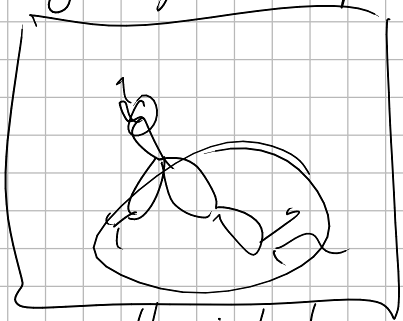
SHP Chaos and Fractals 3-27

Renormalization as a dynamical system on parameter space

Goal: • Understand the renormalization group governs how dynamics of a system change at different length scales

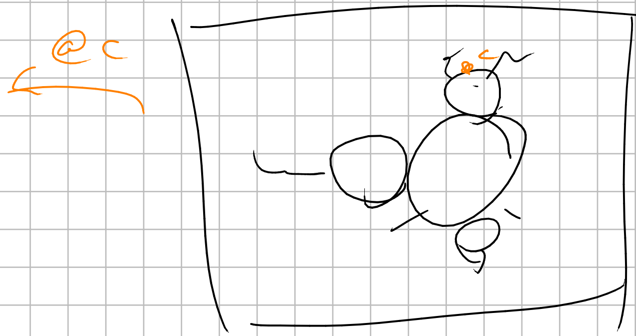
• Understand the RG as a dynamical system D where the states of D are parameters of the original system.

$J \subset$ Dynamical space



self-similarity

$M \subset$ Parameter space



a dynamical system on Parameter space would allow the same argument to conclude M is self-similar

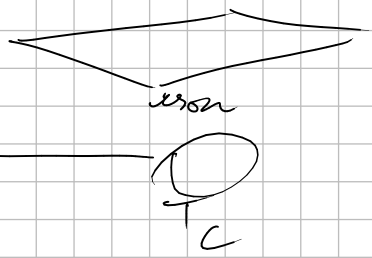
Relationship between a bifurcation | Phase transition

parameter $T \nearrow$

$T_c = 0C$

	bifurcation	Phase transition
# of degrees of freedom	finite e.g. 2	$\to \infty$ (3 for each molecule)
	deterministic systems	Statistical sysems Probability (System is in a given state) governed by Boltzmann's Law we'll see this $P(S_{\text{Energy } E}) \propto e^{-\frac{E}{kT}}$

Main example: 2d Ising model for a ferromagnet



low temperatures ← spontaneous magnetization

high temp → not spontaneously magnetic
need to apply external field for magnetism

ferromagnetism

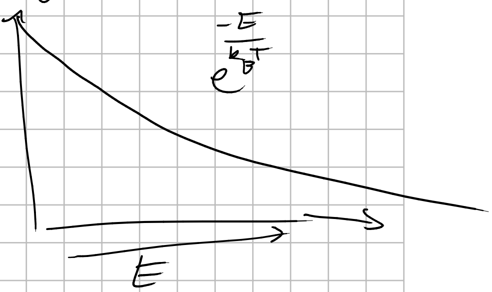
paramagnetism

Ising model: For a statistical mechanical model
ingredients: • Set of states $S = \{S_i\}$
• The energy $E(S_i)$ for each S_i

Boltzmann's law: A system in thermal equilibrium @ temp T ($0 = 0K$)

