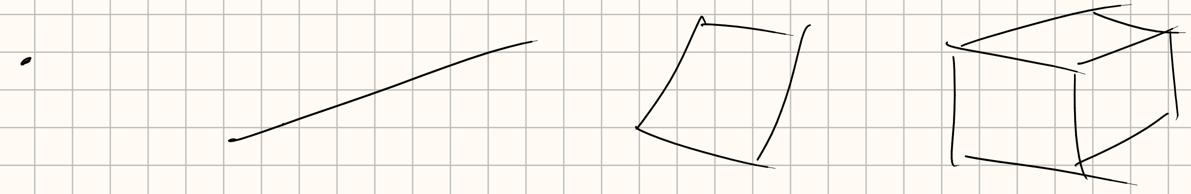
SHP Chaos & Fractals 4-17

Goal: Understand fractal dimension

Intuitive notion of dimension ($=$ ~~dimensional~~ Euclidean space)

dim $= 0$ dim $= 1$ dim $= 2$ dim $= 3$



dim $= \#$ of coordinates

First issue: Cardinality of 1d space $=$ Cardinality of 2d space

$$\text{Cardinality}(\{1,4,6\}) = 3 \quad \text{etc.}$$

Infinite sets have the same cardinality if there is a 1-1 correspondence between them

Eg: $\text{Cardinality}(\{1,4,6\}) = \text{Cardinality}(\{\triangle, \square, \bigcirc\})$

Eg: $\underset{\text{Natural \#s}}{\text{Cardinality}(\mathbb{N})} = \underset{\text{Even \#s}}{\text{Cardinality}(2\mathbb{N})}$

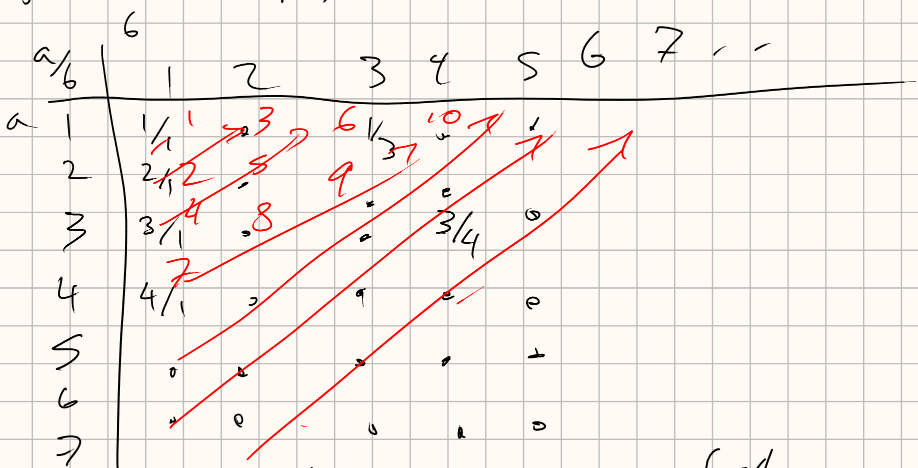
$$\{1, 2, 3, 4, 5, 6, \ldots, n, \ldots\} \qquad \{2, 4, 6, 8, 10, \ldots\}$$

Cardinality $(-) = \#(-)$

Eg: $\#(\mathbb{Z}) = \#(\mathbb{Q})$

$= \{a/b \mid a \in \mathbb{Z},\ b \in \mathbb{Z}\}$

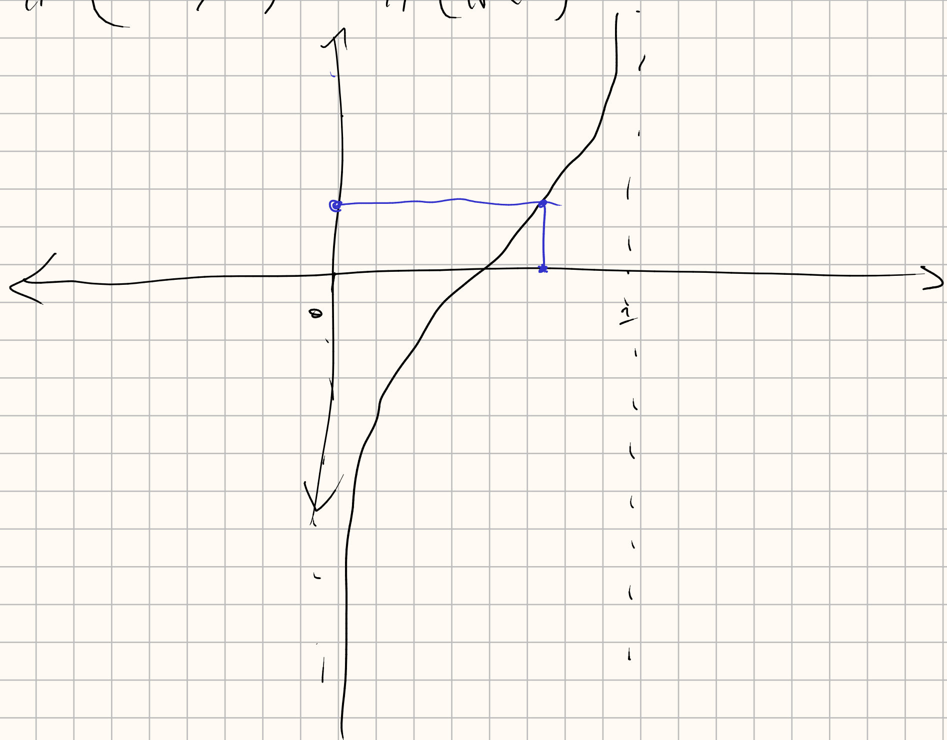
(Skipping a technical pt)



