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The obstacle problem for Monge Ampere equation

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Abstract. We consider the following obstacle problem for Monge-Ampere equation

$$\det D^2 u = f\chi_{\{u>0\}}$$

and discuss the regularity of the free boundary $\partial\{u=0\}$. We prove that $\partial\{u=0\}$ is $C^{1,\alpha}$ if f is bounded away from 0 and ∞, and it is $C^{1,1}$ if $f \equiv 1$.

1. Introduction

With each convex function u defined on Ω we can associate a Borel measure Mu such that, for every measurable set $E \subset \Omega$,

$$Mu(E) = |\nabla u(E)|,$$

where $|\quad|$ represents the n dimensional Lebesgue measure and $\nabla u(E)$ represents the image of the subgradients of u at all points $x \in E$ (see [3]).

In the case $u \in C^2(\Omega)$ then the measure is given by the Jacobian, i.e

$$Mu(E) = \int_E \det D^2 u \quad dx.$$

Given a positive Borel measure μ in Ω, we say that u solves the Monge Ampere equation (in the Alexandrov sense)

$$\det D^2 u = \mu \quad \text{if } Mu = \mu.$$

We consider the following obstacle problem: Given a finite measure μ_0 in Ω we define $\mathcal{D}_{\mu_0}$ the class of nonnegative supersolutions

$$\mathcal{D}_{\mu_0} = \{v : \Omega \to [0,\infty), \quad v \text{ convex}, \quad v = 1 \text{ on } \partial\Omega, \quad Mu \le \mu_0\}.$$

Then, we would like to study the minimization problem

$$(P) \qquad u = \inf_{v \in \mathcal{D}_{\mu_0}} v.$$

Using Perron's method we show in the beginning of the next section

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Proposition 1.1. *The minimizer u of (P) is in the class $\mathcal{D}_{\mu_0}$ and $Mu = \mu_0$ on $\{u > 0\} \cap \Omega$.*

In this paper we are interested in the regularity of the free boundary $\partial\{u = 0\}$ in two cases

1) the measure μ_0 is given by a function f, bounded away from 0 and ∞, i.e

$$d\mu_0 = f dx, \quad 0 < \lambda \le f \le \Lambda < \infty$$

2) μ_0 is the Lebesgue measure, that is $f \equiv 1$ in case 1.

Notice that, in both cases, affine transformations play an important role. If u solves an obstacle problem and $A : \mathbb{R}^n \to \mathbb{R}^n$ is affine, then

$$\bar{u}(x) := (\det A)^{\frac{2}{n}} u(A^{-1}x)$$

solves a similar obstacle problem, and its 0 set is $A\{u = 0\}$. Therefore, we study the regularity in the classes of “normalized" solutions

$$\mathcal{D}_{\lambda,\Lambda,\delta} := \{u : \Omega_u \to \mathbb{R}^+, \quad u = \delta \text{ on } \partial\Omega_u, \quad B_1 \subset \{u = 0\} \subset B_{k(n)},$$
$$u \text{ solves } (P) \text{ with } \lambda dx \le d\mu_0 \le \Lambda dx\}$$

respectively,

$$\mathcal{D}_{1,\delta} := \{u : \Omega_u \to \mathbb{R}^+, \quad u = \delta \text{ on } \partial\Omega_u, \quad B_1 \subset \{u = 0\} \subset B_{k(n)},$$
$$u \text{ solves } (P) \text{ with } d\mu_0 = dx\}.$$

Below we state the main theorems of the paper, Theorem 1.2 and 1.3. In Theorem 1.2 we show that, if we are in the case 1, then $\partial\{u = 0\}$ is $C^{1,\alpha}$ and moreover, it separates polinomially from its tangent plane.

Theorem 1.2. *There exist positive constants α small, C large depending only on n, λ, Λ, and d depending also on δ such that:*

If $u \in \mathcal{D}_{\lambda,\Lambda,\delta}$ and $x_0 \in \partial\{u = 0\}$ then there exists an appropriate system of coordinates centered at x_0 such that inside the ball $|y| < d$

$$\{y_n > C|y'|^{1+\alpha}\} \subset \{u = 0\} \subset \{y_n > C^{-1}|y'|^{\frac{1}{\alpha}}\}.$$

At the end of Sect. 3 we construct a two dimensional example which shows that the result of Theorem 1.2 is optimal.

In Theorem 1.3 we prove that if we are in case 2, then the principal curvatures of the free boundary are bounded away from 0 and ∞.

Theorem 1.3. *There exist positive constants r small, R large depending only on n and δ such that:*

If $u \in \mathcal{D}_{1,\delta}$ then at each point of $\partial\{u = 0\}$ there exist a tangent ball of radius r contained in $\{u = 0\}$, respectively a tangent ball of radius R which contains $\{u = 0\}$.

Next we give two other interpretations of the obstacle problem.

Optimal Transportation with a Dirac δ

Suppose we want to transport the measure $h(y)dy + \delta_0$ supported in B_1 into the density $g(x)dx$ in Ω with the cost function $c(x,y) = |x-y|^2$. Also, assume that the point 0 is transported into the set $K \subset\subset \Omega$, and that h and g are bounded away from 0 and ∞ and satisfy the compatibility condition

$$\int_{B_1} h(y)dy + 1 = \int_{\Omega} g(x)dx.$$

Then, by Brenier's theorem, the map $x \to y(x)$ is given by the gradient image of a convex function $u : \Omega \to \mathbb{R}$, $\{u = 0\} = K$, and since the y image is convex, u satisfies in the weak sense the following Monge-Ampere equation

$$\det D^2 u(x) = \frac{g(x)}{h(\nabla u) + \delta_0(\nabla u)}.$$

This is an obstacle problem for u and our theorems imply regularity of the set K.

Monge Ampere with a cone singularity at 0

Suppose that the function $v \geq 0$, $v(0) = 0$ solves the Monge Ampere equation

$$\det D^2 v = f \quad \text{in } B_1 \setminus \{0\}$$

with $0 < \lambda \leq f \leq \Lambda < \infty$ (or $f \equiv 1$). Also, assume that v has a tangent cone at 0 whose 1 level set is given by a bounded convex set L. We would like to study the regularity of the tangent cone, i.e the regularity of L.

