# SHP Chaos and Fractals 3-13

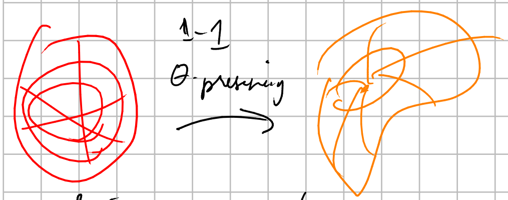
## Solution to problem 2 last week

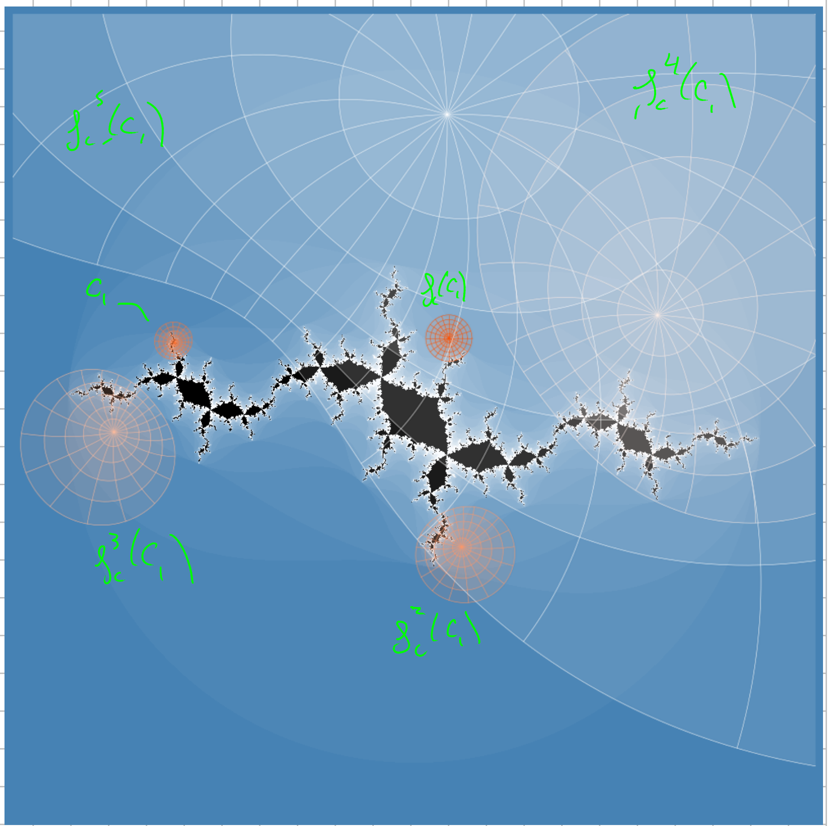
Why are Julia sets self-similar?

### Observe

1: We avoided the critical pt $f_c'(z) = 0$ @ $z = 0$

$\Rightarrow$ $f_c$ maps these circles conformally to each other



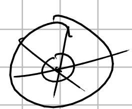




2. Since $C_1$ contains pts not in $FJ(f_c)$

$f_c^n(C_1) \longrightarrow \infty$

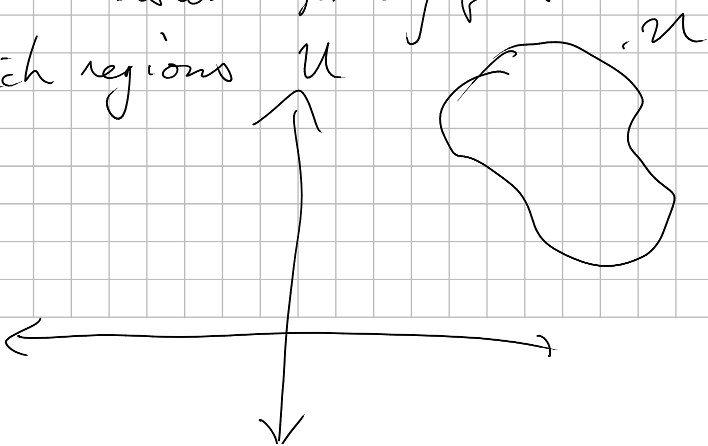
3. $f_c$ maps points in $FJ(f_c)$ to ______ .

## Goal

Understand the Riemann mapping theorem about conformal maps between  and essentially arbitrary regions in $\mathbb{C}$.

i.e. answer the following question:

Which regions $U$ are conformally equivalent to a circle?

