GLOBAL $W^{2,p}$ ESTIMATES FOR THE MONGE-AMPERE EQUATION

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ABSTRACT. We use a localization property of boundary sections for solutions to the Monge-Ampere equation and obtain global $W^{2,p}$ estimates under natural assumptions on the domain and boundary data.

1. INTRODUCTION

Interior $W^{2,p}$ estimates for strictly convex solutions for the Monge-Ampere equation were obtained by Caffarelli in [C] under the necessary assumption of small oscillation of the right hand side. The theorem can be stated as follows.

Theorem 1.1 (Interior $W^{2,p}$ estimates)**.** *Let* $u : \overline{\Omega} \to \mathbb{R}$,

$$u = 0 \quad on\ \partial\Omega, \tag{1.1}$$

be a continuous convex solution to the Monge-Ampere equation

$$\det D^2 u = f(x) \quad in\ \Omega, \qquad 0 < \lambda \le f \le \Lambda, \tag{1.2}$$

for some positive constants λ, Λ.

For any p, $1 < p < \infty$ *there exists* $\varepsilon(p) > 0$ *depending on* p *and* n *such that if*

$$B_\rho \subset \Omega \subset B_{1/\rho}$$

and

$$\sup_{|x-y|\le\rho} |\log f(x) - \log f(y)| \le \varepsilon(p), \tag{1.3}$$

for some small $\rho > 0$, *then*

$$\|u\|_{W^{2,p}(\{u<-\rho\})} \le C,$$

and C *depends on* ρ, λ, Λ, p *and* n.

In this short paper we answer the natural question of whether this interior estimates can be extended up to the boundary. Precisely, we obtain the following global $W^{2,p}$ estimate under natural assumptions on the domain and boundary data.

Theorem 1.2 (Global $W^{2,p}$ estimates)**.** *Let* Ω *be a convex bounded domain and* $u : \overline{\Omega} \to \mathbb{R}$ *be a Lipschitz continuous convex solution of the Monge-Ampere equation* (1.2). *Assume that*

$$u|_{\partial\Omega}, \quad \partial\Omega \in C^{1,1},$$

and there exists $\rho > 0$ *small such that* f *satisfies* (1.3). *If* u *separates quadratically on* $\partial\Omega$ *from its tangent planes i.e*

$$u(y) - u(x) - \nabla u(x) \cdot (y - x) \ge \rho |x - y|^2 \qquad \forall x, y \in \partial\Omega, \tag{1.4}$$

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then

$$\|u\|_{W^{2,p}(\Omega)} \le C,$$

with C depending on $\|\partial\Omega\|_{C^{1,1}}$, $\|u|_{\partial\Omega}\|_{C^{1,1}}$, $\|u\|_{C^{0,1}}$, ρ, λ, Λ, p and n.

Remark 1.3. The gradient $\nabla u(x)$ when $x \in \partial\Omega$ is understood in the sense that

$$y_{n+1} = u(x) + \nabla u(x) \cdot (y - x)$$

is a supporting hyperplane for the graph of $u(y)$ but

$$y_{n+1} = u(x) + (\nabla u(x) - \delta \nu_x) \cdot (y - x)$$

is not a supporting hyperplane for any $\delta > 0$, where ν_x denotes the exterior normal to $\partial\Omega$ at x.

In general the Lipschitz continuity of the solution can be easily obtained from the boundary data by the use of barriers. Also the quadratic separation assumption (1.4) can be checked in several situations directly from the boundary data, see Proposition 3.2 in [S2]. This is the case for example when the boundary data is more regular i.e

$$u|_{\partial\Omega}, \partial\Omega \in C^3, \quad \text{and } \Omega \text{ is uniformly convex.} \tag{1.5}$$

As a consequence of Theorem 1.2 we obtain

Corollary 1.4. *Let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous solution of the Monge-Ampere equation* (1.2) *that satisfies* (1.3) *and* (1.5)*. Then*

$$\|u\|_{W^{2,p}(\Omega)} \le C,$$

with C depending on $u|_{\partial\Omega}$, $\partial\Omega$, ρ, λ, Λ, p and n.

In particular, if u solves the Monge-Ampere equation (1.2) with $f \in C(\overline{\Omega})$ and (1.5) holds, then $u \in W^{2,p}(\Omega)$ for any $p < \infty$.