If u is the Legendre transform of v, then its 0 set is the image of the subgradients of the tangent cone. It is straight forward to check that u solves an obstacle problem with constants Λ^{-1}, λ^{-1} in a neighborhood of its 0 set. The set L is the convex dual of $\{u = 0\}$ and one obtains the same regularity for L as for $\{u = 0\}$ in Theorems 1.2 and 1.3.

We conclude the introduction by considering the radially symmetric case. If u is radially symmetric and $\{u = 0\} = B_1$ then,

$$|\nabla u(B_{1+\varepsilon} \setminus B_1)| \sim |B_{1+\varepsilon} \setminus B_1| \sim \varepsilon$$

thus $|\nabla u|$ on $\partial B_{1+\varepsilon}$ is proportional to $\varepsilon^{\frac{1}{n}}$. This implies

$$u(x) \sim (|x| - 1)^{1+\frac{1}{n}} \qquad \text{for } |x| \text{ close to } 1,\ |x| > 1$$

hence, near B_1, the radial pure second derivatives tend to ∞ and the tangential ones to 0. This shows that our obstacle problem is quite different from the obstacle problem for the uniformly elliptic equations.

Finally, we remark that another obstacle problem for Monge-Ampere equation was considered by K. Lee (see [4]) in which the obstacle is a smooth function above u. In this case $u \in C^{1,1}$ and the problem reduces to the obstacle problem for the uniformly elliptic equations.

2. Preliminaries

In this section we prove Proposition 1.1 and recall some known facts for the Monge Ampere equation. We start with the following results (see [3])

Theorem 2.1. *(Weak convergence of measures)*
If u_k are convex functions in Ω that converge uniformly on compact sets to u then Mu_k converges weakly to Mu, that is

$$\int_\Omega f dMu_k \to \int_\Omega f dMu$$

for every continuous f with compact support in Ω.

Theorem 2.2. *(Solvability of Dirichlet Problem)*
If Ω is open, bounded and strictly convex set, μ is a finite Borel measure and g is continuous on $\partial\Omega$ then, there exists a unique $u \in C(\bar{\Omega})$ convex solution to the Dirichlet problem

$$Mu = \mu, \quad u = g \text{ on } \partial\Omega.$$

Theorem 2.3. *(Alexandrov's Maximum Principle)*
If u convex, $u = 0$ on $\partial\Omega$ then

$$u(x_0) \geq -C\left(dist(x_0, \partial\Omega)Mu(\Omega)\right)^{\frac{1}{n}}.$$

The next lemma is used for Perron's method

Lemma 2.4. *Suppose $Mv_i \leq \mu_0$, $i = 1, 2$. Then w, the convex envelope of $\min\{v_1, v_2\}$ satisfies $Mw \leq \mu_0$.*

Proof. Denote

$$A = \{v_1 = w\}, \quad B = \{v_2 = w\} \setminus A, \quad C = \Omega \setminus (A \cup B).$$

If $p \in \nabla(C)$ then there exist distinct points $x, y \in \Omega$ such that $p \in \nabla w(x) \cap \nabla w(y)$. Then p is not a differentiation point for the Legendre transform z of w,

$$z(p) := \sup_{x \in \Omega}(p \cdot x - w(x))$$

thus, $|\nabla(C)| = 0$. One has

$$Mw(E) = Mw(E \cap A) + Mw(E \cap B) + Mw(E \cap C) \leq$$

$$\leq Mv_1(E \cap A) + Mv_2(E \cap B) \leq \mu_0(E \cap A) + \mu_0(E \cap B) \leq \mu_0(E). \qquad \square$$

Proof of Proposition 1.1. Alexandrov maximum principle implies that the functions in $\mathcal{D}_{\mu_0}$ are uniformly continuous on $\partial\Omega$, therefore locally uniformly Lipschitz in Ω. Using a diagonal subsequence and Lemma 2.4 we find a decreasing sequence $v_n \in \mathcal{D}_{\mu_0}$ which converges uniformly to u in $\bar{\Omega}$, thus

$$Mu(O) \leq \liminf Mv_k(O) \leq \mu_0(O)$$

for any open set $O \subset\subset \Omega$. This implies

$$Mu(E) \leq \mu_0(E)$$

for any Borel set $E \subset \Omega$, hence $u \in \mathcal{D}_{\mu_0}$.

Next we show that $Mu = \mu_0$ on $\{u > 0\}$.

First we prove that for $x \in \{u > 0\}$, there exists $\delta_x > 0$ such that

$$Mu(B_\varepsilon(x)) = \mu_0(B_\varepsilon(x)), \quad \text{for } \varepsilon < \delta_x. \tag{1}$$

Solve in $B_\varepsilon(x)$ the Dirichlet problem $v = u$ on $\partial B_\varepsilon(x)$, $Mv = \mu_0$ and assume that $v < u$ at some point inside $B_\varepsilon(x)$. If δ_x is chosen small, then $v > 0$ from the maximum principle. The convex envelope of $\min\{u, v\}$ is in $\mathcal{D}_{\mu_0}$ and we contradict the minimality of u thus (1) is proved.

Since any Borel set in $\{u > 0\}$ is generated by the σ algebra of $B_\varepsilon(x)$, $\varepsilon < \delta_x$, we conclude that $Mu = \mu_0$ on $\{u > 0\}$ and the Proposition is proved. $\square$

Before we prove the next Proposition we recall the following lemmas from [2] (see lemmas 1, 2 and 3):

Lemma 2.5. *(John's lemma)*

Let Ω be a bounded convex set with center of mass at 0. There exists a constant $k(n)$ depending only on n and an ellipsoid E such that

$$E \subset \Omega \subset kE.$$

Lemma 2.6. *(Centered sections)*

Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a globally defined convex function with $u(0) = 0$. Also, assume u is bounded in a neighborhood of 0 and the graph of u doesn't contain an entire line.

