ODEs 5-4

Goal for today - Start a taxonomy of differential equations

- Identify types of equations that have techniques which can produce solutions
- Solve IVPs

First steps in taxonomy

1. ODE or a PDE ?

   $\frac{\partial}{\partial t}$ only or $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots$

2. Order of the ODE

   = highest order derivative that shows up in the equation

Eg: $\frac{dy}{dt} = f(y,t)$ - The most general 1^{st} Order equation

Eg $\frac{d^3y}{dt^3} + y^{10}\frac{dy}{dt} = e^t$ - 3rd order

Mostly for now study 1^{st} order ODEs (later: reduce higher order & 1st order)

3. Linear and non-linear ODEs

Linear equations generally are easier to solve

Eg: $\frac{dy}{dt} + t^4 y = e^t \cdot \sin(t^{10})$ is linear equation because it is linear in y.

Eg: $\frac{dy}{dt} - y^2 = 0$ is non-linear because is a higher power of y

Eg: $\frac{d^3y}{dt^3} + f(t)\frac{d^2y}{dt^2} + g(t)\frac{dy}{dt} + h(t)y = i(t)$ is linear

Def: A linear ODE is one of the form

$$\sum_{i=0}^{N} f_i(t)\frac{d^iy}{dt^i} = g(t)$$

The most general *first order linear* ODE is of the form

$$\boxed{\frac{dy}{dt} + P(t)y = Q(t)}$$

This has an explicit solution in terms of <u>integrating factors</u> which we will learn on Friday

4. Is the ODE autonomous?

An ODE is autonomous if it depends on y - dependent variable
not on t - independent variable

<u>Eg</u>: $\frac{dy}{dt} = ky$ (natural growth) A quantity y grows proportional to its size.
is an autonomous ODE

<u>Eg</u>: Non-autonomous ODE
$\frac{dy}{dt} = \sin(t)y$ - depends on t

The most general 1st order autonomous ODE is
$$\frac{dy}{dt} = f(y)$$

<u>Eg</u>: The most general 1st order linear autonomous ODE
is $\frac{dy}{dt} + \overset{P(t)}{a}y = \overset{Q(t)}{b}$

i.e. a $\overset{1^{st}\text{ order}}{\vee}$ linear ODE with constant coefficients

<u>How do we solve $y' + ay = b$?</u>

$\left[\frac{dy}{dt} = f(t) \rightsquigarrow y(t) = \int f(t)dt + c\right.$

$$\frac{dy}{dt} + ay = b$$

$$\frac{dy}{dt} = b - ay$$

"ys on one side ts on the other side"

$$\frac{dy}{b-ay} = dt$$

"integrate both sides"

$$\int \frac{dy}{b-ay} = \int dt$$

$$-\frac{1}{a}\log(b-ay) = t + c$$

$$\log(b-ay) = -at + c_1$$

$$b-ay = e^{-at+c_1} = e^{c_1}e^{-at}$$

$$-ay = -b + c_2 e^{-at}$$

$$\underline{y = \frac{b}{a} + c_3 e^{-at}}$$

<u>Check</u>: $y = \frac{b}{a} + ce^{-at}$ solves $\frac{dy}{dt} + ay = b$

$$\frac{dy}{dt} = -ace^{-at}$$

$$\frac{dy}{dt} + ay = -ac\cancel{e^{-at}} + a\left(\frac{b}{a} + c\cancel{e^{-at}}\right) \stackrel{?}{=} b$$

$$= a\frac{b}{a} = b \qquad = b$$

This equation is an example of a type of equations you can solve by integrating $\rightsquigarrow$ <u>separable equations</u>

Separable equations

<u>Def</u>: A 1st order ODE is <u>separable</u> if it can be written as

$$\frac{dy}{dt} = M(y)N(t)$$

<u>Fact</u>: If an equation is separable, it can be solved using separation of variables

Eg: $\frac{dy}{dx} = y^2x - 2yx + x$

$$\frac{dy}{dx} = (y^2-2y+1)x$$
$$= (y-1)^2 x$$ ← separable equation

↕ divided by $(y-1)^2$ pay attention to $y=1$

$$\frac{dy}{(y-1)^2} = x\,dx$$

$$\int \frac{dy}{(y-1)^2} = \int x\,dx$$

$$\frac{-1}{y-1} = \frac{1}{2}x^2 + c$$

$$(y-1) = \frac{-1}{\frac{1}{2}x^2+c}$$

$y = \frac{-1}{\frac{1}{2}x^2+c} + 1$ is almost the general sol'n to the ODE

If $y(x_0) = 1$ then the ODE $\frac{dy}{dx} - (y-1)^2x$

if $y=1$ then this is $\frac{dy}{dx} = 0$ so

$y(x) \equiv 1$ is a solution but not $\frac{-1}{\frac{1}{2}x^2+c}+1$ for $c \in (-\infty,\infty)$

In general: ODE $\frac{dy}{dt} = M(y)N(t)$ has a constant solution

$y(t) \equiv a$ if $M(a) = 0$.

