A LOCALIZATION PROPERTY AT THE BOUNDARY FOR MONGE-AMPERE EQUATION

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1. Introduction

In this paper we study the geometry of the sections for solutions to the Monge-Ampere equation

$$\det D^2 u = f, \qquad u : \overline{\Omega} \to \mathbb{R} \quad \text{convex},$$

which are centered at a boundary point $x_0 \in \partial\Omega$. We show that under natural local assumptions on the boundary data and the domain, the sections

$$S_h(x_0) = \{x \in \overline{\Omega} | \quad u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\}$$

are "equivalent" to ellipsoids centered at x_0, that is, for each $h > 0$ there exists an ellipsoid E_h such that

$$cE_h \cap \overline{\Omega} \subset \; S_h(x_0) - x_0 \subset \; CE_h \cap \overline{\Omega},$$

with c, C constants independent of h.

The situation in the interior is well understood. Caffarelli showed in [C1] that if

$$0 < \lambda \le f \le \Lambda \quad \text{in } \Omega,$$

and for some $x \in \Omega$,

$$S_h(x) \subset\subset \Omega,$$

then $S_h(x)$ is equivalent to an ellipsoid centered at x i.e.

$$kE \subset S_h(x) - x \subset k^{-1}E$$

for some ellipsoid E of volume $h^{n/2}$ and for a constant $k > 0$ which depends only on λ, Λ, n.

This property provides compactness of sections modulo affine transformations. This is particularly useful when dealing with interior $C^{2,\alpha}$ and $W^{2,p}$ estimates of strictly convex solutions of

$$\det D^2 u = f$$

when $f > 0$ is continuous (see [C2]).

Sections at the boundary were also considered by Trudinger and Wang in [TW] for solutions of

$$\det D^2 u = f$$

but under stronger assumptions on the boundary behavior of u and $\partial\Omega$, and with $f \in C^\alpha(\overline{\Omega})$. They proved $C^{2,\alpha}$ estimates up to the boundary by bounding the mixed derivatives and obtained that the sections are equivalent to balls.

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2. Statement of the main Theorem.

Let Ω be a bounded convex set in $\mathbb{R}^n$. We assume throughout this note that

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \geq 0\} \cap B_{\frac{1}{\rho}}, \tag{2.1}$$

for some small $\rho > 0$, that is $\Omega \subset (\mathbb{R}^n)^+$ and Ω contains an interior ball tangent to $\partial\Omega$ at 0.

Let $u : \overline{\Omega} \to \mathbb{R}$ be convex, continuous, satisfying

$$\det D^2 u = f, \quad \lambda \leq f \leq \Lambda \quad \text{in } \Omega. \tag{2.2}$$

We extend u to be ∞ outside $\overline{\Omega}$.

By subtracting a linear function we may assume that

$$x_{n+1} = 0 \text{ is the tangent plane to } u \text{ at } 0, \tag{2.3}$$

in the sense that

$$u \geq 0, \quad u(0) = 0,$$

and any hyperplane $x_{n+1} = \epsilon x_n$, $\epsilon > 0$ is not a supporting hyperplane for u.

In this paper we investigate the geometry of the sections of u at 0 that we denote for simplicity of notation

$$S_h := \{x \in \overline{\Omega} : \quad u(x) < h\}.$$

We show that if the boundary data has quadratic growth near $\{x_n = 0\}$ then, as $h \to 0$, S_h is equivalent to a half-ellipsoid centered at 0.

Precisely, our main theorem reads as follows.

Theorem 2.1. *Assume that Ω, u satisfy (2.1)-(2.3) above and for some $\mu > 0$,*

$$\mu|x|^2 \leq u(x) \leq \mu^{-1}|x|^2 \quad \text{on } \partial\Omega \cap \{x_n \leq \rho\}. \tag{2.4}$$

Then, for each $h < c(\rho)$ there exists an ellipsoid E_h of volume $h^{n/2}$ such that

$$kE_h \cap \overline{\Omega} \subset S_h \subset k^{-1}E_h.$$

Moreover, the ellipsoid E_h is obtained from the ball of radius $h^{1/2}$ by a linear transformation A_h^{-1} (sliding along the $x_n = 0$ plane)

$$A_h E_h = h^{1/2} B_1$$

$$A_h(x) = x - \nu x_n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}, 0),$$

with

$$|\nu| \leq k^{-1}|\log h|.$$

The constant k above depends on μ, λ, Λ, n and $c(\rho)$ depends also on ρ.

Theorem 2.1 is new even in the case when $f = 1$. The ellipsoid E_h, or equivalently the linear map A_h, provides information about the behavior of the second derivatives near the origin. Heuristically, the theorem states that in S_h the tangential second derivatives are bounded from above and below and the mixed second derivatives are bounded by $|\log h|$. This is interesting given that f is only bounded and the boundary data and $\partial\Omega$ are only $C^{1,1}$ at the origin.

Remark. Given only the boundary data φ of u on $\partial\Omega$, it is not always easy to check condition (2.4). Here we provide some examples when (2.4) is satisfied:

1) If φ is constant and the domain Ω is included in a ball included in $\{x_n \geq 0\}$.

2) If the domain $\partial\Omega$ is tangent of order 2 to $\{x_n = 0\}$ and the boundary data φ has quadratic behavior in a neighborhood of 0.

3) φ, $\partial\Omega \in C^3$ at the origin, and Ω is uniformly convex at the origin.

We obtain compactness of sections modulo affine transformations.

Corollary 2.2. *Under the assumptions of Theorem 2.1, assume that*

$$\lim_{x\to 0} f(x) = f(0)$$

and

$$u(x) = P(x) + o(|x|^2) \quad on\ \partial\Omega$$

with P a quadratic polynomial. Then we can find a sequence of rescalings

$$\tilde{u}_h(x) := \frac{1}{h} u(h^{1/2} A_h^{-1} x)$$

which converges to a limiting continuous solution $\bar{u}_0 : \overline{\Omega}_0 \to \mathbb{R}$ with

$$kB_1^+ \subset \Omega_0 \subset k^{-1} B_1^+$$

such that

$$\det D^2 \bar{u}_0 = f(0)$$

and

$$\bar{u}_0 = P \quad on\ \overline{\Omega}_0 \cap \{x_n = 0\},$$
$$\bar{u}_0 = 1 \quad on\ \partial\overline{\Omega}_0 \cap \{x_n > 0\}.$$

In a future work we intend to use the results above and obtain $C^{2,\alpha}$ and $W^{2,p}$ boundary estimates under appropriate conditions on the domain and boundary data.