Then, for each $h > 0$, there exists a "h-section" S_h centered at 0, that is there exists p_h such that the convex set

$$S_h := \{x : \quad u(x) < h + p_h \cdot x\}$$

is bounded and has 0 as center of mass.

Lemma 2.7. *Suppose $v = 0$ on $\partial\Omega$, v convex, 0 is the center of mass of Ω and*

$$0 < \theta Mv(\Omega) \leq Mv(\frac{1}{2}\Omega). \tag{2}$$

If $u(x_0) \leq u(0)$, then Ω is balanced around x_0, that is

$$(1 + \eta)x_0 - \eta\Omega \subset \Omega, \tag{3}$$

where $\eta(\theta, n) > 0$ is small, depending only on n and θ.

Remark. Another way of writing (3) is:
For any line segment $[y_1, y_2]$ passing through x_0 with $y_i \in \partial\Omega$ we have

$$\frac{|y_2 - x_0|}{|y_1 - x_0|} \geq \eta.$$

From now on we consider only measures μ_0 that satisfy

$$(H) \quad \lambda dx \leq d\mu_0 \leq \Lambda dx, \quad 0 < \lambda \leq \Lambda < \infty,$$

and assume that the zero set $K := \{u = 0\}$ has nonempty interior (otherwise u satisfies the standard Monge-Ampere equation).

Notice that, if u solves (P) for μ_0 then, given any affine transformation A, the function

$$u'(x) := (\det A)^{-\frac{2}{n}} u(Ax)$$

solves the problem (P) in the class

$$\{v \text{ convex in } \Omega' = A^{-1}\Omega, \quad v = (\det A)^{-\frac{2}{n}} \text{ on } \partial\Omega', \quad Mv \leq \mu'\},$$

$$\mu'(E) := \frac{1}{\det A}\mu_0(AE),$$

thus μ' satisfies (H) as well.

Next we discuss the regularity of u:

Proposition 2.8. *The minimizer* $u \in C^1(\Omega)$.

Proof. First we prove

$$\nabla u(K) = 0. \tag{4}$$

Assume not, then, after eventually an affine transformation, we have $a > 0$ such that

$$u(0) = 0, \quad u \geq 0, \quad , u \geq a x_n^+, \quad u(te_n) = at + o(t).$$

Consider the functions

$$u_{h,b} := u - (h + bx_n), \quad h > 0, \quad 0 < b < a,$$

fix b close to a and let $h \to 0$. Denote $\Omega_{h,b} = \{u_{h,b} < 0\}$ and notice that the minimum of $u_{h,b}$ occurs at 0.

The half line te_n, $t > 0$ intersects $\partial\Omega_{h,b}$ at distance $h(a-b)^{-1} + o(h)$. Since $\Omega_{h,b} \subset \{x_n \geq -hb^{-1}\}$ and b is close to a we find that the center of mass $x_{h,b}$ of the convex set $\Omega_{h,b}$ satisfies $x_{h,b} \cdot e_n \geq hb^{-1}$. Thus,

$$\frac{1}{2}(x_{h,b} + \Omega_{h,b}) \subset \{x_n \geq 0\}$$

and $u_{h,b}$ satisfies (2) with θ depending only on λ, Λ and n. This contradicts Lemma 2.7 since $\Omega_{h,b}$ is not balanced around 0 and (4) is proved.

Next we prove that $u \in C^1(\Omega \setminus K)$. Using $u = 1$ on $\partial\Omega$ and (4), we find that the supporting plane at $x \in \Omega \setminus K$ coincides with u in a convex set that has extremal points in $\Omega \setminus K$. In this set $Mu = \mu_0$ thus, by Theorem 1 of [1] we conclude that u is strictly convex, therefore $C^{1,\alpha}$.

With this the proposition is proved. $\square$

Remark. Using the same techniques as in [2] one can prove that $u \in C^{1,\alpha}(\Omega)$.

3. Proof of Theorem 1.2

We start with a simple lemma.

Lemma 3.1. *Assume v is convex, $v \geq 0$ in Ω and $Mv \geq \lambda dx$. Then*

$$\sup_{\Omega} v \geq c|\Omega|^{\frac{2}{n}},$$

where $c(\lambda, n) > 0$ is small depending on λ and n.

Proof. From John's lemma we find an affine transformation A such that $B_1 \subset A^{-1}\Omega \subset B_{k(n)}$. The normalized function

$$v'(x) = (\det A)^{-\frac{2}{n}} v(Ax)$$

is defined in $\Omega' = A^{-1}\Omega$ and satisfies $v' \geq 0$, $Mv' \geq \lambda dx$.

If $x \in B_{1/2}$ then,

$$|\nabla v'(x)| \leq 2 \sup_{\Omega'} v'$$

hence,

$$\lambda |B_{1/2}| \leq |\nabla v'(B_{1/2})| \leq C(n)(\sup_{\Omega'} v')^n.$$

Since $\det A \sim |\Omega|$, the lemma is proved. □

Next lemma gives a bound from below for the growth of u away from the free boundary.

Lemma 3.2. *(Bound from below)*

Suppose $0 \in \partial K$ and $K \subset \{x_n > 0\}$. Then, if $y \in \Omega$, $y_n \leq 0$ we have

$$u(y) \geq c|K \cap \{x_n \leq |y_n|\}|^{\frac{2}{n}},$$

with $c(\lambda, n) > 0$, small.

Proof. The convex set

$$\Omega_y := \frac{1}{2} y + \frac{1}{2}(K \cap \{x_n \leq |y_n|\})$$

is included in $\{x_n < 0\}$ and therefore in $\{u > 0\}$. Moreover, each point in Ω_y belongs to a line segment from y to K hence,

$$\sup_{\Omega_y} u \leq u(y),$$

and the result follows from the previous lemma. □

Remark 1. In the radially symmetric case we obtain

$$u(x) \geq c[dist(x, K)]^{\frac{n+1}{n}}$$

which is optimal.

