ODEs 6-2

Comment about homework: Q1 & Q2 are about series solutions to systems

Ansatz: $\vec{y}(t) = \begin{pmatrix} \sum_{n=0}^{\infty} a_n t^n \\ \sum_{n=0}^{\infty} b_n t^n \end{pmatrix}$

$\Rightarrow$ system of recurrence relations

Eg: (series solution to a system) find the first 3 terms in a approximation to the general solution to

$$\vec{y}'(t) = \begin{pmatrix} 1 & \sin(t) \\ 0 & 1 \end{pmatrix} \vec{y}(t)$$

DON'T find eigenvectors & eigenvalues of

Approximate $\sin(t) = t - \frac{t^3}{3} + \cancel{O(t^5)}$

Ansatz. $\vec{y}(t) = \begin{pmatrix} \sum a_n t^n \\ \sum b_n t^n \end{pmatrix}$

$$\vec{y}'(t) = \begin{pmatrix} \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n \\ \sum_{n=0}^{\infty} (n+1) b_{n+1} t^n \end{pmatrix} = \begin{pmatrix} 1 & t - \frac{t^3}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sum^{\infty} a_n t^n \\ \sum_{n=0}^{\infty} b_n t^n \end{pmatrix}$$

$$\begin{pmatrix} \sum (n+1) a_{n+1} t^n \\ \sum (n+1) b_{n+1} t^n \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} a_n t^n + t \sum_{n=0}^{\infty} b_n t^n - \frac{t^3}{3} \sum_{n=0}^{\infty} b_n t^n \\ \sum_{n=0}^{\infty} b_n t^n \end{pmatrix}$$

$$= \begin{pmatrix} \sum a_n t^n + \sum_{n=0}^{\infty} b_n t^{n+1} - \sum_{n=0}^{\infty} \frac{b_n}{3} t^{n+3} \\ \sum b_n t^n \end{pmatrix}$$

$$\begin{pmatrix} \sum (n+1) a_{n+1} t^n \\ \sum (n+1) b_{n+1} t^n \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_{n-1} t^n - \sum_{n=0}^{\infty} \frac{b_{n-3}}{3} t^n \\ \sum_{n=0}^{\infty} b_n t^n \end{pmatrix} \quad (b_k = 0 \text{ if } k < 0)$$

System of recurrence relations

$$a_{n+1} = \left(a_n + b_{n-1} - \frac{b_{n-3}}{3}\right) / (n+1)$$

$$b_{n+1} = b_n / (n+1)$$

a_0 and b_0 can be anything $\quad (b_{-1} = b_{-2} = 0)$

$$a_1 = a_0 + b_{-1} - 0/3 = a_0$$
$$b_1 = b_0$$
$$a_2 = (a_1 + b_0 - 0/3)/2 = \frac{a_0}{2} + b_0/2$$
$$b_2 = b_1/2 = b_0/2$$

$$\vec{y}(t) = \begin{pmatrix} a_0 + (a_0+b_0)t + \frac{1}{2}(a_0+b_0)t^2 + O(t^3) \\ b_0 + b_0 t + b_0/2\, t^2 + O(t^3) \end{pmatrix}$$

$f(t) \in O(t^3)$ if $\lim_{t\to 0} \frac{f(t)}{t^3}$ is finite

Eg: $4t^3 + 8t^4 + \frac{1}{6!}t^5 + \dots$

Goal: Study IVPs using the Laplace transform

Recall: Want to pick a "basis" for functions where $\frac{d}{dt}$ is as nice as possible

Study, instead of $f(t)$ the Laplace transform

$$\boxed{\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt} \qquad s \in \mathbb{C}$$

Table of Laplace Transforms

Time domain $f(t)$	Frequency domain $\mathcal{L}\{f(t)\}(s) = F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^p \quad p > -1$	$\frac{p!}{s^{p+1}}$ if $p \in \mathbb{Z}$ $\qquad \frac{\Gamma(p+1)}{s^{p+1}}$ in general
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

Always: only take $\mathcal{L}\{f(t)\}$ if $f(t) \in O(e^{at})$ $\quad$ [e.g. $f = e^{e^t}$ is bad

Eg: (Laplace transform of $\sin(at)$)

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt = \frac{-1}{a}e^{-st}\cos(at)\Big|_0^\infty - \int_0^\infty \frac{-1}{a}(-s)e^{-st}\cos(at)\,dt$$