The assumptions on the boundary behavior of u and $\partial\Omega$ in Theorem 1.2 and Corollary 1.4 seem to be optimal. Wang in [W] gave examples of solutions u to (1.2) with $f = 1$ and either $u \in C^{2,1}$ or $\partial\Omega \in C^{2,1}$, that do not belong to $W^{2,p}(\Omega)$ for large values of p.

The proof of Theorem 1.2 is based on a localization theorem for the Monge-Ampere equation at boundary points which was proved in [S1], [S2]. It states that under natural local assumptions on the domain and boundary data, the sections

$$S_h(x_0) = \{x \in \overline{\Omega} | \quad u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\},$$

with $x_0 \in \partial\Omega$ are "equivalent" to ellipsoids centered at x_0. We give its precise statement below.

Assume for simplicity that

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \ge 0\} \cap B_{\frac{1}{\rho}}, \tag{1.6}$$

for some small $\rho > 0$, that is $\Omega \subset (\mathbb{R}^n)^+$ and Ω contains an interior ball tangent to $\partial\Omega$ at 0. Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous, convex, satisfying

$$\det D^2 u = f, \qquad 0 < \lambda \le f \le \Lambda \quad \text{in } \Omega. \tag{1.7}$$

After subtracting a linear function we also assume that

$$x_{n+1} = 0 \text{ is the tangent plane to } u \text{ at } 0, \tag{1.8}$$

in the sense that

$$u \ge 0, \quad u(0) = 0,$$

and any hyperplane $x_{n+1} = \delta x_n$, $\delta > 0$, is not a supporting plane for u.

Theorem 1.5 shows that if the boundary data has quadratic growth near $\{x_n = 0\}$ then, each section of u at 0

$$S_h := S_h(0) = \{x \in \overline{\Omega} : \quad u(x) < h\},$$

is equivalent to a half-ellipsoid centered at 0.

Theorem 1.5 (Localization theorem)**.** *Assume that* Ω, u *satisfy* (1.6)-(1.8) *above and,*

$$\rho |x|^2 \le u(x) \le \rho^{-1} |x|^2 \quad \text{on } \partial\Omega \cap \{x_n \le \rho\}.$$

Then, for each $h < c_0$ *there exists an ellipsoid* E_h *of volume* $\omega_n h^{n/2}$ *such that*

$$c_0 E_h \cap \overline{\Omega} \, \subset \, S_h \, \subset \, c_0^{-1} E_h \cap \overline{\Omega}.$$

Moreover, the ellipsoid E_h *is obtained from the ball of radius* $h^{1/2}$ *by a linear transformation* A_h^{-1} *(sliding along the* $x_n = 0$ *plane)*

$$A_h E_h = h^{1/2} B_1$$

$$A_h(x) = x - \nu x_n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}, 0),$$

with

$$|\nu| \le c_0^{-1} |\log h|.$$

The constant $c_0 > 0$ *above depends on* $\rho, \lambda, \Lambda,$*and* n.

We describe the idea of the proof of Theorem 1.2 below.

If the assumptions of Theorem 1.2 hold say for simplicity with $f = 1$ then, by rescaling, one can deduce from the Localization theorem above that the second derivatives can blow-up near $\partial\Omega$ in a logarithmic fashion i.e

$$\|D^2 u(y)\| \le C |\log \ dist(y, \partial\Omega)|^2,$$

thus $D^2 u \in L^p(\Omega)$ for any $p < \infty$. For general f we need to use the interior $W^{2,p}$ estimates instead of the estimate above together with a Vitali covering lemma for sections of u near $\partial\Omega$.

2. Proof of Theorem 1.2

We start by remarking that under the assumptions of Theorem 1.5 above, we obtain that also the section $\{u < h^{1/2} x_n\}$ has the shape of E_h.

Indeed, since

$$S_h \subset c_0^{-1} E_h \subset \{x_n \le c_0^{-1} h^{1/2}\}$$

and $u(0) = 0$, we can conclude from the convexity of u that the set

$$F := \{x \in \overline{\Omega} | \quad u < c_0 h^{1/2} x_n\}$$

satisfies for all small h

$$F \subset S_h \cap \Omega, \tag{2.1}$$

and F is tangent to $\partial\Omega$ at 0. We show that F is equivalent to E_h by bounding its volume from below.

Lemma 2.1. *We have*

$$|F| \ge c |E_h|$$

for some $c > 0$ *small depending on* ρ, λ, Λ, n.