$$P(S_i) \propto e^{-\frac{E(S_i)}{k_B T}}$$

probability in state S_i



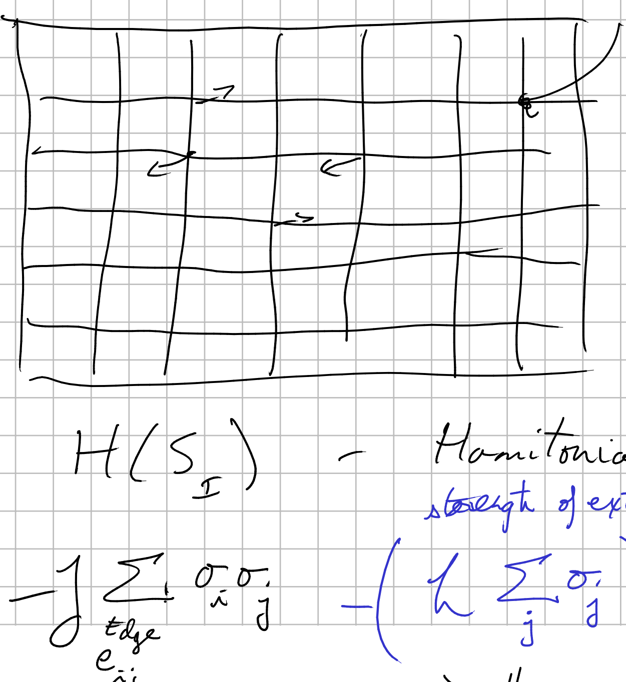
For Ising

• Set of states: Spins on a lattice

$S = \{ S_I = (\sigma_i)_{i \in L} \}$

$|S| = 2^{L^2}$

$\to \infty$ as $L \to \infty$



$i : \sigma_i \in \{1, -1\}$
if site i is spin up or spin down

external field h

• Energy: $H(S_I)$ - Hamiltonian function

strength of external field

$$H(S_I) = -J \sum_{\substack{\text{edge} \\ e_{ij}}} \sigma_i \sigma_j - \left(h \sum_j \sigma_j \right)$$

mostly consider $h = 0$.

External simulation: 1st result when google ising model simulation

Note: Sequence of discrete states

$$S_{I_1} \to S_{I_2} \to S_{I_3} \to$$

Given S_{I_N} — pick random $S_{I_{N+1}}$

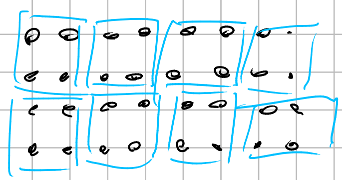
close to S_{I_N}

randomly based on Boltzmann's law

Block Spins & Renormalization

Theory — $\Theta(T,J) = \Theta(\xi T, \xi J) = \Theta(\frac{T}{J},1)$

Θ



between nearby sites

Block-spin renormalization

interaction between neighboring blocks

Note
an exact solution to many quantities was provided by Onsanger ~1930

Nobel prize in chemistry 1968

Up to some approximation a new stat. mech. system

- States S_B $\boxed{}_{B_i}$ average spin $\longrightarrow \{\pm 1\}$ σ_{B_i}
- Hamiltonian governed by an approximation to neighboring block interactions.

$$H(S_B) = -J_B \sum_{\sigma_{B_i}, B_j} \sigma_{B_i}\sigma_{B_j}$$

$$\Theta(T,J) \xrightarrow{BSR} \Theta(T_B, J_B)$$

This $(T,J) \longmapsto BSR(T,J) = (T_B, J_B)$ is a dynamical system in parameter space.

