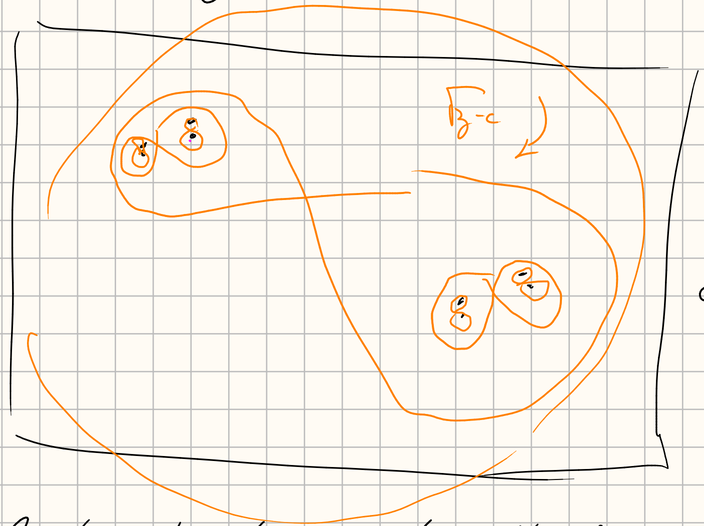
SHP Chaos & Fractals 4-24

Review solution to 4-17 problem + details on Cantor sets

disconnected Julia sets are "Cantor-like" fractals



Maybe J $\lim_{n\to\infty} 2^n$ points

There are an uncountable # of points in J

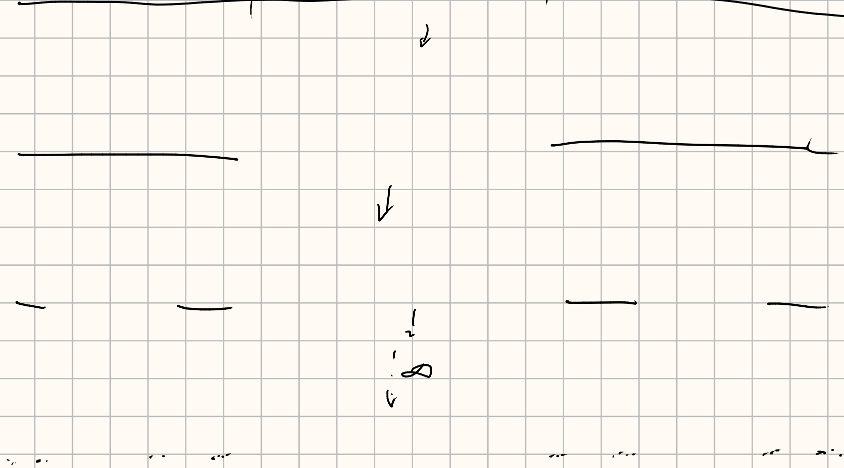
Why are they uncountable?

$[0,1] = \{0.110100\ldots\}$ - infinite binary expansions

A point in J is described by a sequence

LRRLLR...

Cantor set (middle thirds set)



remove $\frac{1}{3}$ of each line segment

What is the fractal dimension? We can use scaling dimension

length $\times 3 \longleftrightarrow$ size $\times 2$

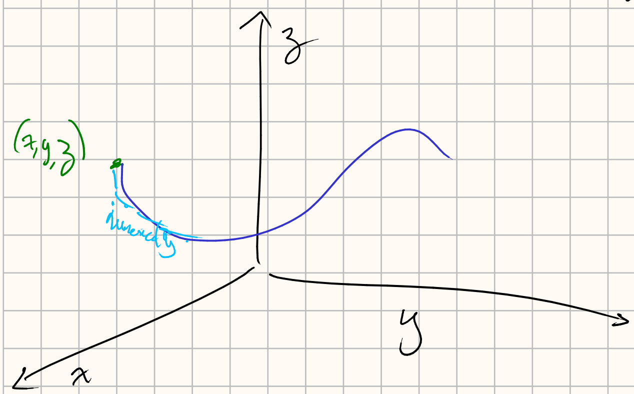
$$D = \frac{\log\left(\text{size factor}\right)}{\log\left(\text{length factor}\right)} = \frac{\log(2)}{\log(3)} \approx 0.631$$

Goal for today

- Chaotic continuous time dynamical systems (i.e. ordinary differential equation)
- Measure for sensitivity to initial conditions

The state of a dynamical system is measured by some coordinates (x, y, z)

& its change over time is described by a differential equation



$$\frac{dx}{dt} = f_x(x,y,z)$$

$$\frac{dy}{dt} = f_y(x,y,z)$$

$$\frac{dz}{dt} = f_z(x,y,z)$$

need to write these in terms of x, y, z (+ external parameters)

$$\dot{x} = \frac{dx}{dt}$$

Eg: Lorenz system (σ, β, ρ are parameters)

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta \cdot z$$

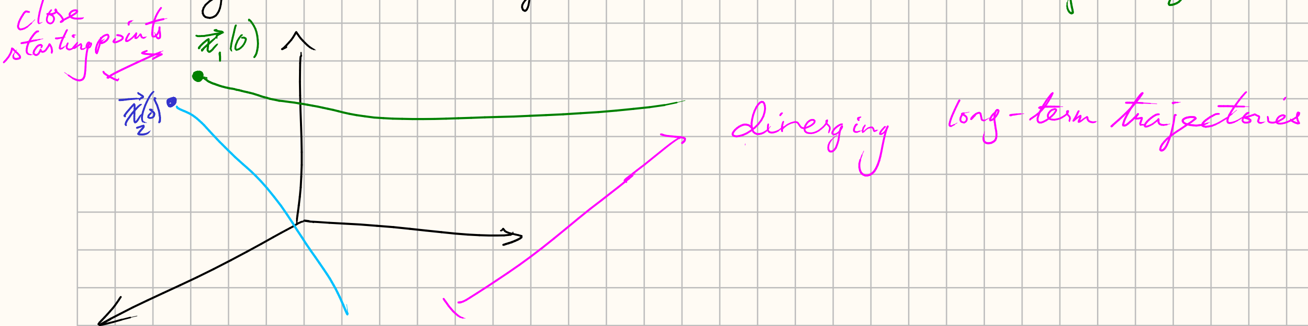
We saw a demo of the time evolution of this system: it has a attracting, oscilatory trajectory that is not periodic & has a fractal structure:

a strange attractor

Next step: Sensitive dependence on initial conditions

We'll measure (i.e. approximate) how quickly nearby trajectories diverge.

$\vec{x}(t) = (x(t), y(t), z(t))$



Suppose we have a trajectory $\vec{x}_1(t)$ and a nearby initial condition $\vec{x}_2(0)$ s.t.

$$|\vec{x}_1(0) - \vec{x}_2(0)| = \delta \qquad 0 < \delta \ll 1$$

then suppose $|\vec{x}_1(t) - \vec{x}_2(t)| \approx e^{\lambda_\delta t}\delta$

i.e. $sep(t) \approx e^{\lambda_\delta t} sep(0)$

Def: λ_δ is the Lyapunov exponent for the trajectory $\vec{x}_1(t)$ in the direction of $\vec{x}_2(t)$ as $\delta \to 0$

i.e. Let λ_δ be the exponent a given δ

the $\boxed{\lambda_{lyapunov} = \lim_{\delta \to 0} \lambda_\delta}$.