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C^1 *regularity for infinity harmonic functions in two dimensions*

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Abstract

A continuous function $u : \Omega \to R$, $\Omega \subset R^n$ is said to be "infinity harmonic" if satisfies the PDE

$$-\triangle_\infty u := -\sum_{i,j=1}^{n} u_i u_j u_{ij} = 0 \qquad \text{in } \Omega \tag{1}$$

in the viscosity sense. In this paper we prove that infinity harmonic functions are continuously differentiable when $n = 2$.

1. Introduction

The equation (1) arises when considering optimal Lipschitz extensions from $\partial\Omega$ to Ω. That is, we want to extend a given Lipschitz function $g : \partial\Omega \to R$ to a function $u : \bar{\Omega} \to R$, $u = g$ on $\partial\Omega$, that satisfies the following "absolute minimizing Lipschitz" (AML) property:

for any open set $U \subset \Omega$ and $v : U \to R$ with $v = u$ on ∂U, we have

$$\|\nabla u\|_{L^\infty(U)} \le \|\nabla v\|_{L^\infty(U)}.$$

Jensen [6] proved the equivalence between the (AML) property and solutions of (1). He also proved that the Dirichlet equation for (1) is uniquely solvable.

Crandall, Evans and Gariepy [3] showed that u is infinity harmonic if and only if u satisfies comparison with cones from above and below. To be more precise, we say that u satisfies comparison with cones from above in Ω if given any open set $U \subset\subset \Omega$, and $a, b \in R$ such that

$$u(x) \le a + b|x - x_0| \quad \text{on } \partial(U \setminus x_0)$$

then

$$u(x) \leq a + b|x - x_0| \quad \text{in } U.$$

Similarly one can define comparison with cones from below.

An interesting question is to determine whether or not infinity harmonic functions are continuously differentiable. A result in this direction was obtained by Crandall and Evans [4] (see also Crandall-Evans-Gariepy [3]) which showed that at small scales u is close to a plane.

Theorem 1. *[Crandall-Evans-Gariepy]*

Let $u : \Omega \to R$, $\Omega \subset R^n$ be infinity harmonic. Then for each $x \in \Omega$ there exist vectors $e_{x,r} \in R^n$ with $|e_{x,r}| = S(x)$ (see section 2 for the definition of S) such that

$$\max_{B_r(x)} \frac{|u(y) - u(x) - e_{x,r} \cdot (y - x)|}{r} \to 0 \quad \text{as } r \to 0.$$

In this paper we prove that in 2 dimensions the vectors $e_{x,r}$ converge as $r \to 0$, and obtain

Theorem 2. *Let $u : \Omega \to R$, $\Omega \subset R^2$ be infinity harmonic. Then $u \in C^1(\Omega)$.*

The idea of the proof is the following. Suppose that

$$u(0) = 0, \quad \|u - e_1 \cdot x\|_{L^\infty(B_1)} \leq \varepsilon.$$

From the theory of elliptic equations in two dimensions (see [5] chapter 12), heuristically we can find a plane $e \cdot x$ (the tangent plane at 0) such that $\{u = e \cdot x\}$ divides R^2 into four connected regions. If e and e_1 are not close to each other then, one connected component of $\{u > e \cdot x\}$ is included in a narrow strip and we are able to derive a contradiction.

Using a compactness argument we prove

Theorem 3. *(Modulus of continuity for the gradient)*

There exists a function

$$\rho : [0,1] \to R^+, \quad \lim_{r \to 0} \rho(r) = 0$$

such that for any infinity harmonic function

$$u : B_1 \subset R^2 \to R, \quad \|\nabla u\|_{L^\infty(B_1)} \leq 1$$

we have

$$|\nabla u(x) - \nabla u(y)| \leq \rho(|x - y|), \quad \text{if } x, y \in B_{1/2}\ .$$

As a consequence of Theorem 3 we obtain the following Liouville type theorem.