Remark 2. If $B_1 \subset \Omega \subset B_{k(n)}$, then the function u grows better than quadratic away from ∂K, i.e

$$u(x) \geq c[dist(x, \partial K)]^2. \tag{5}$$

Indeed, let x_0 be the point where the distance to K is realized. Since K contains the convex set generated by x_0 and B_1 we find

$$|K \cap \{y : (y - x_0) \cdot (x_0 - x) \leq |x - x_0|^2\}| \geq c(n)|x - x_0|^n$$

so,

$$u(x) \geq c(x - x_0)^2.$$

Remark 3. Lemma 3.2 implies that ∂K cannot contain a line segment if $n = 2$. If, say $K \subset \{x_2 \geq 0\}$ and $-e_1, e_1 \in K$, then $u(-te_2) \geq ct$ which contradicts the fact that u is C^1 at 0.

This result was first proved by Pogorelov, which showed that in two dimensions a solution of Monge Ampere equation cannot contain a line segment.

Next we show that ∂K cannot contain a line segment in any dimension. We prove this for solutions that are defined only in a half space. More precisely, we assume u is defined in the bounded domain $\{x_n \leq 1\} \cap \Omega_\sigma$, $u = \sigma$ on $\partial \Omega_\sigma \cap \{x_n \leq 1\}$, $Mu = 0$ on K and $\lambda dx \leq Mu \leq \Lambda dx$ on $\{u > 0\}$.

Lemma 3.3. *(Strict convexity of K)*

∂K cannot contain a line segment with one vertex on $\{x_n = 1\}$ and the other in $\{x_n < 1\}$.

Proof. Assume by contradiction, say

$$K \subset \{x_1 \geq 0\} \tag{6}$$

$$[0, e_n] \in \partial K, \quad K \cap \{x_1 = 0\} \subset \{x_n \geq 0\}. \tag{7}$$

and extend u by $+\infty$ outside the domain of definition.

Let $y = ae_n$, be a point of the segment close to 0 and consider the sections $S_{y,h}$ centered at y (see Lemma 2.6).

Claim: For h small

$$S_{y,h} \subset \Omega_\sigma \cap \{x_n \leq 1/2\}.$$

Notice that $S_{y,h}$ is balanced around y with the ratio $c = k(n)^{-1}$. Hence, if $x \in S_{y,h}$ then $(1 + c)y - cx \in S_{y,h}$ and

$$\frac{c}{1+c}u(x) \leq \frac{c}{1+c}u(x) + \frac{1}{1+c}u((1+c)y - cx) \leq h$$

thus,

$$u(x) \leq \frac{1+c}{c}h, \quad \text{if } x \in S_{y,h}. \tag{8}$$

This implies $S_{y,h} \subset \Omega_\sigma$ for h small.

In order to prove the other inclusion, assume by contradiction that for a sequence of h tending to 0 there exists $z_h \in S_{y,h} \cap \{x_n = 1/2\}$, hence $(1+c)y - cz_h \in S_{y,h}$. If $z_h \to z_0$ as $h \to 0$ then, by (8)

$$u(z_0) = 0, \quad u((1+c)y - cz_0) = 0.$$

From (6) we find

$$(1+c)y - cz_0 \in \{x_1 \le 0\} \cap K \cap \left\{x_n \le (1+c)a - \frac{c}{2}\right\}$$

which contradicts (7), (6) for a small and the claim is proved.

Denote by $[\alpha_h e_n, \beta_h e_n]$ the intersection of the line te_n with $S_{y,h}$, $\alpha_h \le a \le \beta_h$. Obviously, $\alpha_h < 0$ and $\alpha_h \to 0$ as $h \to 0$.

Let l_h denote the linear functional which gives the section $S_{y,h}$. Since

$$\{x_1 < 0\} \cap S_{y,h} \subset \{0 < u < \sigma\},$$

one has that $u - l_h$ satisfies the hypothesis (2) of Lemma 2.7 with θ depending only on λ, Λ and n. This contradicts Lemma 2.7 because $(u - l_h)(0) \le (u - l_h)(y)$ but the set $S_{y,h}$ is not balanced around 0 as $h \to 0$. □

Next we show that the family of solutions defined in a half space for which K is an upper-graph satisfies a compactness property.

Definition. Let E_σ, $\sigma > 0$ small depending on λ, Λ, n, denote the family of functions u that satisfy:

1) $u \ge 0$ is convex, defined in $\Omega_u \cap \{x_n \le 1\}$, $u = \sigma$ on $\partial\Omega_{\sigma,u}$
2) $Mu = 0$ on $\{u = 0\}$, $\lambda dx \le Mu \le \Lambda dx$ in the set $\{u > 0\}$
3) $K \cap \{x_n = 1\}$ is normalized in $\mathbb{R}^{n-1}$, i.e

$$\{|x'| \le 1\} \cap \{x_n = 1\} \subset K \cap \{x_n = 1\} \subset \{|x'| \le k(n-1)\}$$

4) $K \subset \{x_n \ge 0\}$ and $K \cap \{x_n = 0\} \ne \emptyset$
5) K is an upper-graph, i.e if $x \in K$ then $x + te_n \in K$ for $t \ge 0$.

Lemma 3.4. *(Convergence of "graph solutions")*

If $u_k \in E_\sigma$ then, there exists a subsequence that converges uniformly in $\Omega_\infty \cap \{x_n \le 3/4\}$ to a function u_∞, $u_\infty = \sigma/8$ on $\partial\Omega_\infty$, which satisfies property 2 above.

Moreover, in $\{x_n \le 3/4\}$, $\partial\{u_k = 0\}$ converge uniformly to $\partial\{u_\infty = 0\}$ in the sense that they are in a ε neighborhood of the limit for k large.