Up to rescaling: 1-d discrete dynamical system

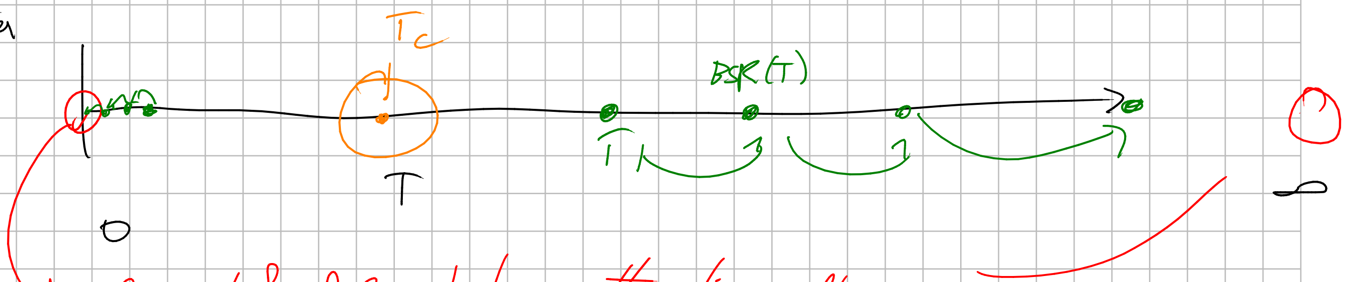
$T,J \longmapsto \xi T, \xi J$

How to

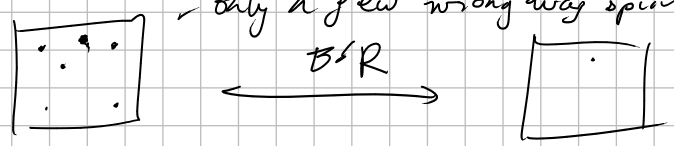
Understand dynamical systems?

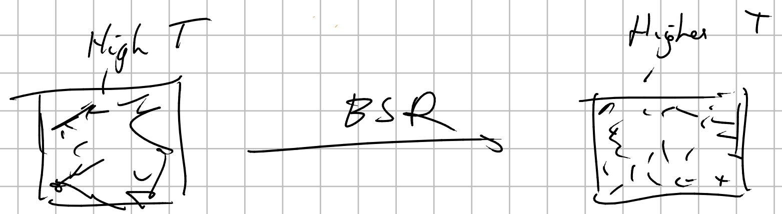
Step 1 What are fixed points?

Parameter space



fixed point of RG, which is attracting b/c





i.e. $T=\infty$ another attracting fixed pt

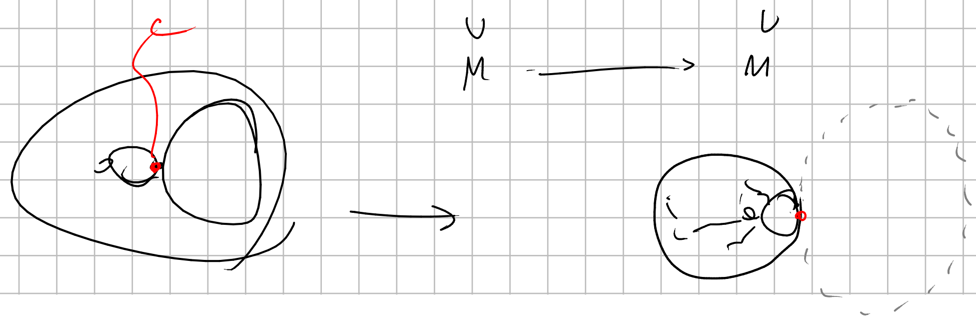
There is one more RG-fixed point T_c
which is repelling

Key fact
The behavior of a system at a fixed point of renormalization is scale invariant i.e. self-similar structures at all lengthscales (fractals)

Where to go from here?

Next time For the mandelbrot set, every parabolic orbit c gives a renormalization map

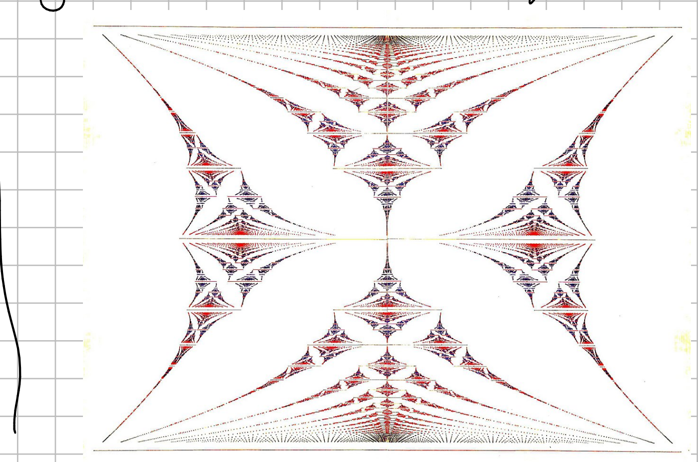
$$\begin{array}{ccc} \mathbb{C}_{\text{parameter}} & \longrightarrow & \mathbb{C}_{\text{parameter}} \\ \cup & & \cup \\ M & \longrightarrow & M \end{array}$$



