SHP Chaos & Fractals 4-10

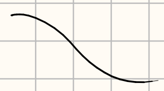
Plan for 4 weeks

Today: Renormalization & the Mandelbrot set

After:

- Strange attractors

periodic orbit



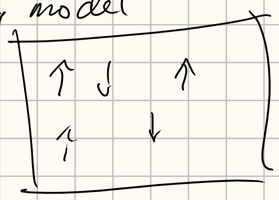
: periodic-like orbit which is a fractal

- Measures of chaos
  - Lyapunov exponents
    (how many years in the future can we predict planetary motion in the solar system?)
  - Fractal dimension + dimension paradoxes

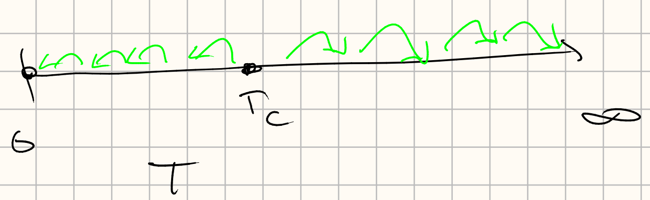
Recall from last time:

Renormalization: a dynamical system on parameter space which describes how our original system changes with varying length scales

Eg: ising model



Dynamical space



Renormalization 4 nobel prizes : K. Wilson

Demo about external rays

Main fact about external rays & parabolic/repelling orbits

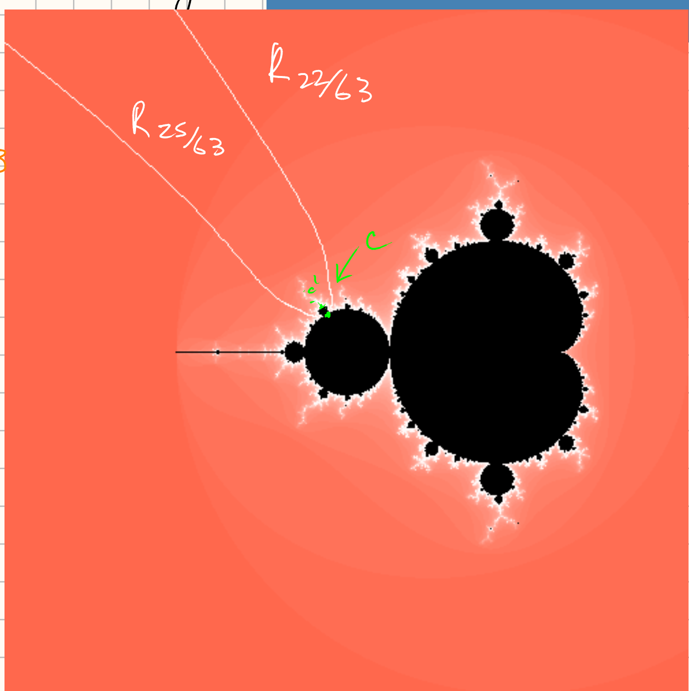
Thm Let θ be a rational angle $\in [0,1)$ (/2π) periodic under doubling with orbit $\{\theta_1, \ldots, \theta_n\}$ $(\theta_i = 2\theta_{i-1})$

1| There exists a unique c where $f_c = z^2+c$ has a parabolic orbit and the rays R_{θ_i} land on that orbit

E.g.: $\theta = \frac{11}{63}$

2| The two angles θ_i and θ_j who cut out the segment containing c (= critical value of f_c) also land on c in the parameter plane in the sense that R_{θ_i} and R_{θ_j} cut $\mathbb{C}$ into two regions @ c.