3. Preliminaries

Next proposition was proved by Trudinger and Wang in [TW]. Since our setting is slightly different we provide its proof.

Proposition 3.1. *Under the assumptions of Theorem 2.1, for all $h \le c(\rho)$, there exists a linear transformation (sliding along $x_n = 0$)*

$$A_h(x) = x - \nu x_n,$$

with

$$\nu_n = 0, \quad |\nu| \le C(\rho) h^{-\frac{n}{2(n+1)}}$$

such that the rescaled function

$$\tilde{u}(A_h x) = u(x),$$

satisfies in

$$\tilde{S}_h := A_h S_h = \{\tilde{u} < h\}$$

the following:

(i) *the center of mass of $\tilde{S}_h$ lies on the x_n-axis;*

(ii)

$$k_0 h^{n/2} \le |\tilde{S}_h| = |S_h| \le k_0^{-1} h^{n/2};$$

(iii) *the part of* $\partial \tilde S_h$ *where* $\{\tilde u < h\}$ *is a graph, denoted by*
$$\tilde G_h = \partial \tilde S_h \cap \{\tilde u < h\} = \{(x', g_h(x'))\}$$
that satisfies
$$g_h \le C(\rho)|x'|^2$$
and
$$\frac{\mu}{2}|x'|^2 \le \tilde u \le 2\mu^{-1}|x'^2| \quad \text{on } \tilde G_h.$$

The constant k_0 *above depends on* μ, λ, Λ, n *and the constants* $C(\rho), c(\rho)$ *depend also on* ρ.

In this section we denote by c, C positive constants that depend on n, μ, λ, Λ. For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion. Constants that depend also on ρ are denote by $c(\rho)$, $C(\rho)$.

Proof. The function
$$v := \mu|x'|^2 + \frac{\Lambda}{\mu^{n-1}}x_n^2 - C(\rho)x_n$$
is a lower barrier for u in $\Omega \cap \{x_n \le \rho\}$ if $C(\rho)$ is chosen large.

Indeed, then
$$v \le u \quad \text{on } \partial\Omega \cap \{x_n \le \rho\},$$
$$v \le 0 \le u \quad \text{on } \Omega \cap \{x_n = \rho\},$$
and
$$\det D^2 v > \Lambda.$$
In conclusion,
$$v \le u \quad \text{in } \Omega \cap \{x_n \le \rho\},$$
hence
$$S_h \cap \{x_n \le \rho\} \subset \{v < h\} \subset \{x_n > c(\rho)(\mu|x'|^2 - h)\}. \tag{3.1}$$

Let x_h^* be the center of mass of S_h. We claim that
$$x_h^* \cdot e_n \ge c_0(\rho)h^\alpha, \quad \alpha = \frac{n}{n+1}, \tag{3.2}$$
for some small $c_0(\rho) > 0$.

Otherwise, from (3.1) and John's lemma we obtain
$$S_h \subset \{x_n \le C(n)c_0h^\alpha \le h^\alpha\} \cap \{|x'| \le C_1h^{\alpha/2}\},$$
for some large $C_1 = C_1(\rho)$. Then the function
$$w = \epsilon x_n + \frac{h}{2}\left(\frac{|x'|}{C_1h^{\alpha/2}}\right)^2 + \Lambda C_1^{2(n-1)}h\left(\frac{x_n}{h^\alpha}\right)^2$$
is a lower barrier for u in S_h if c_0 is sufficiently small.

Indeed,
$$w \le \frac{h}{4} + \frac{h}{2} + \Lambda C_1^{2(n-1)}(C(n)c_0)^2h < h \quad \text{in } S_h,$$
and for all small h,
$$w \le \epsilon x_n + \frac{h^{1-\alpha}}{C_1^2}|x'|^2 + C(\rho)hc_0\frac{x_n}{h^\alpha} \le \mu|x'|^2 \le u \quad \text{on } \partial\Omega,$$

and

$$\det D^2 w = 2\Lambda.$$

Hence

$$w \le u \quad \text{in } S_h,$$

and we contradict that 0 is the tangent plane at 0. Thus claim (3.2) is proved.

Now, define

$$A_h x = x - \nu x_n, \quad \nu = \frac{x_h^{*'}}{x_h^* \cdot e_n},$$

and

$$\tilde{u}(A_h x) = u(x).$$

The center of mass of $\tilde{S}_h = A_h S_h$ is

$$\tilde{x}_h^* = A_h x_h^*$$

and lies on the x_n-axis from the definition of A_h. Moreover, since $x_h^* \in S_h$, we see from (3.1)-(3.2) that

$$|\nu| \le C(\rho) \frac{(x_h^* \cdot e_n)^{1/2}}{(x_h^* \cdot e_n)} \le C(\rho) h^{-\alpha/2},$$

and this proves (i).

If we restrict the map A_h on the set on $\partial\Omega$ where $\{u < h\}$, i.e. on

$$\partial S_h \cap \partial\Omega \subset \{x_n \le \frac{|x'|^2}{\rho}\} \cap \{|x'| < C h^{1/2}\}$$

we have

$$|A_h x - x| = |\nu| x_n \le C(\rho) h^{-\alpha/2} |x'|^2 \le C(\rho) h^{\frac{1-\alpha}{2}} |x'|,$$

and part (iii) easily follows.

Next we prove (ii). From John's lemma, we know that after relabeling the x' coordinates if necessary,

$$D_h B_1 \subset \tilde{S}_h - \tilde{x}_h^* \subset C(n) D_h B_1 \tag{3.3}$$

where

$$D_h = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

Since

$$\tilde{u} \le 2\mu^{-1} |x'|^2 \quad \text{on } \tilde{G}_h = \{(x', g_h(x'))\},$$

we see that the domain of definition of g_h contains a ball of radius $(\mu h/2)^{1/2}$. This implies that

$$d_i \ge c_1 h^{1/2}, \qquad i = 1, \cdots, n-1,$$

for some c_1 depending only on n and μ. Also from (3.2) we see that

$$\tilde{x}_h^* \cdot e_n = x_h^* \cdot e_n \ge c_0(\rho) h^\alpha$$

which gives

$$d_n \ge c(n) \tilde{x}_h^* \cdot e_n \ge c(\rho) h^\alpha.$$

We claim that for all small h,

$$\prod_{i=1}^{n} d_i \geq k_0 h^{n/2},$$

with k_0 small depending only on μ, n, Λ, which gives the left inequality in (ii).