Theorem 4. *Let $u : R^2 \to R$ be a global infinity harmonic function. If u grows at most linearly at ∞, i.e*

$$|u(x)| \le C(1 + |x|),$$

then u is linear.

Theorem 4 follows easily from Proposition 3.

Suppose u satisfies the hypothesis of Theorem 4. We use Proposition 3 for the rescaled function

$$w(x) = \frac{1}{R} u(Rx), \quad x \in B_1$$

and obtain

$$|\nabla u(x_0) - \nabla u(0)| = |\nabla w(x_0 R^{-1}) - \nabla w(0)| \le C\rho(|x_0| R^{-1}).$$

The conclusion follows as we let $R \to \infty$.

2. The Proofs

Notation:

Ω is an open set in R^2

$B_r(x_0)$ denotes the open ball of radius r and center x_0

$B_r = B_r(0)$

$x \cdot y$ represents the Euclidean inner-product.

$\{f < g\}$ denotes $\{x \in R^2| \quad f(x) < g(x)\}$

Suppose that $u : \Omega \to R$ is infinity harmonic. If $B_r(x) \subset \Omega$ we define

$$S^+(x,r) = \max_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{|y - x|}$$

and

$$S^-(x,r) = \min_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{|y - x|}$$

We recall the following result from [3].

Proposition 1. *The function $S^+(x,r)$ is increasing in r and $S^-(x,r)$ is decreasing in r. Moreover,*

$$S(x) := \lim_{r \to 0} S^+(x,r) = -\lim_{r \to 0} S^-(x,r)$$

Our main goal is to prove

Proposition 2. *Suppose u is infinity harmonic in $B_1 \subset R^2$. Given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that if*

$$\|u - e_1 \cdot x\|_{L^\infty(B_1)} \le \delta, \quad |e_1| = 1, \tag{2}$$

then u is differentiable at 0 and

$$|\nabla u(0) - e_1| \le \varepsilon.$$

Theorem 2 clearly follows from Theorem 1 and Proposition 2.

We start with a lemma.

Lemma 1. *Let $u : \Omega \to R$, $\Omega \subset R^2$, be infinity harmonic. Assume that Ω is convex, $u(0,0) = 0$ and for some t_0 small and $c \in R$ we have*

$$u(t,0) \geq ct, \quad \textit{if } t \in [-t_0, t_0]$$

$$u(t_0,0) > ct_0, \quad u(-t_0,0) > -ct_0.$$

Then there exists a plane $P := e \cdot x$, $|e| = S(0)$, such that $(t_0,0)$ and $(-t_0,0)$ belong to distinct connected components of the set $\{u > P\}$.

Proof: From Theorem 1 we can find $r_i \to 0$ and $e = (a_1, a_2) \in R^2$, $|e| = S(0)$ such that

$$\frac{\|u(x) - e \cdot x\|_{L^\infty(B_{r_i})}}{r_i} \to 0 \quad \text{as } i \to \infty \tag{3}$$

Notice that $a_1 = c$.

Assume that $(-t_0, 0)$ and $(t_0, 0)$ can be connected by a polygonal line included in $\{u > P\} \cap \Omega$. Close the polygonal line by connecting $(-t_0, 0)$ and $(t_0, 0)$ by a line segment. Denote this polygonal path by Γ. Without loss of generality we assume that there exists an open set $U \subset\subset \Omega$ such that

$$\Gamma \subseteq \partial U, \quad B_\delta \cap \{x_2 > 0\} \subset U$$

for some $\delta > 0$ small.

If $x \in \partial U$ we can find $\varepsilon > 0$ such that

$$u(x) \geq e \cdot x + (0, \varepsilon) \cdot x;$$

hence the inequality is also true for $x \in U$. This contradicts (3) and the lemma is proved. □

Next we prove

Proposition 3. *Suppose that u is infinity harmonic in $B_{6R} \subset R^2$ and satisfies*

H1)

$$\|u - e_1 \cdot x\|_{L^\infty(B_{6R})} \leq 1, \quad |e_1| = 1$$

H2) There exists a plane $P := e \cdot x$ such that the set $\{u > P\}$ has at least two distinct connected components that intersect B_R.