$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)\,dt$$

$$= \frac{1}{a} - \frac{s}{a}\left[\frac{1}{a}e^{-st}\cancel{\sin(at)}\Big|_0^\infty - \frac{1}{a}\int(-s)e^{-st}\sin(at)\,dt\right]$$

$$= \frac{1}{a} - \frac{s}{a}\,\frac{s}{a}\,\mathcal{L}\{\sin(at)\}$$

$$\mathcal{L}\{\sin(at)\} = \frac{1}{a} - \frac{s^2}{a^2}\mathcal{L}\{\sin(at)\}$$

$$\left(1+\frac{s^2}{a^2}\right)\mathcal{L}\{\sin(at)\} = \frac{1}{a}$$

$$\boxed{\mathcal{L}\{\sin(at)\} = \frac{\frac{1}{a}}{1+\frac{s^2}{a^2}} = \frac{a}{s^2+a^2}}$$

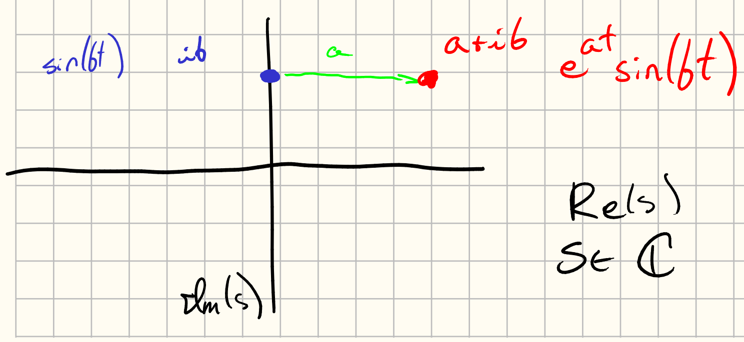
Eg: $\mathcal{L}\{\cos(at)\}$ from above, satisfies

$$\mathcal{L}\{\sin(at)\} = \frac{1}{a} - \frac{s}{a}\mathcal{L}\{\cos(at)\}$$

$$\boxed{\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}}$$

What about $e^{at}\sin(bt)$ etc?

Note: $e^{at}\sin(bt)$ is just a shift of complex frequency



Fact (HW#7) Frequency shift

$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a) \qquad (= F(s-a))$$

Use this to find
$\mathcal{L}\{e^{at}\sin(bt)\}$

$$\mathcal{L}\{e^{at}\sin(bt)\}(s) = \mathcal{L}\{\sin(bt)\}(s-a)$$

$$= \frac{b}{s^2+b^2}\Bigg|_{s\mapsto s-a}$$

$$= \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\cos(bt)\}(s) = \frac{s-a}{(s-a)^2+b^2}$$

All frequency domain functions of type

$$\frac{as+b}{cs^2+ds+e}$$ are Laplace transforms of

solutions to 2nd order constant coefficient linear ODEs

Eg: Find $f(t)$ whose Laplace transform is

$$F(s) = \frac{s+1}{2s^2+4s+4}$$

$\Delta = 4^2 - 4(4\cdot 2) < 0$ we can't write this as

$$\frac{a}{bs+c} + \frac{d}{es+f}$$

$$= \frac{1}{2}\,\frac{s+1}{s^2+2s+2}$$ instead: complete the square

$$F(s) = \frac{1}{2}\,\frac{s+1}{(s+1)^2+1} = \frac{1}{2}\,\frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{2}\mathcal{L}\{e^{-t}\cos(t)\}$$

✓ $(s+1)^2+1$
$= s^2+2s+1+1 = s^2+2s+2$

$$F(s) = \mathcal{L}\left\{\tfrac{1}{2}e^{-t}\cos(t)\right\}$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-t}\cos(t) \quad \checkmark$$

Used:

Fact: Linearity of the Laplace transform

$$\boxed{\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}}$$

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§ Differentiation and the Laplace transform

Time domain differentiation

a) $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

b) $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Pf: a) $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}f(t)\,dt$$

$$= \cancel{\lim_{t\to\infty} e^{-st}f(t)} - f(0) + s\int_0^\infty e^{-st}f(t)\,dt$$

b/c $f \in O(e^{at})$

$$= s\mathcal{L}\{f(t)\} - f(0)$$

b) $\mathcal{L}\{f^{(n)}(t)\} = \mathcal{L}\left\{\frac{d}{dt}f^{(n-1)}(t)\right\} = s\mathcal{L}\{f^{(n-1)}\} - f^{(n-1)}(0)$

keep going

$$= s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

We're "diagonalized" differentiation

Also Frequency derivatives

- $\mathcal{L}\{-t f(t)\} = \frac{d}{ds}\mathcal{L}\{f(t)\}$
- $\mathcal{L}\{(-1)^n t^n f(t)\} = \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$

Now: Use this to solve ODEs, specifically IVPs.

