)}{8}}.$$

Then, as at the end of the proof of the previous lemma, we obtain that

$$u \in Q(t^{\frac{1}{2}} e, C\lambda^{\delta_2}, S^u_{0,t/2}) \quad \text{for } e \geq t^{\frac{\delta_0(\alpha)}{8}},$$

from which the lemma follows, since

$$t^{\frac{1}{2}} t^{\frac{\delta_0(\alpha)}{8}} \leq \lambda^{\frac{2+\alpha}{4}}$$

for δ_1 small and universal (depending only on α). □

The next lemma proves Proposition 7.1 for a point $Y \in S_{X_0,\lambda}$ at distance greater than $\lambda^{1/2}$ from the x_2-axis, assuming the conclusions of Lemmas 7.3 and 7.4.

LEMMA 7.5 *Assume that for* $\lambda^{\alpha/2+1-\delta_1} \leq t \leq \lambda$, *we have*

$$S_{X_0,t} - X_0 \in A_t D_t(\Gamma \pm \lambda^{\delta_2}), \quad |A_t - I| \leq \lambda^{\delta_2}, \tag{7.16}$$

and

$$u \in Q(e, C\lambda^{\delta_2}, S_{X_0,\frac{t}{2}}) \text{ for some } e, \quad \lambda^{\frac{2+\alpha}{4}} \leq e \leq t^{\frac{1}{2}}.$$

If

$$Y = (y_1, y_2) \in S_{X_0,\frac{t}{3}}, \quad 1 \leq |y_1| e^{-\frac{2}{2+\alpha}} \leq 2,$$

then D^2u *is Hölder-continuous in the ball* $B := B(Y, \lambda^{1+\alpha})$, *and for some constant* $0 < \beta < 1$, *it satisfies*

$$\|D^2 u\|_{C^{0,\beta}(B)} \leq C \quad \text{and} \quad |D^2u(Y) - D^2u(0)| \leq C\lambda^{\beta}. \tag{7.17}$$

PROOF: Consider the section S^u_{Y,ce^2} for a small constant c. By Theorem 2.2 there exists a matrix

$$F := \begin{pmatrix} a & 0 \\ d & b \end{pmatrix}, \quad a, b > 0,$$

such that

$$F B_{1/C_0} \subset S^u_{Y,ce^2} - Y \subset F B_1, \quad C_0(\alpha) > 0 \text{ universal.} \tag{7.18}$$

Using the assumptions of the lemma and (7.18), we derive bounds on the coefficients of the matrix F. Clearly

$$\nu := \frac{c^{1/2} e}{b}$$

satisfies the bound

$$\frac{1}{2C_0} \leq \frac{c^{1/2} e}{b} \leq 2. \tag{7.19}$$

Since $e \le t^{1/2}$, the corresponding section for the rescaling v (see (7.9), (7.13)) satisfies

$$S^v_{\tilde{Y},ce^2/t} \subset \{|x_1|^{2+\alpha} + x_2^2 \le \tfrac{3}{4}\},$$

or, more precisely,

$$S^v_{\tilde{Y},ce^2/t} - \tilde{Y} \subset \left(\left(\frac{e^2}{t}\right)^{\frac{1}{2+\alpha}} + \lambda^{\frac{\delta_2}{2(2+\alpha)}}\right) B_1.$$

Thus,

$$D_t^{-1} A_t^{-1} F B_{1/C_0} \subset \left(e^{\frac{2}{2+\alpha}} t^{-\frac{1}{2+\alpha}} + \lambda^{\delta_3}\right) B_1.$$

The last inclusion implies the estimate

$$|d| \le 2C_0\left(e^{\frac{2}{2+\alpha}} t^{\frac{\alpha}{2(2+\alpha)}} + \lambda^{\delta_3} t^{\frac{1}{2}} + a\lambda^{\delta_3}\right) \le 4C_0(e^{\frac{2}{2+\alpha}} + a)\lambda^{\delta_3}. \tag{7.20}$$