Proof. Let $u \in E_\sigma$. If $x \in \{x_n \le 7/8\} \cap \partial\Omega_u$ then, by (5), $dist(x, K) \le C_1\sigma^{\frac{1}{2}}$. Consider the function

$$u^* := (u + \sigma(x_n - 7/8))^-$$

which is well defined since

$$O_{u^*} := \{u < -\sigma(x_n - 7/8)\} \subset \{u < \sigma\}.$$

One has

$$|Mu^*(O_{u^*})| = |Mu^*(O_{u^*} - K)| \leq \Lambda|O_{u^*} - K| \leq C(n)C_1\sigma^{\frac{1}{2}},$$

thus, by Alexandrov's maximum principle, u^* has a modulus of continuity depending only on λ, Λ, n. This implies that u is uniformly Lipschitz in the domain $\{u < \sigma/8\} \cap \{x_n \leq 7/8\}$. If $u_k \in E_\sigma$, then there exists a subsequence that converges uniformly to u_∞.

Obviously, $\{u_k = 0\}$ is is included in a ε neighborhood of $\{u_\infty = 0\}$ for k large. Also, since the functions grow at least quadratically away from their 0 set, we have that $\partial\{u_k = 0\}$ cannot have a point at some fixed distance inside $\{u_\infty = 0\}$ for $k \to \infty$.

Finally, Mu_k converges weakly to Mu_∞ by Lemma 2.1, hence u_∞ satisfies property 2. □

Remark. By the same argument one can prove compactness of the family $\mathcal{D}_{\lambda,\Lambda,\delta}$ defined in the Introduction.

Denote by L_a the projection along e_n of $\partial K \cap \{x_n = a\}$.

Lemma 3.5. *Suppose $u \in E_\sigma$. There exists $c(\lambda, \Lambda, n) > 0$ such that*

$$dist(L_1, L_{1/2}) > c.$$

Proof. Assume by contradiction this is not the case, thus there exists a sequence u_k for which the distance between L_1^k, $L_{1/2}^k$ converges to 0. Then, the limiting function u_∞ has a line segment along e_n direction included in $\partial\{u_\infty = 0\}$ which contradicts Lemma 3.3. □

As a consequence, we obtain that ∂K, viewed as a graph in the e_n direction, is balanced around the minimum.

Corollary. Let $x_0 \in K \cap \{x_n = 0\}$. Then

$$dist(x_0, L_1) > c.$$

We denote by φ_a the function in $\mathbb{R}^{n-1}$ whose graph coincides with the cone generated by x_0 and $\partial K \cap \{x_n = a\}$.

Lemma 3.6. *There exists $c(\lambda, \Lambda, n) > 0$ small, such that*

$$2 - c \geq \varphi_1 \varphi_{1/2}^{-1} \geq 1 + c.$$

Proof. The left inequality follows from the lemma above. The right inequality is proved also by compactness. Otherwise, ∂K of the limiting function u_∞ contains a line segment with endpoints on $\{x_n = 1/2\}$ respectively $\{x_n = 0\}$ and again we contradict Lemma 3.3. □

Proposition 3.7. *(Regularity of "graph solutions")*

There exist positive constants α small C large, depending only on λ, Λ ,n such that if $u \in E_\sigma$, $x_0 \in K \cap \{x_n = 0\}$ then

$$\{x_n \leq 1\} \cap \{x_n > C|x' - x_0|^{1+\alpha}\} \subset K \subset \{x_n > C^{-1}|x' - x_0|^{\frac{1}{\alpha}}\}.$$

Proof. Assume $x_0 = 0$. Notice that the expression $\varphi_1 \varphi_{1/2}^{-1}$ remains invariant under transformations

$$\bar{u}(x', x_n) := (r \det B)^{-\frac{2}{n}} u(Bx', rx_n),$$

where B is an affine transformation in $\mathbb{R}^{n-1}$ such that $\{x_n = 1\} \cap \{\bar{u} = 0\}$ is normalized. Moreover, if $r|\det B| \leq 1$ then $\bar{u} \in E_\sigma$ (we restrict $\bar{u}$ to $\{\bar{u} < \sigma\}$). Now the result follows by iterating Lemma 3.6. □

Theorem 1.2 is a consequence of the above proposition and the next lemma.

Lemma 3.8. *Let $u \in \mathcal{D}_{\lambda,\Lambda,\delta}$ and for each point $x_0 \in \partial K$ denote by ν_{x_0} the interior normal to a supporting hyperplane of K at x_0. There exists $d(\lambda, \Lambda, n, \delta) > 0$ such that*

$$K \cap \{(x - x_0) \cdot \nu_{x_0} \leq d\}$$

is an upper graph in the ν_{x_0} direction.

Proof. Assume by contradiction this is not the case. Then, since $\mathcal{D}_{\lambda,\Lambda,\delta}$ is a compact family, we can find a function $u \in \mathcal{D}_{\lambda,\Lambda,\delta}$ such that K has two supporting planes at a point $x_0 \in \partial K$.

Let $\tilde{\nu}$ be a unit vector in the interior of the tangent cone generated by the supporting hyperplanes at x_0. Since ∂K cannot contain a line segment, we conclude that K is an upper graph in the $\tilde{\nu}$ direction in $\{(x - x_0) \cdot \tilde{\nu} \leq \varepsilon\}$, for some $\varepsilon > 0$. This contradicts proposition 3.7 and the lemma is proved. □

We conclude this section by constructing an example in two dimensions where $K = \{y \leq -|x|^\alpha\}$ for $1 < \alpha < \infty$. Since $(x, y) \to (\lambda x, \lambda^\alpha y)$ leaves $\{y < -|x|^\alpha\}$ invariant, we look for functions u that satisfy the homogeneity condition

$$\frac{1}{\lambda^{\alpha+1}} u(\lambda x, \lambda^\alpha y) = u(x, y).$$

If $u(1, t) = u(-1, t) = g(t)$ for $t \geq -1$, $g(-1) = 0$, g convex, then

$$u(x, y) = |x|^{\alpha+1} g(y|x|^{-\alpha}),$$

and one can compute

$$\begin{aligned} \det D^2 u(x, y) &= P(\alpha, g(t)), \quad t = y|x|^{-\alpha}, \\ P(\alpha, g) &:= \alpha \left((\alpha + 1)g - (\alpha - 1)tg'\right) g'' - g'^2. \end{aligned}$$