Eg: $W_{22/63, 25/63}$



3| $f_{c'}$ has

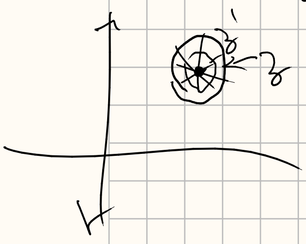
a) A parabolic orbit with these angles $\{\theta_i\}$ only @ $c' = c$

b) A repelling orbit with the same angles $\{\theta_i\}$ landing on that orbit $\Leftrightarrow$ c' is in the sector cut out by those rays $W_{22/63, 25/63}$

Recall: f has a periodic orbit of order n

$$z \mapsto f(z) \mapsto \cdots \mapsto f^n(z) = z$$

then z is a fixed point of f^n since $f^n(z) = z$



think of $g = f^n$ as a function of z' s.t.

$$g(0) = 0 \qquad g(z') = \sum_{n=0}^{\infty} a_n (z')^n$$

$$\Downarrow$$

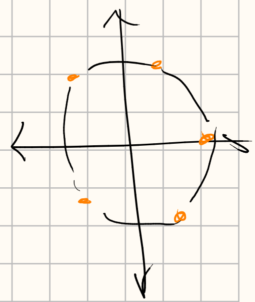
$$a_0 = 0$$

so $g(z') = a_1 z' + a_2 z'^2 + \cdots$ $\quad a_1 \neq 0$

$$= a_1 z'(1 + b_1 z' + b_2 z'^2 + \cdots)$$

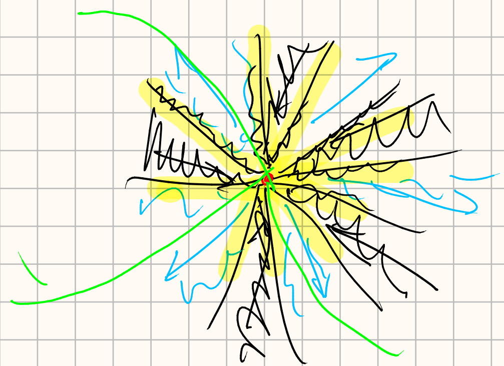
- $|a_1| < 1$ attracting orbit
- $|a_1| > 1$ repelling orbit
- $|a_1| = 1$ neutral orbit / if $a_1 = \sqrt[k]{1}$

then the orbit is parabolic.



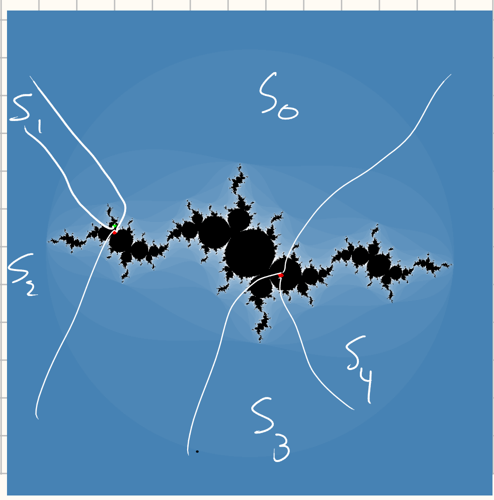
Nearby a point on a parabolic orbit (i.e. a fixed point of f^n) structure looks like

R_{θ_i}



Obs 1 Since f_c sends R_{θ_i} to $R_{\theta_{i+1}}$

f_c also sends $S_i \longrightarrow$ some union of S_j



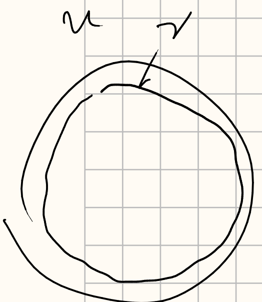
Obs 2: (maybe problems for next week)

If we iterate f_c n-times (n = order of the parabolic orbit)

then this maps S_1 (= sector containing c) to itself.

Key theory: (~200pg book Cplx dynamics & Renormalization)

If we have some region $\mathcal{U}$ and a family of maps $f_{c'}: \underset{\mathcal{U}}{\hat{V}} \to \mathcal{U}$ s.t.



- there is a unique critical pt
- Other than the critical value, a point x in $\mathcal{U}$ has 2 values for $f_{c'}^{-1}(x) \in V$.