To this aim we consider the barrier,

$$w = \epsilon x_n + \sum_{i=1}^{n} ch \left(\frac{x_i}{d_i} \right)^2 .$$

We choose c sufficiently small depending on μ, n, Λ so that for all $h < c(\rho)$,

$$w \leq h \quad \text{on } \partial \tilde{S}_h,$$

and on the part of the boundary $\tilde{G}_h$, we have $w \leq \tilde{u}$ since

$$\begin{aligned} w &\leq \epsilon x_n + \frac{c}{c_1^2}|x'|^2 + ch\left(\frac{x_n}{d_n}\right)^2 \\ &\leq \frac{\mu}{4}|x'|^2 + chC(n)\frac{x_n}{d_n} \\ &\leq \frac{\mu}{4}|x'|^2 + ch^{1-\alpha}C(\rho)|x'|^2 \\ &\leq \frac{\mu}{2}|x'|^2. \end{aligned}$$

Moreover, if our claim does not hold, then

$$\det D^2 w = (2ch)^n (\prod d_i)^{-2n} > \Lambda,$$

thus $w \leq \tilde{u}$ in $\tilde{S}_h$. By definition, $\tilde{u}$ is obtained from u by a sliding along $x_n = 0$, hence 0 is still the tangent plane of $\tilde{u}$ at 0. We reach again a contradiction since $\tilde{u} \geq w \geq \epsilon x_n$ and the claim is proved.

Finally we show that

$$|\tilde{S}_h| \leq C h^{n/2}$$

for some C depending only on λ, n. Indeed, if

$$v = h \quad \text{on } \partial \tilde{S}_h,$$

and

$$\det D^2 v = \lambda$$

then

$$v \geq u \geq 0 \quad \text{in } \tilde{S}_h.$$

Since

$$h \geq h - \min_{\tilde{S}_h} v \geq c(n, \lambda) |\tilde{S}_h|^{2/n}$$

we obtain the desired conclusion.

□

In the proof above we showed that for all $h \leq c(\rho)$, the entries of the diagonal matrix D_h from (3.3) satisfy

$$d_i \geq c h^{1/2}, \quad i = 1, \ldots n-1$$

$$d_n \geq c(\rho) h^{\alpha}, \quad \alpha = \frac{n}{n+1}$$

$$c h^{n/2} \leq \prod d_i \leq C h^{n/2}.$$

The main step in the proof of Theorem 2.1 is the following lemma that will be proved in the remaining sections.

Lemma 3.2. *There exist constants c, $c(\rho)$ such that*

$$d_n \geq c h^{1/2}, \tag{3.4}$$

for all $h \leq c(\rho)$.

Using Lemma 3.2 we can easily finish the proof of our theorem.

Proof of Theorem 2.1. Since all d_i are bounded below by $ch^{1/2}$ and their product is bounded above by $Ch^{n/2}$ we see that

$$C h^{1/2} \geq d_i \geq c h^{1/2} \qquad i = 1, \cdots, n$$

for all $h \leq c(\rho)$. Using (3.3) we obtain

$$\tilde{S}_h \subset C h^{1/2} B_1.$$

Moreover, since

$$\tilde{x}_h^* \cdot e_n \geq d_n \geq c h^{1/2}, \qquad (\tilde{x}_h^*)' = 0,$$

and the part $\tilde{G}_h$ of the boundary $\partial \tilde{S}_h$ contains the graph of $\tilde{g}_h$ above $|x'| \leq c h^{1/2}$, we find that

$$c h^{1/2} B_1 \cap \tilde{\Omega} \subset \tilde{S}_h,$$

with $\tilde{\Omega} = A_h \Omega$, $\tilde{S}_h = A_h S_h$. In conclusion

$$c h^{1/2} B_1 \cap \tilde{\Omega} \subset A_h S_h \subset C h^{1/2} B_1.$$

We define the ellipsoid E_h as

$$E_h := A_h^{-1}(h^{1/2} B_1),$$

hence

$$c E_h \cap \overline{\Omega} \subset S_h \subset C E_h.$$

Comparing the sections at levels h and $h/2$ we find

$$c E_{h/2} \cap \overline{\Omega} \subset C E_h$$

and we easily obtain the inclusion

$$A_h A_{h/2}^{-1} B_1 \subset C B_1.$$

If we denote

$$A_h x = x - \nu_h x_n$$

then the inclusion above implies

$$|\nu_h - \nu_{h/2}| \leq C,$$

which gives the desired bound

$$|\nu_h| \leq C |\log h|$$

for all small h.

□

We introduce a new quantity $b(h)$ which is proportional to $d_n h^{-1/2}$ and which is appropriate when dealing with affine transformations.

Notation. Given a convex function u we define

$$b_u(h) = h^{-1/2} \sup_{S_h} x_n.$$

Whenever there is no possibility of confusion we drop the subindex u and use the notation $b(h)$.

Below we list some basic properties of $b(h)$.

1) If $h_1 \le h_2$ then

$$\left(\frac{h_1}{h_2}\right)^{\frac{1}{2}} \le \frac{b(h_1)}{b(h_2)} \le \left(\frac{h_2}{h_1}\right)^{\frac{1}{2}}.$$

2) A rescaling

$$\tilde{u}(Ax) = u(X)$$

given by a linear transformation A which leaves the x_n coordinate invariant does not change the value of b, i.e

$$b_{\tilde{u}}(h) = b_u(h).$$

3) If A is a linear transformation which leaves the plane $\{x_n = 0\}$ invariant the values of b get multiplied by a constant. However the quotients $b(h_1)/b(h_2)$ do not change values i.e

$$\frac{b_{\tilde{u}}(h_1)}{b_{\tilde{u}}(h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.$$

4) If we multiply u by a constant, i.e.

$$\tilde{u}(x) = \beta u(x)$$

then

$$b_{\tilde{u}}(\beta h) = \beta^{-1/2} b_u(h),$$

and

$$\frac{b_{\tilde{u}}(\beta h_1)}{b_{\tilde{u}}(\beta h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.$$

From (3.3) and property 2 above,

$$c(n) d_n \le b(h) h^{1/2} \le C(n) d_n,$$

hence Lemma 3.2 will follow if we show that $b(h)$ is bounded below. We achieve this by proving the following lemma.

Lemma 3.3. *There exist c_0, $c(\rho)$ such that if $h \le c(\rho)$ and $b(h) \le c_0$ then*

$$\frac{b(th)}{b(h)} > 2, \tag{3.5}$$

for some $t \in [c_0, 1]$.