Also, if $u(t, 1) = f(t)$, f convex, then

$$u(x, y) = y^{\frac{\alpha+1}{\alpha}} f(xy^{-\frac{1}{\alpha}}), \quad y > 0,$$

and

$$\begin{aligned} \alpha^2 \det D^2 u &= H(\alpha, f(t)), \quad t = xy^{-\frac{1}{\alpha}}, \\ H(\alpha, f) &= \left((\alpha + 1)f + (\alpha - 1)tf'\right) f'' - \alpha^2 f'^2, \quad t = xy^{-\frac{1}{\alpha}}. \end{aligned}$$

Notice that for the particular choice

$$g(t) = \frac{\alpha}{\alpha+1}(s_0+t)^{\frac{\alpha+1}{\alpha}}$$

we have

$$P(\alpha, g) = s_0(\alpha-1)(s_0+t)^{\frac{2}{\alpha}-1}.$$

We are going to modify this function near $g = 0$ and ∞ such that $P(\alpha, g)$ remains bounded away from 0 and ∞.

Define

$$g_1(t) = \begin{cases} c(1+t)^{\frac{3}{2}}, & -1 \le t \le t_0 \\ \frac{\alpha}{\alpha+1}(s_0+t)^{\frac{\alpha+1}{\alpha}}, & t \ge t_0 \end{cases}$$

where t_0 is close to -1,

$$s_0 := 1 + \frac{(1+t_0)(2-\alpha)}{3\alpha},$$

and the constant c is chosen such that g_1 is continuous. Notice that g_1 is well defined since $s_0 + t_0 > 0$. Also $g \in C^{1,1}$ because at t_0

$$\lim_{t \to t_0^-} \frac{g_1'}{g_1} = \frac{3}{2(1+t_0)} = \frac{\alpha+1}{\alpha(s_0+t_0)} = \lim_{t \to t_0^+} \frac{g_1'}{g_1}.$$

Moreover, on $[-1, t_0]$

$$\left(\frac{2}{3c}\right)^2 P(\alpha, g_1) = \left(\frac{\alpha(\alpha+1)}{3} - 1\right)(1+t) - \frac{\alpha(\alpha-1)}{2} t$$

which is between two positive constants if t_0 is chosen close to -1.

If u_1 is the function generated by g_1, then near $t = 0$,

$$f_1(t) = u_1(t,1) = \frac{\alpha}{\alpha+1}(s_0 t^\alpha + 1)^{\frac{\alpha+1}{\alpha}}, \quad t > 0.$$

We modify f_1 in the following way

$$f_2(t) = \begin{cases} a(b+t^2), & 0 \le t \le t_1 \\ f_1(t), & t \ge t_1 \end{cases}$$

where b is chosen such that

$$\lim_{t \to t_1^-} \frac{f_2'}{f_2} = \frac{2t_1}{b+t_1^2} = \frac{(\alpha+1)s_0 t_1^{\alpha-1}}{s_0 t_1^\alpha + 1} = \lim_{t \to t_1^+} \frac{f_2'}{f_2} \tag{9}$$

and a such that f_2 is continuous.

Notice that b is positive if t_1 is small. On $[0, t_1]$ we have

$$a^{-2} H(\alpha, f) = 2(\alpha+1)(b+t^2) - 4(\alpha^2 - \alpha + 1)t^2.$$

The last expression is linear in t^2 and $H(\alpha, f) > 0$ at $t = 0$. When $t = t_1$ we use (9) and obtain

$$a^{-2} H(\alpha, f) = 4t_1^2 \left((s_0^{-1} t_1^{-\alpha} + 1) - (\alpha^2 - \alpha + 1)\right) > 0, \quad \text{if } t_1 \text{ small}.$$

In conclusion we constructed a global u with $K = \{y \le -|x|^\alpha\}$ with $Mu = 0$ on K and $\lambda(\alpha)dx \le Mu \le \Lambda(\alpha)dx$ on $\{u > 0\}$.

4. Proof of Theorem 1.3

First we prove the existence of $R(\delta)$. We are going to use the following estimate of Pogorelov.

Proposition 4.1. *(Pogorelov's estimate)*
If w is convex,

$$\det D^2 w = c_0 > 0 \quad \textit{in } \Omega, \quad w = 0 \textit{ on } \partial\Omega$$

then

$$w_{11}|w| \leq C(n, \max_{\Omega} |w_1|).$$

Remark. The constant $C(n, \max_\Omega |w_1|)$ doesn't depend on c_0.

Proof. Write

$$\log \det D^2 w = \log c_0$$

and differentiate with respect to x_1

$$w^{ij} w_{1ij} = 0, \tag{10}$$

where $[w^{ij}] = [D^2 w]^{-1}$ and we use the index summation convention. Differentiating once more, we have

$$w^{ij} w_{11ij} - w^{ik} w^{jl} w_{1ij} w_{1kl} = 0. \tag{11}$$

Suppose the maximum of

$$\log w_{11} + \log |w| + \frac{1}{2}|w_1|^2 = M \tag{12}$$

occurs at the origin. One can also assume that $D^2 u(0)$ is diagonal since the transformation

$$\bar{w}(x_1, .., x_n) := w(x_1 - \alpha_2 x_2 - .. - \alpha_n x_n, x_2, .., x_n), \quad \alpha_i = \frac{w_{1i}(0)}{w_{11}(0)} \tag{13}$$

doesn't affect the equation or the maximum in (12). Thus, at 0

$$\frac{w_{11i}}{w_{11}} + \frac{w_i}{w} + w_1 w_{1i} = 0 \tag{14}$$

$$\frac{w_{11ii}}{w_{11}} - \frac{w_{11i}^2}{w_{11}^2} + \frac{w_{ii}}{w} - \frac{w_i^2}{w^2} + w_{1i}^2 + w_1 w_{1ii} \leq 0 \tag{15}$$

Multiply (15) by u_{ii}^{-1} and add

$$\frac{w_{11ii}}{w_{11} w_{ii}} - \frac{w_{11i}^2}{w_{ii} w_{11}^2} + \frac{n}{w} - \frac{w_i^2}{w_{ii} w^2} + \frac{w_1 w_{1ii}}{u_{ii}} + w_{11} \leq 0.$$