Given $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ large such that if $R \geq C(\varepsilon)$ then

$$|e - e_1| \leq \varepsilon.$$

Proof: Denote

$$f := e_1 - e$$

and assume that $|f| > \varepsilon$. We have

$$\{f \cdot x < -1\} \cap B_{6R} \subset \{u < P\}$$

$$\{f \cdot x > 1\} \cap B_{6R} \subset \{u > P\}.$$

Thus, from H2, we can find a connected component U of $\{u > P\}$ that intersects B_R and is included in the strip $\mathcal{S} := \{|f \cdot x| \leq 1\}$ of width $2|f|^{-1} < 2\varepsilon^{-1}$.

Notice that we cannot have $U \subset\subset B_{6R}$ since otherwise we contradict the comparison principle. Consider a polygonal line inside U that starts in B_R and exits B_{6R}. By shifting the origin a distance $3R$ in the direction perpendicular to f, one can assume

H1')

$$\|u - e_1 \cdot x\|_{L^\infty(B_{2R})} \leq 1, \quad |e_1| = 1$$

H2') The set $\{u > P\} \cap B_{2R}$ has a connected component $U \subset \mathcal{S}$ that contains a polygonal line connecting the two arcs of $\mathcal{S} \cap \partial B_R$.

Proposition 3 will follow from the next two lemmas.

Let $\alpha \in [0, \frac{\pi}{2}]$ denote the angle between the directions of e and f.

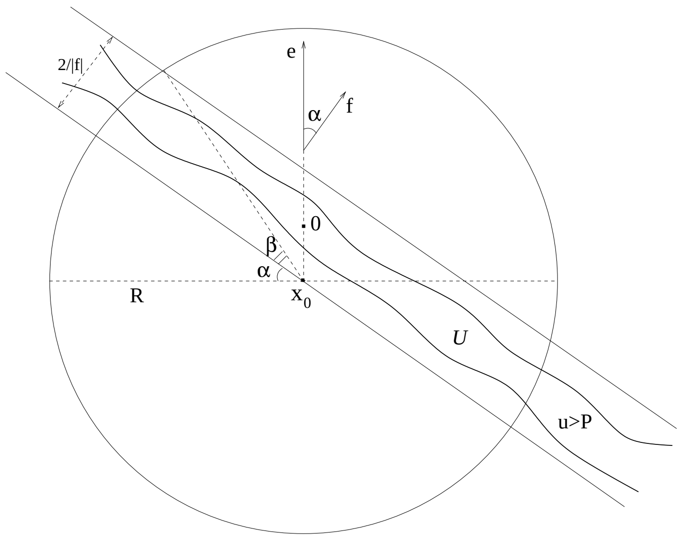
Lemma 2. *Fix $\delta_1 > 0$ small. If $|e| \geq \delta_1$ and $R \geq C(\varepsilon, \delta_1)$, then*

$$\alpha \geq \frac{\pi}{2} - \delta_1.$$

Proof: Assume that $\alpha < \frac{\pi}{2} - \delta_1$. Denote by x_0 the intersection of the half line $\{-te, \quad t \geq 0\}$ with $\partial\mathcal{S}$. Clearly,

$$|u - e \cdot x| \leq |u - e_1 \cdot x| + |(e_1 - e) \cdot x| \leq 2 \quad \text{in } U \cap B_R(x_0),$$

$$u = e \cdot x \quad \text{on } \partial U \cap B_R(x_0). \tag{4}$$



On the set $U \cap \partial B_R(x_0)$, we have

$$u(x) \le e \cdot x + 2 \le \sup_{S \cap \partial B_R(x_0)} e \cdot x + 2 \tag{5}$$

$$\le e \cdot x_0 + |e| R \sin(\alpha + \beta) + 2$$

$$= e \cdot x_0 + |e| R \left(\sin(\alpha + \beta) + \frac{2}{|e| R} \right),$$

where

$$\sin \beta = \frac{2}{R|f|}.$$

If $R \ge C(\varepsilon, \delta_1)$ is large, then since $\sin \alpha < 1$, we deduce from (4), (5) that

$$u(x) \le e \cdot x_0 + |e||x - x_0| \quad \text{on } \partial(U \cap B_R(x_0)).$$