Other examples of renormalization

Eg 1: Hofstadter butterfly : model of electrons in a non-contract lattice w/ magnetic field

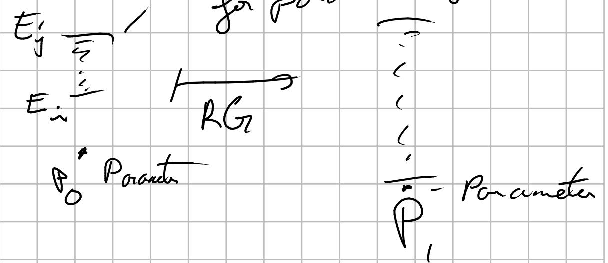
Energy levels of an electron



Parameters of the theory

- There is a renormalization map

Particular band of energy levels for parameter P_0



$\therefore$ RG $\rightsquigarrow$ Parameter space phase diagram is self-similar & fractal

Eg 2 Renormalization in Quantum Chromodynamics
QCD

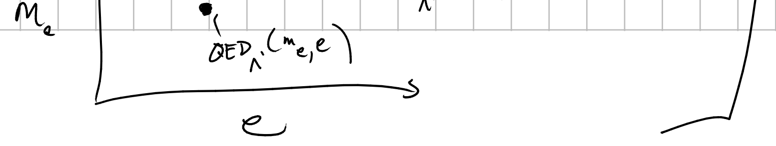
the theory of quarks / gluons & strong interaction
(quarks: pieces of nucleons; gluons: "quark photons" - carry strong force between quarks etc.)

2004 Nobel Prize D. Gross, F. Wilczek, D. Politzer

Quantum field theories depend on parameters (like J from Ising model)
- Eg: electric charge e is a parameter for QED $\leftarrow$ photons, electrons
- One can take a QFT at a very high energy Λ^1 & average all fluctuations of energy E $\Lambda^2 < E < \Lambda^1$ and get a theory @ energy Λ^2.
  Rescale everything by $\xi = \frac{\Lambda^1}{\Lambda^2}$ then we get a new theory at scale Λ^1.

RG for a QFT: again we get a dynamical system on

Parameter space of QFTs



- in QFTs one needs to do this to get finite values for m, e

A different formulation of RG for QFTs using a continuous dynamical system:

$$QFT_{\Lambda'}(g) \xrightarrow{RG_{\Lambda^2 < E < \Lambda'}} QFT_{\Lambda^2 \cdot \frac{\Lambda'}{\Lambda^2}}(g') = QFT_{\Lambda'}(g')$$

$$g \longmapsto g' = RG(g)$$

take $\Lambda^2 = \Lambda' - \varepsilon$ for $\varepsilon \ll 1$ we can get a continuous time dynamical system

$$\boxed{\frac{dg}{dt} = \beta(g)}$$

Beta function – The change in parameters of a QFT wrt length scale as a function of the current parameter

$(t = \log \Lambda)$

Eg: e in QED

$$\frac{de}{d\log(\Lambda)} = \beta_{QED}(e) \propto e^3$$

Eg: QCD a parameter g ($\sim e$) describes how strong strong interaction is

$$\frac{dg}{d\log(\Lambda)} = \beta_{QCD}(g) = -\left(11 - \frac{N}{6} - \frac{2n_g}{3}\right)\frac{g^3}{16\pi^2}$$

where $N = 3$ (# gluons)

$n_g = 6$ (# flavors of quarks)

$\beta < 0$ for these values of N, n_g

$$\frac{dg}{d\log\Lambda} < 0$$

RG-fixed point where $g = 0$ $\Rightarrow$ no interactions between quarks @ gluons & at very high energy (i.e. short distances)