From (14)

$$\frac{w_i}{w} = -\frac{w_{11i}}{w_{11}}, \quad i \neq 1,$$

which together with (10), (11) gives

$$\sum_{i,j\neq 1} \frac{w_{1ij}^2}{w_{11}w_{ii}w_{jj}} + \frac{n}{w} - \frac{w_1^2}{w_{11}w^2} + w_{11} \le 0,$$

thus,

$$e^{2M} - ne^{\frac{1}{2}w_1^2}e^M \le w_1^2 e^{w_1^2}$$

and the result follows. □

If $u \in \mathcal{D}_{1,\delta}$ (see Introduction) then, from (5)

$$dist(K, \partial\{u = \delta\}) \le c(n)\delta^{\frac{1}{2}},$$

hence any $|p| < \delta_1(\delta)$ is the subgradient of u at some point in $\{u < \delta\}$.

Thus, if v is the Legendre transform of u, then

$$\det D^2 v = 1, \quad \text{in } B_{\delta_1} \setminus \{0\}.$$

The graph of

$$v^*(p) = \sup_{x\in K} p \cdot x$$

represents the tangent cone of v at 0. Let L_v, L_{v^*} denote the δ_2 level sets of v, respectively v^*, where $\delta_2(\delta) > 0$ such that $L_{v^*} \subset B_{\delta_1}$. Using that $\mathcal{D}_{1,\delta}$ is a compact family one can prove that

$$L_v \subset (1-\delta_3)L_{v^*}, \quad \delta_3(\delta) > 0 \tag{16}$$

Lemma 4.2. *If* $p_0 \in \partial L_{v^*}$ *then* $\|D^2 v^*(p_0)\| \le C(n,\delta)$.

Proof. Consider the function

$$\bar{v} := v - (1+\delta_4)\nabla v^*(p_0)\cdot p.$$

If $\delta_4(\delta) > 0$ is small then, by (16), $\{\bar{v} < 0\} \subset L_{v^*}$.

We apply Pogorelov's estimate to the functions

$$\bar{v}^t(p) = \frac{1}{t}\bar{v}(pt), \quad t > 0$$

and obtain

$$|\bar{v}^t|\|D^2\bar{v}^t\| \le C(n, \max_{\{\bar{v}^t<0\}} |\nabla \bar{v}^t|) = C(n, \max_{\{\bar{v}<0\}} |\nabla \bar{v}|) \le C(n,\delta) \tag{17}$$

As $t \to 0$, $\bar{v}^t$ converges uniformly to

$$v^* - (1+\delta_4)\nabla v^*(p_0)\cdot p$$

and the lemma follows from (16) and (17). □

From the lemma we find that the set L_{v^*} has at all its boundary points a tangent interior ball with radius depending only on δ and n. It is easy to check that this implies the existence of $R(\delta)$ in Theorem 1.3.

For the existence of $r(\delta)$ we prove a Pogorelov type estimate.

Suppose the convex function u is decreasing in the e_n direction in the section $S = \{u < p \cdot x\}$, and denote by $w(x_1, .., x_{n-1}, s)$ the graph in the e_n direction of the s level set, i.e

$$u(x_1, .., x_{n-1}, w(x_1, .., x_{n-1}, s)) = s.$$

In the same way define l_p the graph in the e_n direction of the s level set of $p \cdot x$. The next proposition gives a bound for the curvatures of the level sets of u.

Proposition 4.3. *If*

$$\det D^2 u = f(u) \quad \textit{in } S$$

then

$$w_{11}(l_p - w) \leq C\left(n, \max_S \left|\frac{u_n}{p_n}\right|, \left|\frac{p_1}{p_n}\right| + \max_S \left|\frac{u_1}{u_n}\right|\right).$$

Remark. The constant C does not depend on f. Also, $\frac{u_1}{u_n} = -w_1$.

The normal map to the graph of u at $X = (x_1, .., x_n, x_{n+1}) = (x_1, .., x_n, u(x))$ is given by

$$\nu = (\nu_1, .., \nu_{n+1}) = (1 + |\nabla u|^2)^{-\frac{1}{2}}(-u_1, .., -u_n, 1).$$

The Gauss curvature of the graph of u at X equals

$$\begin{aligned} g(X, \nu) &= \det D_i\left(u_j(1 + |\nabla u|^2)^{-\frac{1}{2}}\right) = \\ &= (1 + |\nabla u|^2)^{-\frac{n+2}{2}} \det D^2 u = (\nu_{n+1})^{n+2} f(x_{n+1}). \end{aligned}$$

The graph of u can be viewed as the graph of w in the e_n direction,

$$(x_1, .., x_n, u(x)) = X = (x_1, .., x_{n-1}, w(x_1, .., x_{n-1}, x_{n+1}), x_{n+1})$$

thus

$$g(X, \nu) = (\nu_n)^{n+2} \det D^2 w,$$

or

$$\det D^2 w = f(x_{n+1})\left(\frac{\nu_{n+1}}{\nu_n}\right)^{n+2} = f(x_{n+1})(-w_{e_{n+1}})^{n+2}.$$

By relabeling the coordinates $(-x_{n+1} \to x_n)$ we have

$$\det D^2 w = f(x_n)(w_n)^{n+2}, \quad w_n > 0.$$

In order to prove Proposition 4.3 it suffices to prove the next estimate.

Lemma 4.4. *Suppose w satisfies*

$$\det D^2 w = f(x \cdot \eta)(w_\xi + c)^{n+2}, \quad w_\xi + c > 0$$

in the bounded set $\{w < 0\}$, where η, ξ are vectors. If $e_1 \cdot \eta = 0$ then

$$w_{11}|w| \leq C\left(n, \max_{\{w<0\}} |w_1|, \max_{\{w<0\}} \frac{c}{w_\xi + c}\right).$$

Proof. Assume the maximum of

$$\log w_{11} + \log |w| + \frac{\sigma}{2} w_1^2 \tag{18}$$

occurs at the origin, where $\sigma > 0$ is small, depending only on $\max |w_1|$.