Today: Solve equations we already know how to solve

Eg: (IVP solution using Laplace transform)

Solve the IVP

$$y'' + \omega^2 y = 0 \qquad y(0)=1 \quad y'(0)=1$$

Method: Take $\mathcal{L}\{\ \}$ of both sides

$$\mathcal{L}\{y'' + \omega^2 y\} = \mathcal{L}\{0\}$$

$\mathcal{L}\{\}$ linearity

$$\mathcal{L}\{y''\} + \omega^2\mathcal{L}\{y\} = 0$$

time domain derivative

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \omega^2\mathcal{L}\{y\} = 0$$

Let $Y(s) = \mathcal{L}\{y\}$

$$s^2 Y(s) - s1 - 1 + \omega^2 Y(s) = 0$$

$$(s^2+\omega^2)Y(s) = 1+s$$

$$Y(s) = \frac{1+s}{s^2+\omega^2}$$

$$= \frac{1}{\omega}\frac{\omega}{s^2+\omega^2} + \frac{s}{s^2+\omega^2}$$

find $\mathcal{L}\{e^{at}\cos(bt)\}$ & $\mathcal{L}\{e^{at}\sin(bt)\}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{\omega}\frac{\omega}{s^2+\omega^2} + \frac{s}{s^2+\omega^2}\right\}$$

$$= \boxed{\frac{1}{\omega}\sin(\omega t) + \cos(\omega t)}.$$

Eg: (Solutions w/ e^{at}, not $e^{at}\sin(bt)$, $\Delta \geq 0$)

Solve the IVP

$$y'' - y' - 2y = 0 \qquad y(0) = 1 \quad y'(0) = 0$$

$$\mathcal{L}\{y'' - y' - 2y\} = 0$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

$$s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = 0$$

$$s^2 Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

Δ of denominator is
$(-1)^2 - 4(1)\cdot(-2) = 8 + 1 = 9 > 0$

$$= \frac{s-1}{(s-2)(s+1)}$$

since $\Delta > 0$

$$= \frac{a}{s-2} + \frac{b}{s+1}$$

$$= \frac{as + a + bs - 2b}{(s+1)(s-2)} = \frac{s-1}{s^2 - s - 2}$$

$a + b = 1$
$a - 2b = -1$

$$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{-1}{3}\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$Y(s) = \frac{1}{3}\,\frac{1}{s-2} + \frac{2}{3}\,\frac{1}{s+1}$$

$$y(s) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{y(s) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}$$

Eg: (Using $\mathcal{L}\{-\}$ to solve inhomogeneous IVPs)

Solve the IVP

$$y'' + y = \sin(2t) \qquad y(0) = 2 \quad y'(0) = 1$$

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{\sin(2t)\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$Y(s) = \frac{2}{s^2+4}\cdot\frac{1}{s^2+1} + \frac{2s+1}{s^2+1}$$

$$= \frac{2 + (s^2+4)(2s+1)}{(s^2+4)(s^2+1)}$$

Partial fractions

$$= \frac{as+b}{(s^2+4)} + \frac{cs+d}{s^2+1}$$

$$= \frac{as^3 + as + bs^2 + b + cs^3 + 4cs + ds^2 + 4d}{(s^2+4)(s^2+1)} = \frac{2s^3 + 8s + 6 + s^2}{(s^2+4)(s^2+1)}$$

$$\begin{aligned} s^3 \quad a + c &= 2 \\ d + b &= 1 \\ a + 4c &= 8 \\ b + 4d &= 6 \end{aligned}$$

$$\begin{array}{c} s^3 \\ s^2 \\ s \\ 1 \end{array}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 8 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} b \\ d \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$Y(s) = \frac{as}{s^2+4} + \frac{b}{s^2+4} + \frac{cs}{s^2+1} + \frac{d}{s^2+1}$$

$$= \frac{-1}{3}\,\frac{2}{s^2+4} + 2\,\frac{s}{s^2+1} + \frac{5}{3}\,\frac{1}{s^2+1}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{-1}{3}\sin(2t) + 2\cos(t) + \frac{5}{3}\sin(t)}$$