Proof. From Theorem 1.5, there exists $y \in \partial S_{\theta h}$ such that $y_n \geq c_0(\theta h)^{1/2}$. We evaluate

$$v := u - c_0 h^{1/2} x_n,$$

at y and find

$$v(y) \leq \theta h - c_0 h^{1/2} c_0 (\theta h)^{1/2} \leq -\delta h,$$

for some $\delta > 0$ provided that we choose θ small depending on c_0. Since $v = 0$ on ∂F and

$$\det D^2 v \geq \lambda$$

we have

$$|\inf_F v| \leq C(\lambda)|F|^{2/n},$$

hence

$$ch^{n/2} \leq |F|.$$

□

Next we prove Theorem 1.2. We denote by c, C positive constants that depend on ρ, λ, Λ, p, n and $\|\partial\Omega\|_{C^{1,1}}$, $\|u|_{\partial\Omega}\|_{C^{1,1}}$, $\|u\|_{C^{0,1}}$. For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion.

Given $y \in \Omega$ we denote by

$$S_h(y) := \{x \in \overline{\Omega} | \quad u(x) < u(y) + \nabla u(y) \cdot (x - y) + h\}$$

and let $\bar{h}(y)$ be the maximal value of h such that $S_h(y) \subset \Omega$, i.e

$$\bar{h}(y) := \max\{h \geq 0 | \quad S_h(y) \subset \Omega\}.$$

Lemma 2.2. *Let $y \in \Omega$ and denote for simplicity $\bar{h} = \bar{h}(y)$. If $\bar{h} \leq c_1$ then*

$$S_{\bar{h}}(y) \subset D_{C\bar{h}^{1/2}} := \{x \in \overline{\Omega} | \quad dist(x, \partial\Omega) \leq C\bar{h}^{1/2}\},$$

and

$$\int_{S_{\bar{h}/2}(y)} \|D^2 u\|^p \, dx \leq C|\log \bar{h}|^{2p} \, \bar{h}^{n/2}.$$

Proof. Without loss of generality assume

$$0 \in \partial S_{\bar{h}}(y) \cap \partial\Omega.$$

After subtracting a linear function and relabeling ρ if necessary we may also assume that u satisfies the conditions of Theorem 1.5 at the origin. Since u separates quadratically from 0 on $\partial\Omega$ we deduce that $\nabla u(y) = \alpha e_n$ thus

$$S_{\bar{h}}(y) = \{x \in \overline{\Omega} | \quad u < \alpha x_n\}, \qquad \alpha > 0,$$

and $\bar{h} > 0$. Since (see [C])

$$c\bar{h}^{n/2} \leq |S_{\bar{h}}(y)| \leq C\bar{h}^{n/2},$$

we obtain from (2.1) and Lemma 2.1

$$c\bar{h}^{1/2} \leq \alpha \leq C\bar{h}^{1/2},$$

and that $S_{\bar{h}}(y)$ is equivalent to an ellipsoid $\tilde{E} := E_{C\bar{h}}$. This implies that

$$y + c\tilde{E} \subset S_{\bar{h}}(y) \subset C\tilde{E},$$

with

$$\tilde{E} = \bar{h}^{1/2} A^{-1} B_1, \quad Ax := x - \nu x_n,$$

and

$$\nu_n = 0, |\nu| \le C|\log \bar{h}|.$$

The inclusion above implies

$$S_{\bar{h}}(y) \subset \{x_n \le C\bar{h}^{1/2}\} \subset D_{C\bar{h}^{1/2}}.$$

The rescaling $\tilde{u} : \tilde{\Omega} \to \mathbb{R}$ of u

$$\tilde{u}(x) := \frac{1}{\bar{h}} u(y + \bar{h}^{1/2} A^{-1} x),$$

satisfies

$$\tilde{\Omega} = \bar{h}^{-1/2} A(\Omega - y), \qquad \tilde{S}_1(0) = \bar{h}^{-1/2} A(S_{\bar{h}}(y) - y) \subset \tilde{\Omega},$$

and

$$B_c \subset \tilde{S}_1(0) \subset B_C,$$

where $\tilde{S}_1(0)$ represents the section of $\tilde{u}$ at the origin at height 1. Moreover,

$$\det D^2 \tilde{u}(x) = \tilde{f}(x) := f(y + \bar{h}^{1/2} A^{-1} x)$$

and (1.3) implies

$$|\log \tilde{f}(x) - \log \tilde{f}(z)| \le \varepsilon(p) \qquad \forall x, z \in \tilde{S}_1(0)$$

since

$$|\bar{h}^{1/2} A^{-1}(x - z)| \le C\bar{h}^{1/2} |\log \bar{h}| \le \rho.$$

The interior $W^{2,p}$ estimate for $\tilde{u}$ in $\tilde{S}_0(1)$ gives

$$\int_{\tilde{S}_{1/2}(0)} \|D^2 \tilde{u}\|^p \, dx \le C,$$

hence

$$\int_{S_{\bar{h}/2}(y)} \|D^2 u\|^p \, dx = \int_{\tilde{S}_{1/2}(0)} \|A^T \, D^2 \tilde{u} \, A\|^p \, \bar{h}^{n/2} \, dx \le C |\log \bar{h}|^{2p} \, \bar{h}^{n/2}.$$

□

We also need the following covering lemma.