Again we can assume that $D^2 w(0)$ is diagonal. Indeed, using transformation (13) we find

$$\det D^2 \bar{w} = f(x \cdot \eta)(\bar{w}_{\bar{\xi}} + c)^{n+2}, \quad \bar{w}_{\bar{\xi}}(x) = w_\xi(\bar{x}),$$

thus, the hypothesis and the conclusion remain invariant under this transformation.

We write

$$\log \det D^2 w = \log f(x \cdot \eta) + (n+2) \log(w_\xi + c).$$

Taking derivatives in the e_1 direction we find

$$\frac{w_{1ii}}{w_{ii}} = (n+2) \frac{w_{1\xi}}{w_\xi + c} \tag{19}$$

$$\frac{w_{11ii}}{w_{ii}} - \frac{w_{ij1}^2}{w_{ii} w_{jj}} = (n+2) \frac{w_{11\xi}}{w_\xi + c} - (n+2) \frac{w_{1\xi}^2}{(w_\xi + c)^2} \tag{20}$$

On the other hand, from (18) we obtain at 0

$$\frac{w_{11i}}{w_{11}} + \frac{w_i}{w} + \sigma w_1 w_{1i} = 0, \tag{21}$$

$$\frac{w_{11ii}}{w_{11}} - \frac{w_{11i}^2}{w_{11}^2} + \frac{w_{ii}}{w} - \frac{w_i^2}{w^2} + \sigma w_1 w_{1ii} + \sigma w_{1i}^2 \leq 0. \tag{22}$$

Multiply (22) by w_{ii}^{-1} and add, then use (20), (19)

$$\begin{aligned} &\frac{1}{w_{11}} \left(\frac{w_{ij1}^2}{w_{ii} w_{jj}} + (n+2) \frac{w_{11\xi}}{w_\xi + c} - (n+2) \frac{w_{1\xi}^2}{(w_\xi + c)^2} \right) - \\ &\quad - \frac{w_{11i}^2}{w_{ii} w_{11}^2} + \frac{n}{w} - \frac{w_i^2}{w_{ii} w^2} + (n+2) \sigma w_1 \frac{w_{1\xi}}{w_\xi + c} + \sigma w_{11} \leq 0. \end{aligned} \tag{23}$$

One has

$$\begin{aligned} &\sum_{i,j \neq 1} \frac{w_{ij1}^2}{w_{ii} w_{jj}} - (n+2) \frac{w_{1\xi}^2}{(w_\xi + c)^2} \geq \\ &\quad - \frac{w_{111}^2}{w_{11}^2} + \frac{1}{n} \left(\sum_1^n \frac{w_{1ii}}{w_{ii}} \right)^2 - (n+2) \frac{w_{1\xi}^2}{(w_\xi + c)^2} \geq \\ &\quad - \frac{w_{111}^2}{w_{11}^2} + \frac{(n+2)^2}{n} \frac{w_{1\xi}^2}{(w_\xi + c)^2} - (n+2) \frac{w_{1\xi}^2}{(w_\xi + c)^2} \geq \\ &\geq - \frac{w_{111}^2}{w_{11}^2}. \end{aligned} \tag{24}$$

From (21)

$$\frac{w_i}{w} = -\frac{w_{11i}}{w_{11}}, \quad \text{for } i \neq 1$$

which together with (24) gives us in (23)

$$\frac{n+2}{w_\xi + c}\left(\frac{w_{11\xi}}{w_{11}} + \sigma w_1 w_{1\xi}\right) - \frac{w_{111}^2}{w_{11}^3} + \tag{25}$$
$$+\frac{n}{w} - \frac{w_1^2}{w_{11}w^2} + \sigma w_{11} \leq 0.$$

From (21)

$$\frac{w_{11\xi}}{w_{11}} + \sigma w_1 w_{1\xi} = -\frac{w_\xi}{w} \tag{26}$$

and also

$$\frac{w_{111}}{w_{11}} = -\frac{w_1}{w} - \sigma w_1 w_{11}$$

thus,

$$\frac{w_{111}^2}{w_{11}^2} \leq \frac{w_1^2}{w^2} + \sigma^2 w_1^2 w_{11}^2. \tag{27}$$

We use (26), (27) in (25)

$$-\frac{(n+2)w_\xi}{(w_\xi + c)w} + \frac{n}{w} - 2\frac{w_1^2}{w_{11}w^2} + \sigma(1 - \sigma w_1^2)w_{11} \leq 0.$$

Multiplying by $w_{11}w^2$ we have

$$\sigma(1 - \sigma w_1^2)(ww_{11})^2 + \left(\frac{(n+2)c}{w_\xi + c} - 2\right) ww_{11} \leq 2w_1^2$$

and the result follows if σ is chosen such that $\sigma w_1^2 < 1/2$. □

Below we show that Proposition 4.3 implies the existence of $r(\delta)$ in Theorem 1.3.

For this we solve the equation

$$\det D^2 u^\varepsilon = f_\varepsilon(u^\varepsilon),$$

$$f_\varepsilon(t) = \begin{cases} 1, & t > 0 \\ \varepsilon, & t \leq 0 \end{cases}$$

One has $u^\varepsilon < u$ and u^ε converges uniformly to u as $\varepsilon \to 0$.

Suppose $ae_n \in \partial K$, $a < 0$ and let ν be the interior normal to K at this point. If x is close to $\partial\{u^\varepsilon < 0\}$ then

$$u^\varepsilon(x) \geq \sup_{B_{1/2}} u^\varepsilon.$$

This implies that there exists $\delta_5(\delta), \delta_6(\delta) > 0$ small, such that u^ε is decreasing in the e_n direction in the section

$$\{u^\varepsilon < \delta_5 \nu \cdot ((a + \delta_6)e_n - x)\}.$$

Now we apply Proposition 4.3 to u^ε and the desired result follows as $\varepsilon \to 0$.

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