This lemma states that if the value of $b(h)$ on a certain section is less than a critical value c_0, then we can find a lower section at height still comparable to h where the value of b doubled. Clearly Lemma 3.3 and property 1 above imply that $b(h)$ remains bounded for all h small enough.

The quotient in (3.5) is the same for $\tilde{u}$ which is defined in Proposition 3.1. We normalize the domain $\tilde{S}_h$ and $\tilde{u}$ by considering the rescaling

$$v(x) = \frac{1}{h}\tilde{u}(h^{1/2}Ax)$$

where A is a multiple of D_h (see (3.3)), $A = \gamma D_h$ such that

$$\det A = 1.$$

Then

$$ch^{-1/2} \leq \gamma \leq Ch^{-1/2},$$

and the diagonal entries of A satisfy

$$a_i \geq c, \qquad i = 1, 2, \cdots, n-1,$$

$$cb_u(h) \leq a_n \leq Cb_u(h).$$

The function v satisfies

$$\lambda \leq \det D^2 v \leq \Lambda,$$

$$v \geq 0, \quad v(0) = 0,$$

is continuous and it is defined in $\bar{\Omega}_v$ with

$$\Omega_v := \{v < 1\} = h^{-1/2}A^{-1}\tilde{S}_h.$$

Then

$$x^* + cB_1 \subset \Omega_v \subset CB_1^+,$$

for some x^*, and

$$ct^{n/2} \leq |S_t(v)| \leq Ct^{n/2}, \quad \forall t \leq 1,$$

where $S_t(v)$ denotes the section of v. Since

$$\tilde{u} = h \quad \text{in} \quad \partial\tilde{S}_h \cap \{x_n \geq C(\rho)h\},$$

then

$$v = 1 \quad \text{on } \partial\Omega_v \cap \{x_n \geq \sigma\}, \quad \sigma := C(\rho)h^{1-\alpha}.$$

Also, from Proposition 3.1 on the part G of the boundary of $\partial\Omega_v$ where $\{v < 1\}$ we have

$$\frac{1}{2}\mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq v \leq 2\mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2. \tag{3.6}$$

In order to prove Lemma 3.3 we need to show that if σ, a_n are sufficiently small depending on n, μ, λ, Λ then the function v above satisfies

$$b_v(t) \geq 2b_v(1) \tag{3.7}$$

for some $1 > t \geq c_0$.

Since $\alpha < 1$, the smallness condition on σ is satisfied by taking $h < c(\rho)$ sufficiently small. Also a_n being small is equivalent to one of the a_i, $1 \leq i \leq n-1$ being large since their product is 1 and a_i are bounded below.

In the next sections we prove property (3.7) above by compactness, by letting $\sigma \to 0$, $a_i \to \infty$ for some i. First we consider the 2D case and in the last section the general case.

4. The 2 dimensional case.

In order to fix ideas, we consider first the 2 dimensional case.

We study the following class of solutions to the Monge-Ampere equation. Fix $\mu > 0$ small, λ, Λ. We denote by $\mathcal{D}_\sigma$ the set of convex, continuous functions

$$u : \overline{\Omega} \to \mathbb{R}$$

such that

$$\lambda \le \det D^2 u \le \Lambda; \tag{4.1}$$

$$0 \in \partial\Omega, \quad B_\mu(x_0) \subset \Omega \subset B^+_{1/\mu} \quad \text{for some } x_0; \tag{4.2}$$

$$\mu h^{n/2} \le |S_h| \le \mu^{-1} h^{n/2}; \tag{4.3}$$

$$u = 1 \quad \text{on } \partial\Omega \setminus G, \qquad 0 \le u \le 1 \quad \text{on } G, \quad u(0) = 0, \tag{4.4}$$

with G a closed subset of $\partial\Omega$ included in B_σ,

$$G \subset \ \partial\Omega \cap B_\sigma.$$

Proposition 4.1. *Assume $n = 2$. For any $M > 0$ there exists c_0 small depending on M, μ, λ, Λ, such that if $u \in \mathcal{D}_\sigma$ and $\sigma \le c_0$, then*

$$b(h) := (\sup_{S_h} x) h^{-1/2} > M$$

for some $h \ge c_0$.

Property (3.7) easily follows from the proposition above. Indeed, by choosing

$$M = 2\mu^{-1} > 2b(1)$$

we prove the existence of a section $h \ge c_0$ such that

$$b(h) \ge 2b(1).$$

Also, the function v of the previous section satisfies $v \in \mathcal{D}_{c_0}$ (after renaming the constant μ) provided that σ is sufficiently small and a_1 sufficiently large.

We prove Proposition 4.1 by compactness. First we discuss briefly the compactness of bounded solutions to Monge-Ampere equation. For this we need to introduce solutions with possibly discontinuous boundary data.

Let $u : \Omega \to \mathbb{R}$ be a convex function with $\Omega \subset \mathbb{R}^n$ bounded and convex. We denote by

$$\Gamma_u := \{(x, x_{n+1}) \in \Omega \times \mathbb{R} | \quad x_{n+1} \ge u(x)\}$$

the upper graph of u.

Definition 4.2. We define the values of u on $\partial\Omega$ to be equal to φ i.e

$$u|_{\partial\Omega} = \varphi,$$

if the upper graph of $\varphi : \partial\Omega \to \mathbb{R} \cup \{\infty\}$

$$\Phi := \{(x, x_{n+1}) \in \partial\Omega \times \mathbb{R} | \quad x_{n+1} \ge \varphi(x)\}$$

is given by the closure of Γ_u restricted to $\partial\Omega \times \mathbb{R}$,

$$\Phi := \overline{\Gamma}_u \cap (\partial\Omega \times \mathbb{R}).$$

From the definition we see that φ is always lower semicontinuous. The following comparison principle holds: if $w : \overline{\Omega} \to \mathbb{R}$ is continuous and

$$\det D^2 w \geq \Lambda \geq \det D^2 u, \qquad w|_{\partial\Omega} \leq u|_{\partial\Omega},$$

then

$$w \leq u \quad \text{in } \Omega.$$

Indeed, from the continuity of w we see that for any $\varepsilon > 0$, there exists a small neighborhood of $\partial\Omega$ where $w - \varepsilon < u$. This inequality holds in the interior from the standard comparison principle, hence $w \leq u$ in Ω.

Since the convex functions are defined on different domains we use the following notion of convergence.