The rescaling

$$w(x) := \frac{1}{b^2} u(Y + Fx)$$

satisfies

$$\det D^2 w = \frac{a^2}{b^2} f(x)^{\frac{\alpha}{2}}, \quad B_{1/C_0} \subset S^w_{0,\nu^2} \subset B_1, \tag{7.21}$$

with

$$f(x) := (y_1 + ax_1)^2 + (y_2 + dx_1 + bx_2)^2 \tag{7.22}$$

and

$$|w - P'_{x_1}(x_2)| \le \lambda^{\delta_3} \quad \text{in } S^w_{0,\nu^2}. \tag{7.23}$$

We claim that if c is chosen small and universal, then

$$2a \le e^{\frac{2}{2+\alpha}} \le |y_1|. \tag{7.24}$$

Indeed, otherwise from (7.21), we would deduce that

$$\det D^2 w \ge a^{2+\alpha} b^{-2} \left(x_1 + \frac{y_1}{a}\right)^{\alpha}$$

with

$$(2a)^{2+\alpha} b^{-2} \ge e^2 b^{-2} \ge c^{-1}\nu^2$$

and for small c we would contradict $B_{1/C_0} \subset S^w_{0,\nu^2}$, since ν is bounded.

From (7.19), (7.20), (7.24), and $|y_2| \leq 4\lambda^{1/2}$ we obtain that $f(x)/y_1^2$ is bounded away from 0 and ∞ by universal constants, and also its derivatives are bounded by universal constants. From (7.21) we find that

$$c_1 \leq \frac{a^2|y_1|^\alpha}{b^2} \leq C_1,$$

which implies that $a^{2+\alpha}$, $|y_1|^{2+\alpha}$, b^2, and e^2 are all comparable. Moreover, using also (7.23), we have

$$\|D^2 w\|_{C^{0,1}} \leq C, \quad |w_{22} - 2| \leq \lambda^{\delta_4} \text{ in } \frac{S_{0,\nu^2}}{2}. \tag{7.25}$$

Hence

$$|w_{22}(x) - w_{22}(y)| \leq C\lambda^{\frac{\delta_4}{2}}|x-y|^{\frac{1}{2}} \quad \text{for } x, y \in \frac{S_{0,\nu^2}}{2}. \tag{7.26}$$

Also, we have

$$D^2u(Y + Fx) = b^2(F^{-1})^T D^2 w(x) F^{-1}$$

with

$$b\, F^{-1} = \begin{pmatrix} \frac{b}{a} & 0 \\ -\frac{d}{a} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + O(\lambda^{\delta_3}),$$

which together with (7.25) implies the second part of the conclusion (7.17).

Finally, since

$$|Fx| \geq \frac{b|x|}{2} \geq \lambda^{1+\alpha}|x|,$$

we obtain from (7.25) and (7.26) the estimate

$$|D^2u(Y + Fx) - D^2u(Y + Fy)| \leq C\lambda^{\frac{\delta_4}{2}}|x-y|^{\frac{1}{2}} \leq C|Fx - Fy|^\beta.$$

This finishes the proof of the lemma. □

The next lemma proves Hölder continuity when Y is $\lambda^{1/2}$ close to the x_2-axis.

LEMMA 7.6 *Assume that* (7.16) *holds for* $t = \lambda^{\alpha/2+1-\delta_1}$,

$$u \in Q(e, \lambda^{\delta_2}, S_{X_0,\frac{t}{2}}) \quad \textit{for } e = \lambda^{\frac{2+\alpha}{4}}$$

and

$$|x_0| \geq \frac{\lambda^{1/2}}{2}, \quad Y \in S_{X_0,\frac{t}{3}}, \quad |y_1| \leq e^{\frac{2}{2+\alpha}}.$$

Then the conclusion of Lemma 7.5 still holds.

PROOF: The proof is very similar to that of Lemma 7.5. The only difference is that now the second term of f in (7.22) dominates the sum.