Lemma 2.3 (Vitali covering)**.** *There exists a sequence of disjoint sections*

$$S_{\delta \bar{h}_i}(y_i), \qquad \bar{h}_i = \bar{h}(y_i),$$

such that

$$\Omega \subset \bigcup_{i=1}^{\infty} S_{\bar{h}_i/2}(y_i),$$

where $\delta > 0$ is a small constant that depends only on λ, Λ and n.

Proof. We choose δ such that if

$$S_{\delta \bar{h}(y)}(y) \cap S_{\delta \bar{h}(z)}(z) \ne \emptyset \qquad \text{and} \quad 2\bar{h}(y) \ge \bar{h}(z),$$

then

$$S_{\delta \bar{h}(z)}(z) \subset S_{\bar{h}(y)/2}(y).$$

The existence of δ follows from the engulfing properties of sections of solutions to Monge-Ampere equation (1.2) with bounded right hand side (see [CG]).

Now the proof is identical to the proof of Vitali's covering lemma for balls. We choose $S_{\delta \bar h_1}(y_1)$ from all the sections $S_{\delta \bar h(y)}(y)$, $y \in \Omega$ such that

$$\bar h(y_1) \ge \frac 12 \sup_y \bar h(y),$$

then choose $S_{\delta \bar h_2}(y_2)$ as above but only from the remaining sections $S_{\delta \bar h(y)}(y)$ that are disjoint from $S_{\delta \bar h_1}(y_1)$, then $S_{\delta \bar h_3}(y_3)$ etc. We easily obtain

$$\Omega = \bigcup_{y \in \Omega} S_{\delta \bar h(y)}(y) \subset \bigcup_{i=1}^\infty S_{\bar h_i/2}(y_i).$$

$\square$

End of proof of Theorem 1.2

$$\int_\Omega \|D^2 u\|^p \, dx \le \sum_i \int_{S_{\bar h_i/2}(y_i)} \|D^2 u\|^p \, dx.$$

There are a finite number of sections with $\bar h_i \ge c_1$ and, by the interior $W^{2,p}$ estimate, in each such section we have

$$\int_{S_{\bar h_i/2}(y_i)} \|D^2 u\|^p \, dx \le C.$$

Next we consider the family $\mathcal F_d$ of sections $S_{\bar h_i/2}(y_i)$ such that

$$d/2 < \bar h_i \le d$$

for some constant $d \le c_1$. By Lemma 2.2 in each such section

$$\int_{S_{\bar h_i/2}(y_i)} \|D^2 u\|^p \, dx \le C |\log d|^{2p} |S_{\delta \bar h_i}(y_i)|,$$

and since

$$S_{\delta \bar h_i}(y_i) \subset D_{C d^{1/2}}$$

are disjoint we find

$$\sum_{i \in \mathcal F_d} \int_{S_{\bar h_i/2}(y_i)} \|D^2 u\|^p \, dx \le C |\log d|^{2p} d^{1/2}.$$

We add these inequalities for the sequence $d = c_1 2^{-k}$, $k = 0, 1, \ldots$ and obtain the desired bound.

References

[C] Caffarelli L., Interior $W^{2,p}$ estimates for solutions of Monge-Ampere equation, *Ann. of Math.* **131** (1990), 135-150.

[CG] Caffarelli L., Gutierrez C., Properties of solutions of the linearized Monge-Ampere equations, *Amer. J. Math.* **119** (1997), 423-465.

[S1] Savin O., A localization property at the boundary for Monge-Ampere equation, Preprint arxiv:1010.1745.

[S2] Savin O, Pointwise $C^{2,\alpha}$ estimates at the boundary for Monge-Ampere equation, Preprint arxiv:1101.5436.

[W] Wang X.-J., Regularity for Monge-Ampere equation near the boundary, *Analysis* **16** (1996) 101-107.

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