Definition 4.3. We say that the convex functions $u_m : \Omega_m \to \mathbb{R}$ converge to $u : \Omega \to \mathbb{R}$ if the upper graphs converge

$$\overline{\Gamma}_{u_m} \to \overline{\Gamma}_u \quad \text{in the Hausdorff distance.}$$

Similarly, we say that the lower semicontinuous functions $\varphi_m : \partial\Omega_m \to \mathbb{R}$ converge to $\varphi : \partial\Omega \to \mathbb{R}$ if the upper graphs converge

$$\Phi_m \to \Phi \quad \text{in the Hausdorff distance.}$$

Clearly if u_m converges to u, then u_m converges uniformly to u in any compact set of Ω, and $\Omega_m \to \Omega$ in the Hausdorff distance.

Remark: When we restrict the Hausdorff distance to the nonempty closed sets of a compact set we obtain a compact metric space. Thus, if Ω_m, u_m are uniformly bounded then we can always extract a subsequence m_k such that $u_{m_k} \to u$ and $u_{m_k}|_{\partial\Omega_{m_k}} \to \varphi$.

Next lemma gives the relation between the boundary data of the limit u and φ.

Lemma 4.4. *Let $u_m : \Omega_m \to \mathbb{R}$ be convex functions, uniformly bounded, such that*

$$\lambda \leq \det D^2 u_m \leq \Lambda$$

and

$$u_m \to u, \quad u_m|_{\partial\Omega_m} \to \varphi.$$

Then

$$\lambda \leq \det D^2 u \leq \Lambda,$$

and the boundary data of u is given by φ^* the convex envelope of φ on $\partial\Omega$.

Proof. Clearly $\Phi \subset \overline{\Gamma}_u$, hence $\Phi^* \subset \overline{\Gamma}_u$. It remains to show that the convex set K generated by Φ contains $\overline{\Gamma}_u \cap (\partial\Omega \times \mathbb{R})$.

Indeed consider a hyperplane

$$x_{n+1} = l(x)$$

which lies strictly below K. Then, for all large m

$$\{u_m - l \leq 0\} \subset \Omega_m,$$

and by Alexandrov estimate we have that

$$u_m - l \geq -C d_m^{1/n}$$

where $d_m(x)$ represents the distance from x to $\partial\Omega_m$. By taking $m \to \infty$ we see that

$$u - l \geq -Cd^{1/n}$$

thus no point on $\partial\Omega$ below l belongs to $\overline{\Gamma}_u$.

□

In view of the lemma above we introduce the following notation.

Definition 4.5. Let $\varphi : \partial\Omega \to \mathbb{R}$ be a lower semicontinuous function. When we write that a convex function u satisfies

$$u = \varphi \quad \text{on } \partial\Omega$$

we understand

$$u|_{\partial\Omega} = \varphi^*$$

where φ^* is the convex envelope of φ on $\partial\Omega$.

Whenever φ^* and φ do not coincide we can think of the graph of u as having a vertical part on $\partial\Omega$ between φ^* and φ.

It follows easily from the definition above that the boundary values of u when we restrict to the domain

$$\Omega_h := \{u < h\}$$

are given by

$$\varphi_h = \varphi \quad \text{on} \quad \partial\Omega \cap \{\varphi \leq h\} \subset \partial\Omega_h$$

and $\varphi_h = h$ on the remaining part of $\partial\Omega_h$.

The comparison principle still holds. Precisely, if $w : \overline{\Omega} \to \mathbb{R}$ is continuous and

$$\det D^2 w \geq \Lambda \geq \det D^2 u, \qquad w|_{\partial\Omega} \leq \varphi,$$

then

$$w \leq u \quad \text{in } \Omega.$$

The advantage of introducing the notation of Definition 4.5 is that the boundary data is preserved under limits.

Proposition 4.6 (Compactness)**.** *Assume*

$$\lambda \leq \det D^2 u_m \leq \Lambda, \quad u_m = \varphi_m \quad \text{on } \partial\Omega_m,$$

and Ω_m, φ_m uniformly bounded.

Then there exists a subsequence m_k such that

$$u_{m_k} \to u, \quad \varphi_{m_k} \to \varphi$$

with

$$\lambda \leq \det D^2 u \leq \Lambda, \quad u = \varphi \quad \text{on } \partial\Omega.$$

Indeed, we see that we can also choose m_k such that $\varphi^*_{m_k} \to \psi$. Since $\varphi_{m_k} \to \varphi$ we obtain

$$\varphi \geq \psi \geq \varphi^*,$$

and the conclusion follows from Lemma 4.4.

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. If c_0 does not exist we can find a sequence of functions $u_m \in \mathcal{D}_{1/m}$ such that

$$b_{u_m}(h) \leq M, \quad \forall h \geq \frac{1}{m}.$$

By Proposition 4.6 there is a subsequence which converges to a limiting function u satisfying (4.1)-(4.2)-(4.3) and (see Definition 4.5) $u = \varphi$ on $\partial\Omega$ with

$$\varphi = 1 \quad \text{on } \partial\Omega \setminus \{0\}, \qquad \varphi(0) = 0, \tag{4.5}$$

and moreover u has an obstacle by below in Ω

$$u \geq \frac{1}{M^2} x_2^2. \tag{4.6}$$

We consider the barrier

$$w := \delta(|x_1| + \frac{1}{2}x_1^2) + \frac{\Lambda}{\delta}x_2^2 - Nx_2$$

with δ small depending on μ, and N large so that

$$\frac{\Lambda}{\delta}x_2^2 - Nx_2 \leq 0 \quad \text{in} \quad B^+_{1/\mu}.$$

Then

$$w \leq \varphi \quad \text{on } \partial\Omega,$$

and

$$\det D^2 w > \Lambda.$$

Hence

$$w \leq u \quad \text{in } \Omega$$

which gives

$$u \geq \delta|x_1| - Nx_2.$$

Next we construct another explicit subsolution v such that whenever v is above the two obstacles

$$\delta|x_1| - Nx_2, \quad \frac{1}{M^2}x_2^2,$$

we have

$$\det D^2 v > \Lambda \quad \text{and} \quad v \leq 1.$$

Then we can conclude that

$$u \geq v,$$

and we show that this contradicts the lower bound on $|S_h|$.

We look for a function of the form

$$v := rf(\theta) + \frac{1}{2M^2}x_2^2,$$

where r, θ represent the polar coordinates in the x_1, x_2 plane.