Hence comparison with cones implies

$$u(x) \le e \cdot x_0 + |e||x - x_0| \quad \text{on } U \cap B_R(x_0).$$

We obtain

$$u(x) \le e \cdot x = P \quad \text{on } \{x_0 + te, \quad t \ge 0\} \cap U$$

or

$$\{x_0 + te, \quad t \ge 0\} \cap U = \emptyset.$$

This contradicts H2'. With this Lemma 2 is proved. $\square$

Lemma 3. *Fix $\delta_2 > 0$ small. If $R \ge C(\delta_2)$, then*

$$|e| \ge 1 - \delta_2.$$

Proof: Assume that $|e| < 1 - \delta_2$ and notice that $f \cdot e_1 > \delta_2$.

Denote by $y_0 := -4\delta_2^{-1} e_1$, and let y_1 be the intersection of the half line $\{te_1, \quad t \geq 0\}$ with the line $\{f \cdot x = 1\}$.

Consider the family of cones with vertex at $(y_0, e_1 \cdot y_0 + 1)$ and slope c; that is,

$$V_{y_0,c}(x) := e_1 \cdot y_0 + 1 + c|x - y_0|.$$

Notice that the vertex of $V_{y_0,c}$ is above the graph of u and below the graph of P.

For $c > |e|$ we denote by E_c the ellipse which is the intersection of $V_{y_0,c}$ with P, i.e

$$E_c := \{x| \quad V_{y_0,c}(x) = e \cdot x\}.$$

One has

$$c_0 := 1 - \frac{2}{|y_1 - y_0|} \geq 1 - \frac{\delta_2}{2} > |e|,$$

and

$$V_{y_0,c_0}(y_1) = e_1 \cdot y_0 + 1 + |y_1 - y_0| - 2$$

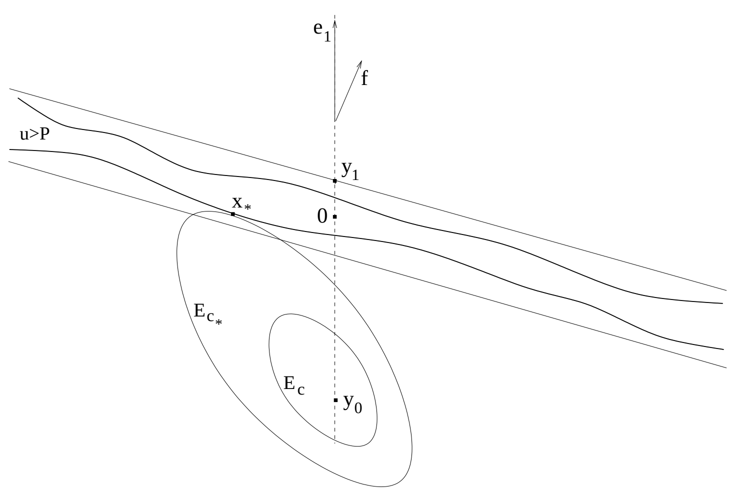
$$= e_1 \cdot y_1 - 1 = e_1 \cdot y_1 - f \cdot y_1 = e \cdot y_1.$$

Hence

$$y_1 \in E_{c_0}. \tag{6}$$

Take c large and decrease c continuously until E_c touches for the first time $\partial(\{u < P\} \cap B_{2R})$. Let the first value be c_* and let

$$x_* \in E_{c_*} \cap \partial(\{u < P\} \cap B_{2R}).$$



From (6) one can conclude that $c_* \geq c_0$. If R is large, then $x_* \in B_R$ and (see Proposition 1)

$$S(x_*) \geq c_* \geq c_0 \geq 1 - \frac{\delta_2}{2}. \tag{7}$$

On the other hand we claim that $S(x_*) \leq |e| + 2R^{-1}$.