Eg: Not true that

$\#(\mathbb{Z}) = \#(\mathbb{R})$ real #s

Eg: $\#((0,1)) = \#(\mathbb{R})$



Eg: First dimension paradox

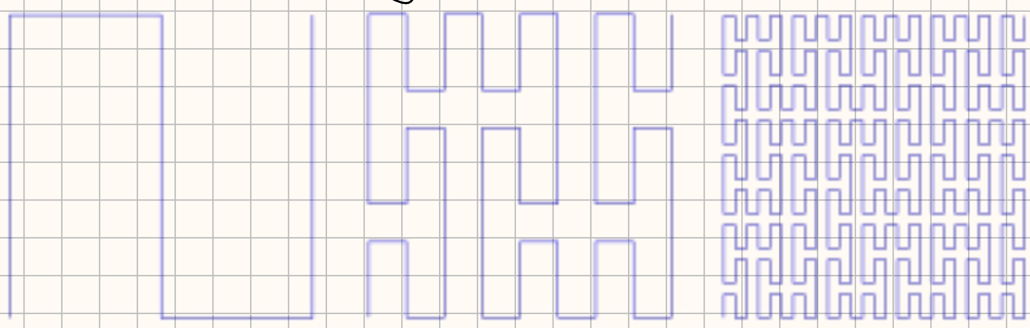
$\#([0,1]) = \#([0,1]\times[0,1])$

(This was known to Cantor)

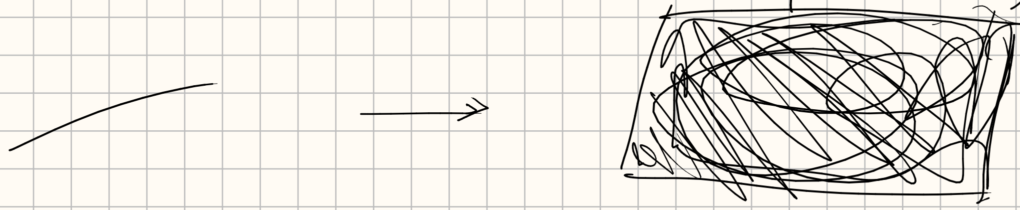
Paradox b/c we can specify a point in □ w/ 1 coordinate

2nd Dimension paradox

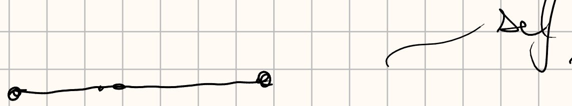
(Space filling curve)



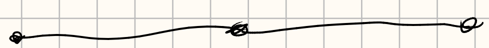
→ ...