The domain of definition of v is the angle

$$K := \{\theta_0 \leq \theta \leq \pi - \theta_0\}$$

with θ_0 small so that

$$\frac{1}{2M^2}x_2^2 \leq \frac{1}{2}(\delta|x_1| - Nx_2) \quad \text{on } \partial K \cap B_\mu.$$

In the set

$$\{v \geq \frac{1}{M^2}x_2^2\}$$

i.e. where

$$\frac{1}{r} \geq \frac{\sin^2\theta}{2M^2 f}$$

we have

$$\det D^2 v = \frac{1}{r}(f''+f)\frac{\sin^2\theta}{M^2} \geq \frac{1}{f}(f''+f)\frac{\sin^4\theta_0}{2M^4}. \tag{4.7}$$

We let

$$f(\theta) = \sigma e^{C_0|\frac{\pi}{2}-\theta|},$$

where C_0 is large depending on θ_0, M, Λ so that (see (4.7))

$$\det D^2 v > \Lambda$$

in the set where

$$\{v \geq \frac{1}{M^2}x_2^2\}.$$

On the other hand we can choose σ small so that

$$v \leq \delta|x_1| - Nx_2 \quad \text{on } \partial K \cap B_\mu$$

and

$$v \leq 1 \quad \text{on the set } \{v \geq \frac{1}{M^2}x_2^2\}.$$

In conclusion

$$u \geq v \geq \epsilon x_2,$$

hence

$$u \geq \max\{\epsilon x_2, \delta|x_1| - Nx_2\}.$$

This implies

$$|S_h| \leq Ch^2$$

for all small h and we contradict that

$$|S_h| \geq \mu h, \quad \forall h \in [0,1].$$

□

5. The higher dimensional case

In higher dimensions it is more difficult to construct an explicit barrier as in Proposition 4.1 in the case when in (3.6) only one a_i is large and the others are bounded. We prove our result by induction depending on the number of large eigenvalues a_i.

Fix μ small and λ, Λ. For each increasing sequence

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1}$$

with

$$\alpha_1 \geq \mu,$$

we consider the family of solutions

$$\mathcal{D}_\sigma^\mu(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$$

of convex, continuous functions $u : \overline{\Omega} \to \mathbb{R}$ that satisfy

$$\lambda \leq \det D^2 u \leq \Lambda \quad \text{in } \Omega, \quad u \geq 0 \text{ in } \overline{\Omega}; \tag{5.1}$$

$$0 \in \partial\Omega, \quad B_\mu(x_0) \subset \Omega \subset B^+_{1/\mu} \quad \text{for some } x_0; \tag{5.2}$$

$$\mu h^{n/2} \leq |S_h| \leq \mu^{-1} h^{n/2}; \tag{5.3}$$

$$u = 1 \quad \text{on } \partial\Omega \setminus G; \tag{5.4}$$

and

$$\mu \sum_1^{n-1} \alpha_i^2 x_i^2 \leq u \leq \mu^{-1} \sum_1^{n-1} \alpha_i^2 x_i^2 \qquad \text{on } G, \tag{5.5}$$

where G is a closed subset of $\partial\Omega$ which is a graph in the e_n direction and is included in boundary in $\{x_n \leq \sigma\}$.

For convenience we would like to add the limiting solutions when $\alpha_{k+1} \to \infty$ and $\sigma \to 0$. We denote by

$$\mathcal{D}_0^\mu(\alpha_1, \dots, \alpha_k, \infty, \infty, \dots, \infty)$$

the class of functions $u : \Omega \to \mathbb{R}$ that satisfy properties (5.1)-(5.2)-(5.3) and (see Definition 4.5) $u = \varphi$ on $\partial\Omega$ with

$$\varphi = 1 \quad \text{on } \partial\Omega \setminus G; \tag{5.6}$$

$$\mu \sum_1^k \alpha_i^2 x_i^2 \leq \varphi \leq \min\{1, , \mu^{-1} \sum_1^k \alpha_i^2 x_i^2\} \quad \text{on } G, \tag{5.7}$$

where G is a closed set

$$G \subset \partial\Omega \cap \{x_i = 0, \quad i > k\},$$

and if we restrict to the space generated by the first k coordinates then

$$\{\, \mu^{-1} \sum_1^k \alpha_i^2 x_i^2 \leq 1 \,\} \subset G \subset \{\, \mu \sum_1^k \alpha_i^2 x_i^2 \leq 1 \,\}.$$

We extend the definition of $\mathcal{D}_\sigma^\mu(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ to include also the pairs with

$$\mu \leq \alpha_1 \leq \ldots \leq \alpha_k < \infty, \quad \alpha_{k+1} = \cdots = \alpha_{n-1} = \infty$$

for which $\sigma = 0$ i.e. $\mathcal{D}_0^\mu(\alpha_1, \alpha_2, \dots, \alpha_k, \infty, \dots, \infty)$.

Proposition 4.6 implies that if

$$u_m \in D_{\sigma_m}^\mu(a_1^m, \dots, a_{n-1}^m)$$

is a sequence with

$$\sigma_m \to 0 \quad \text{and} \quad a_{k+1}^m \to \infty$$

for some fixed $0 \leq k \leq n-2$, then we can extract a convergent subsequence to a function u with

$$u \in D_0^\mu(a_1, .., a_l, \infty, .., \infty) \quad ,$$

for some $l \leq k$ and $a_1 \leq \ldots \leq a_l$.

Proposition 5.1. *For any $M > 0$ and $1 \leq k \leq n-1$ there exists C_k depending on $M, \mu, \lambda, \Lambda, n, k$ such that if $u \in \mathcal{D}_\sigma^\mu(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ with*

$$\alpha_k \geq C_k, \quad \sigma \leq C_k^{-1}$$

then

$$b(h) = (\sup_{S_h} x_n) h^{-1/2} \geq M$$

for some h with $C_k^{-1} \leq h \leq 1$.

As we remarked in the previous section, property (3.7) and therefore Lemma 3.3 follow from Proposition 5.1 by taking $k = n-1$ and $M = 2\mu^{-1}$.

We prove the proposition by induction on k.

Lemma 5.2. *Proposition 5.1 holds for $k = 1$.*

Proof. By compactness we need to show that there does not exist $u \in \mathcal{D}_0^{\mu}(\infty, \ldots, \infty)$ with $b(h) \le M$ for all h.

The proof is almost identical to the 2 dimensional case. One can see as before that

$$u \ge \max\{\delta|x'| - Nx_n, \frac{1}{M^2}x_n^2\}$$

and then construct a barrier of the form

$$v = rf(\theta) + \frac{1}{2M^2}x_n^2, \quad \theta_0 \le \theta \le \frac{\pi}{2}$$

where $r = |x|$ and θ represents the angle in $[0, \pi/2]$ between the ray passing through x and the $\{x_n = 0\}$ plane.