To see this, choose a small $r > 0$ and let U' be the open set defined as the union of all connected components of $\{u > P\} \cap B_R(x_*)$ that intersect $B_r(x_*)$.

If $U' = \emptyset$ then the claim is obvious. So assume $U' \neq \emptyset$ and from H2' we find $U' \subset \mathcal{S}$ provided that r is chosen small enough.

One has

$$u = e \cdot x \quad \text{on } \partial U' \cap B_R(x_*);$$

and for $x \in U' \cap \partial B_R(x_*)$,

$$u(x) \leq e \cdot x + 2 \leq e \cdot x_* + |e|R + 2.$$

This implies

$$u(x) \leq e \cdot x_* + \left(|e| + \frac{2}{R}\right)|x - x_*| \quad \text{on } \partial(U' \cap B_R(x_*)).$$

Hence the inequality is valid also in $U' \cap B_R(x_*)$. Now it is clear that

$$S(x_*) \leq |e| + 2R^{-1} \leq 1 - \delta_2 + 2R^{-1}.$$

This contradicts (7) if $R \geq C(\delta_2)$ is large. With this Lemma 3 is proved. □

Proposition 3 now follows from Lemma 2 and Lemma 3 by choosing $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ small and $R \geq C(\varepsilon)$ large enough. □

By rescaling Proposition 3 we obtain

Corollary 1. *Suppose $u : B_r \to R$, $B_r \subset R^2$ is infinity harmonic and*

$$\|u - e_1 \cdot x\|_{L^\infty(B_r)} \leq \delta r |e_1|.$$

Suppose also that there exists a plane $P := e \cdot x$ such that $\{u > P\}$ has at least two distinct connected components that intersect $B_{r/6}$. Then, given ε, there exits $\delta(\varepsilon)$ such that if $\delta \leq \delta(\varepsilon)$ we have

$$|e - e_1| \leq \varepsilon |e_1|.$$

Corollary 1 follows by noticing that the rescaled function

$$w(x) := \frac{R}{r|e_1|} u(\frac{rx}{R}), \quad R = 6C(\varepsilon)$$

satisfies the hypothesis of Proposition 3 if $\delta R \leq 1$. □

Proof of Proposition 2: First we show

$$\limsup_{r\to 0} |e_{0,r} - e_1| \leq \varepsilon \quad \text{if } \delta \leq \delta(\varepsilon) \tag{8}$$

Case 1: Suppose that u is not identical to a plane in a neighborhood of 0.

Then there exists a line segment $[z_1, z_2]$ in $B_{r/6}$ where u is not linear when restricted to it. On this segment we can find a linear function l of slope

$$\frac{u(z_2) - u(z_1)}{|z_2 - z_1|}$$

and an interior point $y \in (z_1, z_2)$ such that either

$$u \geq l \quad \text{on } [z_1, z_2]$$

$$u(y) = l(y), \quad u(z_1) > l(z_1), \quad u(z_2) > l(z_2)$$

or

$$u \leq l \quad \text{on } [z_1, z_2]$$

$$u(y) = l(y), \quad u(z_1) < l(z_1), \quad u(z_2) < l(z_2).$$

Without loss of generality assume the first situation holds. Then, by Lemma 1, there exists e_y such that the set

$$\{u > u(y) + e_y \cdot (x - y)\}$$

has two distinct connected components in B_1.