+ some other conditions

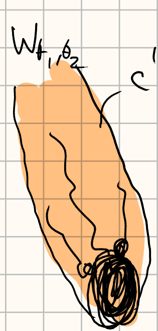
Then our family $f_{c'}$ is equivalent to a family of quadratic maps $f_c = z^2 + c$ for some continuous assignment $c' \mapsto c$

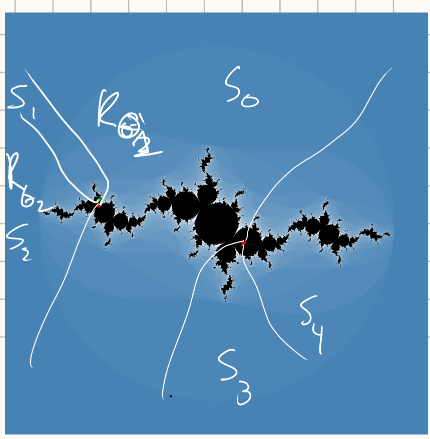
$\{ z^2 + c \mid c \in \mathbb{C} \}$

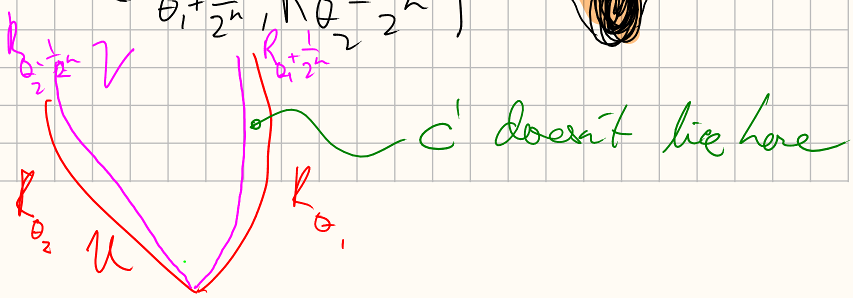
Obs 3 = Obs 2 + Key theory — Says that the theory of quadratic maps shows up again in the sector S_1.

Let $f_{c'}$ for $c' \in W_{\theta_1, \theta_2}$ be the family

assume that c' does not lie outside $(R_{\theta_1 + \frac{1}{2^n}}, R_{\theta_2 - \frac{1}{2^n}})$







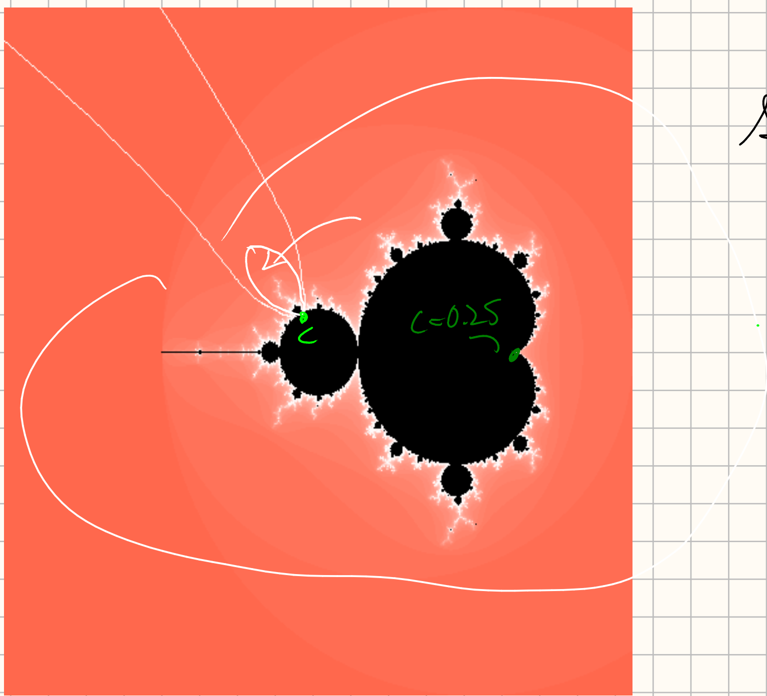
Theory of renormalization $\Longrightarrow$

Thm: There exists a map (the renormalization map for the parabolic point c cut by R_{θ_1} & R_{θ_2})

$$\rho_c : M \longrightarrow W_{\theta_1,\theta_2}$$

$$f_{c'} \longmapsto \text{Renormalized map } f_{c''} \text{ s.t. } f_{c''}^{n} : V \to U$$

is equivalent to $f_{c'}$.



Sends $c = 0.25$

$\downarrow$

c parabolic point