Now,

$$\det D^2 v = \frac{f'' + f}{r}\left(\frac{f\cos\theta - f'\sin\theta}{r\cos\theta}\right)^{n-2}\frac{\sin^2\theta}{M^2}.$$

We have

$$\frac{f}{r} > \frac{\sin^2\theta}{2M^2} \quad \text{on the set } \{v > \frac{1}{M^2}x_n^2\}$$

and we choose a function of the form

$$f(\theta) := \nu e^{C_0(\frac{\pi}{2} - \theta)}$$

which is decreasing in θ.

Then

$$\det D^2 v > \frac{f'' + f}{f}\left(\frac{\sin^2\theta_0}{2M^2}\right)^{n-1} > \Lambda$$

if C_0 is chosen large.

We obtain as before that

$$u \ge \max\{\delta|x'| - Nx_n, \epsilon x_n\}$$

which gives

$$|S_h| \le Ch^n$$

and we reach a contradiction.

□

Now we prove Proposition 5.1 by induction on k.

Proof of Proposition 5.1. In this proof we denote by c, C positive constants that depend on $M, \mu, \lambda, \Lambda, n$ and k.

We assume that the statement holds for k and we prove it for $k + 1$.

It suffices to show the existence of C_{k+1} only in the case when $\alpha_k < C_k$, otherwise we use the induction hypothesis.

If no C_{k+1} exists then we can find a limiting solution

$$u \in \mathcal{D}_0^{\tilde{\mu}}(1, 1, \ldots, 1, \infty, \ldots, \infty)$$

with

$$b(h) < Mh^{1/2}, \quad \forall h > 0 \tag{5.8}$$

where $\tilde{\mu}$ depends on μ and C_k.

We show that such a function u does not exist.

Denote

$$x = (y, z, x_n), \quad y = (x_1, \dots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \dots, x_{n-1}) \in \mathbb{R}^{n-1-k}.$$

On the $\partial\Omega$ plane we have

$$\varphi \geq w := \delta|x'|^2 + \delta|z| + \frac{\Lambda}{\delta^{n-1}} x_n^2 - N x_n$$

for some small δ depending on $\tilde{\mu}$, and N large so that

$$\frac{\Lambda}{\delta^{n-1}} x_n^2 - N x_n \leq 0 \quad \text{on} \quad B^+_{1/\tilde{\mu}}.$$

Since

$$\det D^2 w > \Lambda,$$

we obtain $u \geq w$ on Ω hence

$$u(x) \geq \delta|z| - N x_n. \tag{5.9}$$

We look at the section S_h of u. From (5.8)-(5.9) we see that

$$S_h \subset \{x_n > \frac{1}{N}(\delta|z| - h)\} \cap \{x_n \leq M h^{1/2}\}. \tag{5.10}$$

We notice that an affine transformation $x \to Tx$,

$$Tx := x + \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_{n-k-1} z_{n-k-1} + \nu_{n-k} x_n$$

with

$$\nu_1, \nu_2, \dots, \nu_{n-k} \in span\{e_1, \dots, e_k\}$$

i.e a *sliding along the* y *direction*, leaves the z, x_n coordinate invariant together with the subspace $(y, 0, 0)$.

The section $\tilde{S}_h := T S_h$ of the rescaling

$$\tilde{u}(Tx) = u(x)$$

satisfies (5.10) and $\tilde{u} = \tilde{\varphi}$ on $\partial\tilde{S}_h$ with

$$\tilde{\varphi} = \varphi \quad \text{on } \tilde{G} := \{\varphi \leq h\} \subset G,$$

$$\tilde{\varphi} = h \quad \text{on } \partial\tilde{S}_h \setminus \tilde{G}.$$

From John's lemma we know that S_h is equivalent to an ellipsoid E_h. We choose T an appropriate sliding along the y direction, so that TE_h becomes symmetric with respect to the y and (z, x_n) subspaces, thus

$$\tilde{x}_h^* + c(n)|\tilde{S}_h|^{1/n} A B_1 \subset \tilde{S}_h \subset C(n)|\tilde{S}_h|^{1/n} A B_1, \quad \det A = 1$$

and the matrix A leaves the y and the (z, x_n) subspaces invariant.

By choosing an appropriate system of coordinates in the y and z variables we may assume

$$A(y, z, x_n) = (A_1 y, A_2(z, x_n))$$

with

$$A_1 = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{pmatrix}$$

with $0 < \beta_1 \leq \cdots \leq \beta_k$, and

$$A_2 = \begin{pmatrix} \gamma_{k+1} & 0 & \cdots & 0 & \theta_{k+1} \\ 0 & \gamma_{k+2} & \cdots & 0 & \theta_{k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{n-1} & \theta_{n-1} \\ 0 & 0 & \cdots & 0 & \theta_n \end{pmatrix}$$

with γ_j, $\theta_n > 0$.

Next we use the induction hypothesis and show that $\tilde{S}_h$ is equivalent to a ball.

Lemma 5.3. *There exists C_0 such that*

$$\tilde{S}_h \subset C_0 h^{n/2} B_1^+.$$

Proof. Using that

$$|\tilde{S}_h| \sim h^{n/2}$$

we obtain

$$\tilde{x}_h^* + ch^{1/2} A B_1 \subset \tilde{S}_h \subset Ch^{1/2} A B_1.$$

We need to show that

$$\|A\| \le C.$$

Since $\tilde{S}_h$ satisfies (5.10) we see that

$$\tilde{S}_h \subset \{|(z, x_n)| \le Ch^{1/2}\},$$

which together with the inclusion above gives $\|A_2\| \le C$ hence

$$\gamma_j, \theta_n \le C, \quad |\theta_j| \le C.$$

Also $\tilde{S}_h$ contains the set

$$\{(y, 0, 0)| \quad |y| \le \tilde{\mu}^{1/2} h^{1/2}\} \subset \tilde{G},$$

which implies

$$\beta_i \ge c > 0, \quad i = 1, \cdots, k.$$

We define the rescaling

$$w(x) = \frac{1}{h} \tilde{u}(h^{1/2} A x)$$

which is defined in a domain $\Omega_w := h^{-1/2} A^{-1} \tilde{S}_h$ such that

$$B_c(x_0) \subset \Omega_w \subset B_C^+, \quad 0 \in \partial\Omega_w,$$

and $w = \varphi_w$ on $\partial\Omega_w$ with

$$\varphi_w = 1 \quad \text{on } \partial\Omega_w \setminus G_w,$$

$$\tilde{\mu} \sum \beta_i^2 x_i^2 \le \varphi_w \le \min\{1, \, \tilde{\mu}^{-1} \sum \beta_i^2 x_i^2\} \quad \text{on } G_w,$$

where $G_w := h^{-1/2} A^{-1} \tilde{G}$.