Justify separation of variables using change of variables $\frac{du}{dx}dx = du$

$\frac{dy}{dx} = M(y)\,N(x)$ we want to say

sometimes this is as far as the method goes

$\int \frac{1}{M(y)}dy = \int N(x)dx + C$ but we can say

$$\int \frac{1}{M(u)}du = \int \frac{1}{M(u)}\frac{dy}{dx}dx = \int \frac{1}{M(y(x))}M(y)N(x)\,dx = \int N(x)\,dx.$$

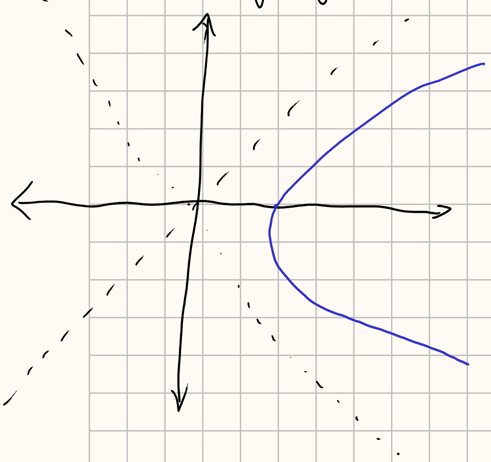
changed vars

$u(x) = y(x)$

$du = \frac{dy}{dx}dx$

Implicit solutions and IVPs

It's frequently the case that isolating y eliminates part of the function



Consider the ODE

$$\frac{dy}{dx} = \frac{x}{y}.$$ It is separable

$$y\,dy = x\,dx$$

$$\int y\,dx = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

$$y^2 - x^2 = c$$

$$y^2 = c + x^2$$

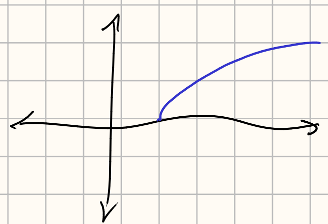
$$y = \sqrt{c + x^2}$$ "General solution"

IVP = Initial Value Problem

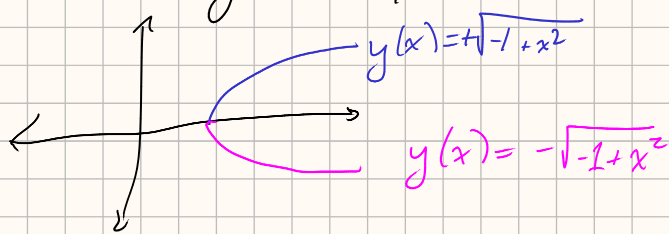
Solve IVP $y(1) = 0$, i.e. find specific solution satisfying this initial condition.

$$y(1) = \sqrt{c+1} = 0 \Rightarrow c = -1$$

$$y(x) = \sqrt{-1 + x^2}$$



- If our solution is required to be a function, there are two solutions



- $y^2 = -1 + x^2$ is the implicit solution

Solve the IVP for different initial conditions

$$y^2 = c + x^2$$

Solve IVP $y(x_0) = y_0$ for $(x_0, y_0) \in \begin{cases} (1,0), & A \text{ already done} \\ (0,1), & B \\ (2,\sqrt{3}) & C \end{cases}$

B $y(0)^2 = c + 0^2$

$(0,1)$ $\quad 1^2 = c \Rightarrow c = 1$

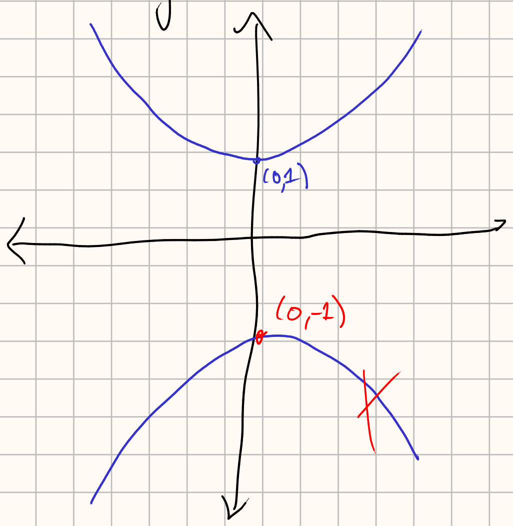
Implicit solution is

$$y^2 = 1 + x^2$$

The implicit solution is not really the solution to the IVP

We want the solution

$$y = +\sqrt{1+x^2}$$



C $x_0 = 2 \quad y_0 = \sqrt{3}$ $\qquad y^2 = c + x^2$

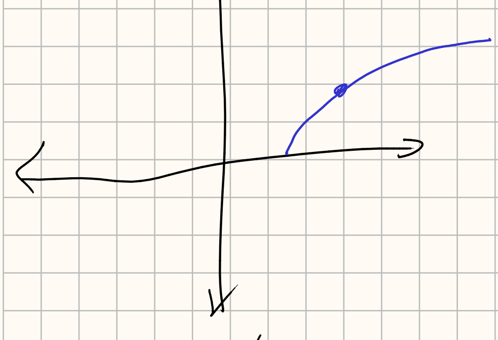
$$\sqrt{3}^2 = y_0(x_0)^2 = c + 2^2$$

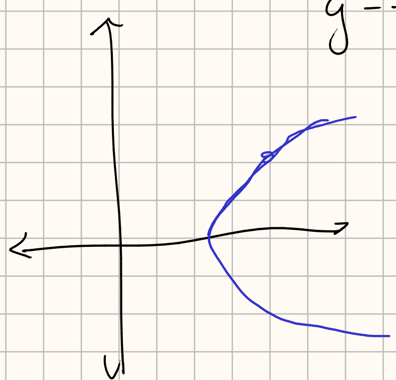
$$3 = c + 4 \qquad c = -1$$

Now the answer depends on if y is a function or can be multivalued

$$y^2 = -1 + x^2$$

$$y = +\sqrt{-1+x^2}$$





y is function $\qquad\qquad$ y can be multi-valued