Indeed, since $\lambda^{1/2} \geq |y_1|$ and $|y_2| \geq \lambda^{1/2}/4$, the function $f(x)/y_2^2$ is bounded away from 0 and ∞ by universal constants, and also its derivatives are bounded by universal constants. Hence $a^{2+\alpha}$, $y_2^{2+\alpha}$, b^2, and e^2 are all comparable and the rest of the proof is the same. □

PROOF OF PROPOSITION 7.1: For $Y \in \Omega_\lambda$ we consider the section $S^u_{Y,\sigma}$ that becomes tangent to the x_2-axis at $X_0 = (0, x_0)$. Since $|x|^\alpha\, dx$ is doubling, there exists a universal constant C_1 such that

$$Y \in S^u_{X_0,t/3}, \quad |x_0| \le 2\lambda^{\frac{1}{2}}, \ \ t := C_1\sigma \le C_2\lambda.$$

We distinguish the following three cases:

(1) If $t \ge t_0 := \lambda^{\alpha/2+1-\delta_1}$, then the proposition follows from Lemmas 7.3 and 7.5 with
$$e = |y_1|^{\frac{2+\alpha}{2}} \ge c_1 t^{\frac{1}{2}}.$$
(2) If $t \le t_0$ and $|y_1| \ge \lambda^{1/2}$, then we apply Lemmas 7.4 and 7.5 for S_{X_0,t_0} with e defined as above.
(3) If $t \le t_0$ and $|y_1| \le \lambda^{1/2}$, then we apply Lemma 7.4 and Lemma 7.6. We remark that the hypothesis $|x_0| \ge \lambda^{1/2}/2$ is satisfied because $Y \in \Omega_\lambda$.

□

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Bibliography

[1] Brenier, Y. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math*. **44** (1991), no. 4, 375–417.

[2] Caffarelli, L. A. Some regularity properties of solutions of Monge Ampère equation. *Comm. Pure Appl. Math*. **44** (1991), no. 8-9, 965–969.

[3] Caffarelli, L. A. Boundary regularity of maps with convex potentials. *Comm. Pure Appl. Math*. **45** (1992), no. 9, 1141–1151.

[4] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer, Berlin, 2001.

[5] Guan, P. Regularity of a class of quasilinear degenerate elliptic equations. *Adv. Math*. **132** (1997), no. 1, 24–45.

[6] Guan, P.; Li, Y.Y. The Weyl problem with nonnegative Gauss curvature. *J. Differential Geom*. **39** (1994), no. 2, 331–342.

[7] Guan, P.; Li, Y. $C^{1,1}$ estimates for solutions of a problem of Alexandrov. *Comm. Pure Appl. Math*. **50** (1997), no. 8, 789–811.

[8] Guan, P.; Sawyer, E. Regularity of subelliptic Monge-Ampère equations in the plane. *Trans. Amer. Math. Soc*., forthcoming. Available online at: `http://www.math.mcgill.ca/guan/pb.html`

[9] Gutiérrez, C. E. *The Monge-Ampère equation*. Progress in Nonlinear Differential Equations and Their Applications, 44. Birkhäuser, Boston, 2001.

[10] Lewy, H. On the existence of a closed convex surface realizing a given Riemannian metric. *Proc. Nat. Acad. Sci*. **24** (1938), no. 2, 104–106. Available online at: `http://www.jstor.org/sici?sici=0027-8424(19380215)24%3A2%3C104%3AOTEOAC%3E2.0.CO%3B2-1`

[11] Lin, C. S. The local isometric embedding in R^3 of 2-dimensional Riemannian manifolds with nonnegative curvature. *J. Differential Geom.* **21** (1985), no. 2, 213–230.

[12] Nirenberg, L. The Weyl and Minkowski problems in differential geometry in the large. *Comm. Pure Appl. Math.* **6** (1953), 337–394.

[13] Weyl, H. Über die Bestimmung einer geschlossen konvexen Fläche durch ihr Linienelement. *Vierteljahrsschrift der Naturforschenden Gessellschaft Zürich* **61** (1916), 40–72; also found in *Gesammelt Abhandlungen*, vol. 1, 614–644. Springer, Berlin, 1968.

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