This implies that

$$w \in \mathcal{D}_0^{\bar{\mu}}(\beta_1, \beta_2, \ldots, \beta_k, \infty, \ldots, \infty)$$

for some value $\bar{\mu}$ depending on $\mu, M, \lambda, \Lambda, n, k$.

We claim that

$$b_u(h) \ge c_\star.$$

First we notice that

$$b_u(h) = b_{\tilde{u}}(h) \sim \theta_n.$$

Since

$$\theta_n \prod \beta_i \prod \gamma_j = \det A = 1$$

and

$$\gamma_j \le C,$$

we see that if $b_u(h)$ (and therefore θ_n) becomes smaller than a critical value c_* then

$$\beta_k \ge C_k(\bar{\mu}, \bar{M}, \lambda, \Lambda, n),$$

with $\bar{M} := 2\bar{\mu}^{-1}$, and by the induction hypothesis

$$b_w(\tilde{h}) \ge \bar{M} \ge 2b_w(1)$$

for some $\tilde{h} > C_k^{-1}$. This gives

$$\frac{b_u(h\tilde{h})}{b_u(h)} = \frac{b_w(\tilde{h})}{b_w(1)} \ge 2,$$

which implies $b_u(h\tilde{h}) \ge 2b_u(h)$ and our claim follows.

Next we claim that γ_j are bounded below by the same argument. Indeed, from the claim above θ_n is bounded below and if some γ_j is smaller than a small value $\tilde{c}_*$ then

$$\beta_k \ge C_k(\bar{\mu}, \bar{M}_1, \lambda, \Lambda, n)$$

with

$$\bar{M}_1 := \frac{2M}{\bar{\mu} c_\star}.$$

By the induction hypothesis

$$b_w(\tilde{h}) \ge \bar{M}_1 \ge \frac{2M}{c_\star} b_w(1),$$

hence

$$\frac{b_u(h\tilde{h})}{b_u(h)} \ge \frac{2M}{c_\star}$$

which gives $b_u(h\tilde{h}) \ge 2M$, contradiction. In conclusion θ_n, γ_j are bounded below which implies that β_i are bounded above. This shows that $\|A\|$ is bounded and the lemma is proved.

□

Next we use the lemma above and show that the function u has the following property.

Lemma 5.4. *If for some $p, q > 0$,*

$$u \ge p(|z| - qx_n), \qquad q \le q_0$$

then

$$u \ge p'(|z| - (q - \eta)x_n)$$

for some $p' \ll p$, and with $\eta > 0$ depending on q_0 and $\mu, M, \lambda, \Lambda, n, k$.

Proof. From Lemma 5.3 we see that after performing a linear transformation T (siding along the y direction) we may assume that

$$S_h \subset C_0 h^{1/2} B_1.$$

Let

$$w(x) := \frac{1}{h} u(h^{1/2} x)$$

for some small $h \ll p$.

Then

$$S_1(w) := \Omega_w = h^{-1/2} S_h \subset B^+_{C_0}$$

and our hypothesis becomes

$$w \geq \frac{p}{h^{1/2}}(|z| - qx_n), \tag{5.11}$$

Moreover the boundary values φ_w of w on $\partial\Omega_w$ satisfy

$$\varphi_w = 1 \quad \text{on } \partial\Omega_w \setminus G_w$$

$$\tilde{\mu}|y|^2 \leq \varphi_w \leq \min\{1, \tilde{\mu}^{-1}|y|^2\} \quad \text{on} \quad G_w,$$

where $G_w := h^{-1/2}\{\varphi \leq h\}$.

Next we show that $\varphi_w \geq v$ on $\partial\Omega_w$ where v is defined as

$$v := \delta|x|^2 + \frac{\Lambda}{\delta^{n-1}}(z_1 - qx_n)^2 + N(z_1 - qx_n) + \delta x_n,$$

and δ is small depending on $\tilde{\mu}$ and C_0, and N is chosen large such that

$$\frac{\Lambda}{\delta^{n-1}} t^2 + Nt$$

is increasing in the interval $|t| \leq (1 + q_0)C_0$.

From the definition of v we see that

$$\det D^2 v > \Lambda.$$

On the part of the boundary $\partial\Omega_w$ where $z_1 \leq qx_n$ we use that $\Omega_w \subset B_{C_0}$ and obtain

$$v \leq \delta(|x|^2 + x_n) \leq \varphi_w.$$

On the part of the boundary $\partial\Omega_w$ where $z_1 > qx_n$ we use (5.11) and obtain

$$1 = \varphi_w \geq C(|z| - qx_n) \geq C(z_1 - qx_n)$$

with C arbitrarily large provided that h is small enough. We choose C such that the inequality above implies

$$\frac{\Lambda}{\delta^{n-1}}(z_1 - qx_n)^2 + N(z_1 - qx_n) < \frac{1}{2}.$$

Then

$$\varphi_w = 1 > \frac{1}{2} + \delta(|x|^2 + x_n) \geq v.$$

In conclusion $\varphi_w \geq v$ on $\partial\Omega_w$ hence the function v is a lower barrier for w in Ω_w. Then

$$w \geq N(z_1 - qx_n) + \delta x_n$$

and, since this inequality holds for all directions in the z-plane, we obtain

$$w \geq N(|z| - (q - \eta)x_n), \qquad \eta := \frac{\delta}{N}.$$

Scaling back we get

$$u \geq p'(|z| - (q - \eta)x_n) \qquad \text{in } S_h.$$

Since u is convex and $u(0) = 0$, this inequality holds globally, and the lemma is proved.

□

We remark that Lemma 5.4 can be used directly to prove Proposition 4.1 and Lemma 5.2.

End of the proof of Proposition 5.1. From (5.9) we obtain an initial pair (p, q_0) which satisfies the hypothesis of Lemma 5.4. We apply this lemma a finite number of times and obtain that

$$u \geq \epsilon(|z| + x_n),$$

and we contradict that $\tilde{S}_h$ is equivalent to a ball of radius $h^{1/2}$.

□

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