By Corollary 1 we have

$$|e_y - e_1| \leq \frac{\varepsilon}{4} \tag{9}$$

if δ is small. From Theorem 1

$$\|u - u(0) - e_{0,r} \cdot x\|_{L^\infty(B_r)} \leq r\sigma(r) \tag{10}$$

$$\sigma(r) \to 0 \quad \text{as } r \to 0,$$

and we find

$$|e_y| = S(y) \leq \max_{\partial B_{r/2}(y)} \frac{|u(x) - u(y)|}{r/2} \leq |e_{0,r}| + 4\sigma(r).$$

Similarly one obtains

$$|e_{0,r}| = S(0) \leq 1 + 2\delta.$$

The above inequalities and (9) imply

$$1 - \varepsilon/4 - 4\sigma(r) \leq |e_{0,r}| \leq 1 + 2\delta.$$

Now we apply Corollary 1 in B_r and obtain

$$|e_y - e_{0,r}| \leq |e_{0,r}|\frac{\varepsilon}{4} \leq \frac{\varepsilon}{2} \tag{11}$$

provided that r is small enough. Now (8) follows from (9), (11).

Case 2: Suppose that u is identical to a plane $P = e \cdot x$ in a neighborhood of 0. Denote by U the interior of the set $\{u = P\}$. If $dist(0, \partial U) > 1/2$, then (8) is obvious. If not, let $x_0 \in \partial U$ be a point where the distance from 0 to ∂U is realized. From case 1 applied to $B_{1/2}(x_0)$ we find

$$\limsup_{r \to 0} |e_{x_0,r} - e_1| \leq \varepsilon$$

hence

$$|e - e_1| \leq \varepsilon.$$

In conclusion (8) is proved.

It remains to prove that $e_{0,r}$ converges as $r \to 0$.

Let $r_i \to 0$ be an arbitrary sequence. By rescaling (8) to the balls B_{r_j} we find that for each ε there exists j large such that

$$\limsup_{i \to \infty} |e_{0,r_i} - e_{0,r_j}| \leq \varepsilon$$

Thus e_{0,r_i} is a Cauchy sequence and Proposition 2 is proved. □

Proof of Theorem 3:

The proof is by compactness. Assume by contradiction the statement is false. Then we can find $\varepsilon_0 > 0$, functions u_k satisfying the hypothesis of Proposition 3 and points $x_k \to 0$ such that

$$|\nabla u_k(x_k) - \nabla u_k(0)| \geq \varepsilon_0 \quad \text{as } k \to \infty.$$

We consider the rescaled functions

$$v_k(x) := \frac{u_k(|x_k|x) - u_k(0)}{|x_k|}.$$

The functions v_k are infinity harmonic, defined on $B_{|x_k|^{-1}}$, $\|\nabla v_k\|_{L^\infty} \leq 1$ and

$$|\nabla v_k(x_k|x_k|^{-1}) - \nabla v_k(0)| \geq \varepsilon_0. \tag{12}$$

By the Arzela Ascoli Theorem there exists a subsequence (we still denote it by v_k) that converges uniformly on compact sets to a function v_∞. Obviously v_∞ is infinity harmonic, defined on R^2 with

$$\|\nabla v_\infty\|_{L^\infty} \leq 1.$$

As a consequence of Theorem 1 one can find $e \in R^2$ and $R_i \to \infty$ such that

$$\|v_\infty - e \cdot x\|_{L^\infty(B_{R_i})} \leq R_i \sigma(R_i)$$
$$\sigma(R_i) \to 0 \quad \text{as } i \to \infty.$$

Thus, for every fixed ball B_{R_i} we have

$$\limsup_{k \to \infty} \|v_k - e \cdot x\|_{L^\infty(B_{R_i})} \leq R_i \sigma(R_i). \tag{13}$$

If $e = 0$, then for large k and $x \in B_1$ we have (see Proposition 1 and(13))

$$|\nabla v_k(x)| \leq S^+(v_k, x) \leq 2\sigma(R_i)$$

which contradicts (12) if R_i is chosen large enough.

If $e \neq 0$, then there exists R_i large such that

$$2|e|^{-1}\sigma(R_i) \leq \delta(\varepsilon_0/4).$$

From (13) and Proposition 2 (rescaled to $B_{R_i/2}(x)$) we find

$$|\nabla v_k(x) - e| \leq |e|\varepsilon_0/4$$

for all $x \in B_1$ and k large. This contradicts (12) and the theorem is proved.
□

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