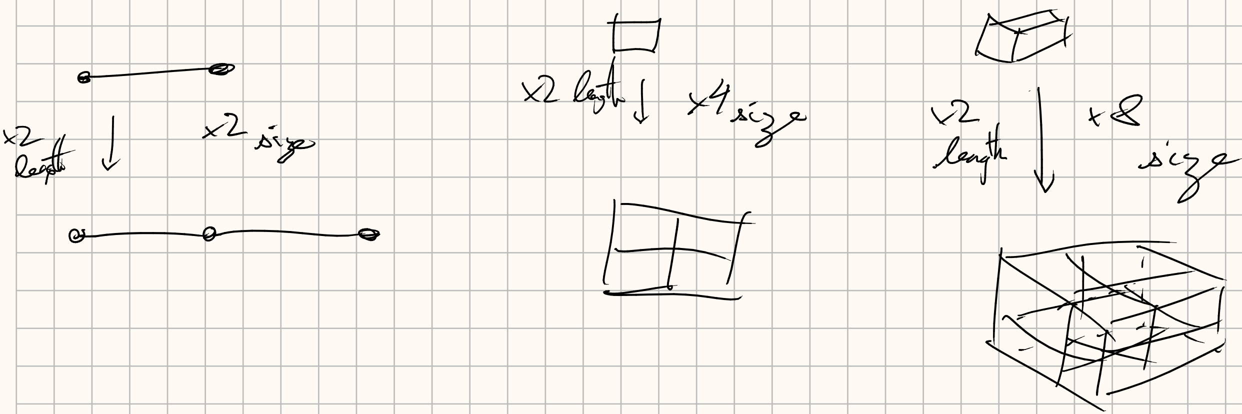


Start defining dimension from a different point: instead of # of coordinates we will study behavior under scaling



If we double the length scale of our object its size also doubles





Def: Scaling dimension

Scale the length by a factor k

Size of the object changes by a factor S

$$\text{scaling dimension} = \frac{\log(S)}{\log(k)}$$

Eg

$SD(—) = 1$

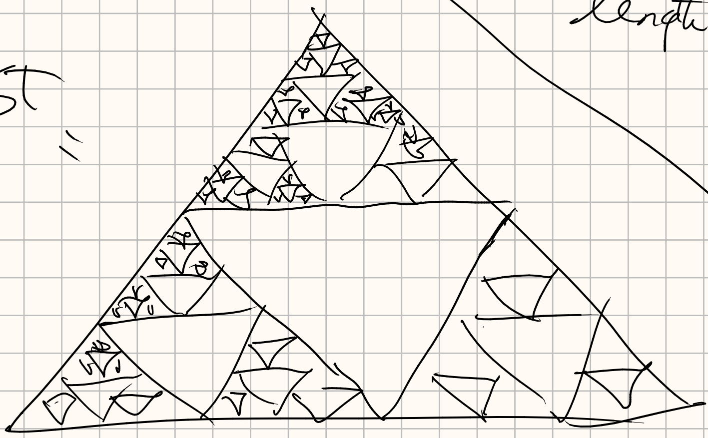
$SD(\square) = 2$

$SD(\text{cube}) = \frac{\log 8}{\log 2} = 3$

Eg: Sierpinski triangle

length scale doubles

ST =



$\times 3$ copies of ST

$$\text{Scaling dim} = \frac{\log 3}{\log 2}$$

There are related + frequently equivalent defin of fractal dimension

1 Scaling dimension only for truly self similar objects

2: (Minkowski - Bouligand) = box-counting dimension

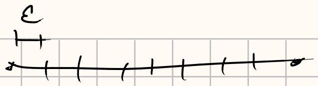
3 Hausdorff - dimension (nicest theoretical properties)

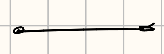
Box-counting dimension

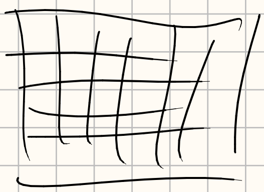
Count how the # of boxes it takes to cover an object changes as we change the size of the box

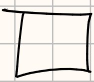
BCD(—) = 1

BCD($\square$) = 2 etc



it takes $\frac{1}{\varepsilon}$ boxes of size length ε to cover 



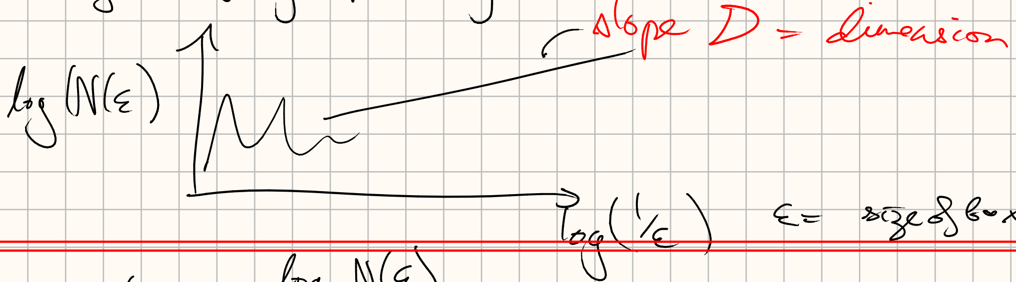
it takes $\left(\frac{1}{\varepsilon}\right)^2$ boxes of length ε to cover 

If in dim D it takes

$N(\varepsilon) = k\left(\frac{1}{\varepsilon}\right)^D$ boxes then

$$\log(N(\varepsilon)) = \log k + D \log\left(\frac{1}{\varepsilon}\right)$$

def: Box counting dimension is the asymptotic slope of a log-log plot of $N(\varepsilon)$ = # of boxes required



ε = size of box

$$\boxed{D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \leftarrow \text{Box counting dimension}}$$