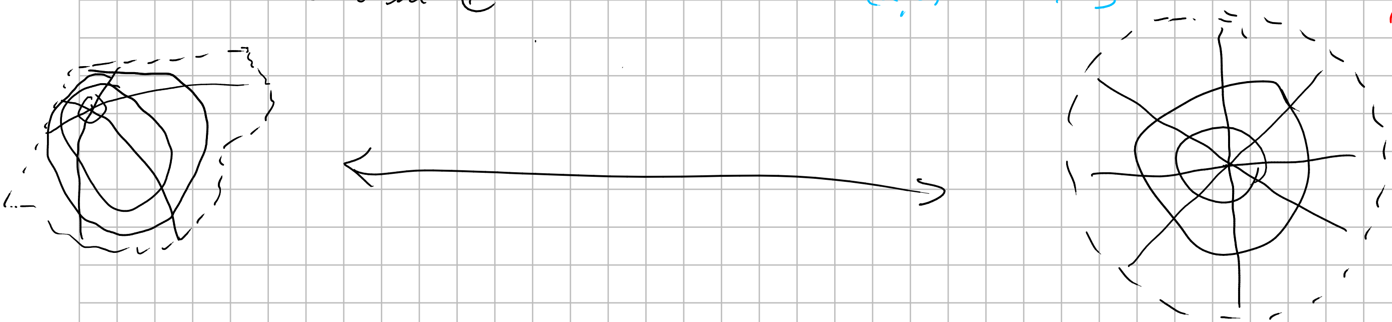


Answer

**Thm** (Riemann mapping theorem)

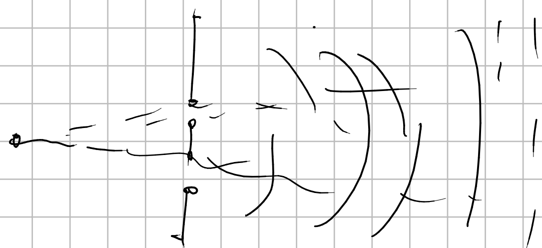
Let $U$ be any
- bounded — contained in some $C_R$, $R < \infty$
- simply connected — loops are contractible
- open — $(a,b)$ not $[c,d]$

domain in $\mathbb{C}$

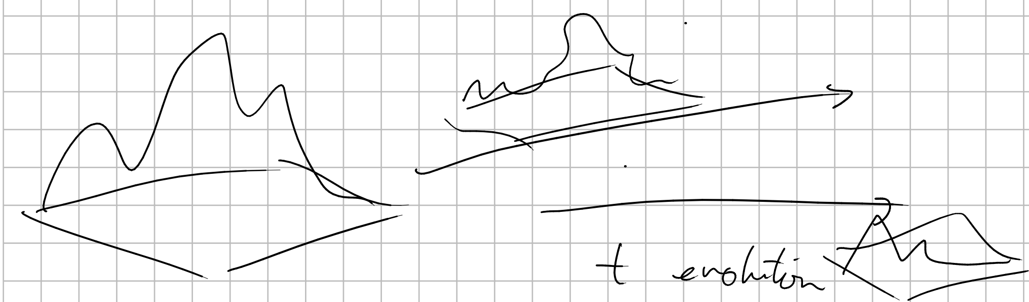


**Aside** RMThm is the reason that quantum gravity is tractable in 2-dimensions (but very very hard in $n$d, $n>2$)

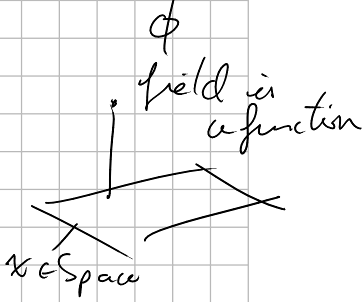
- Quantum particles take all possible paths and combine them



- Quantum fields do the same thing



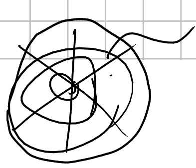

This is tractable to compute if our quantum field is



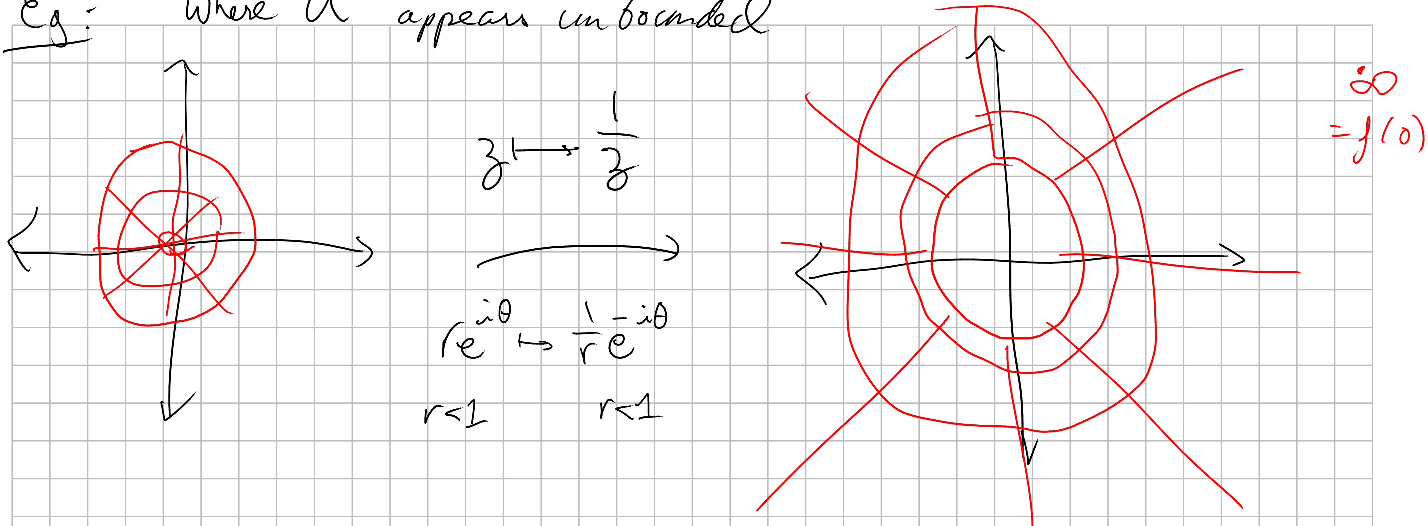

RMThm turns gravitational fields into functions

$\phi$ = scale factor

$\phi(x)$ = how much to stretch space @ $x$

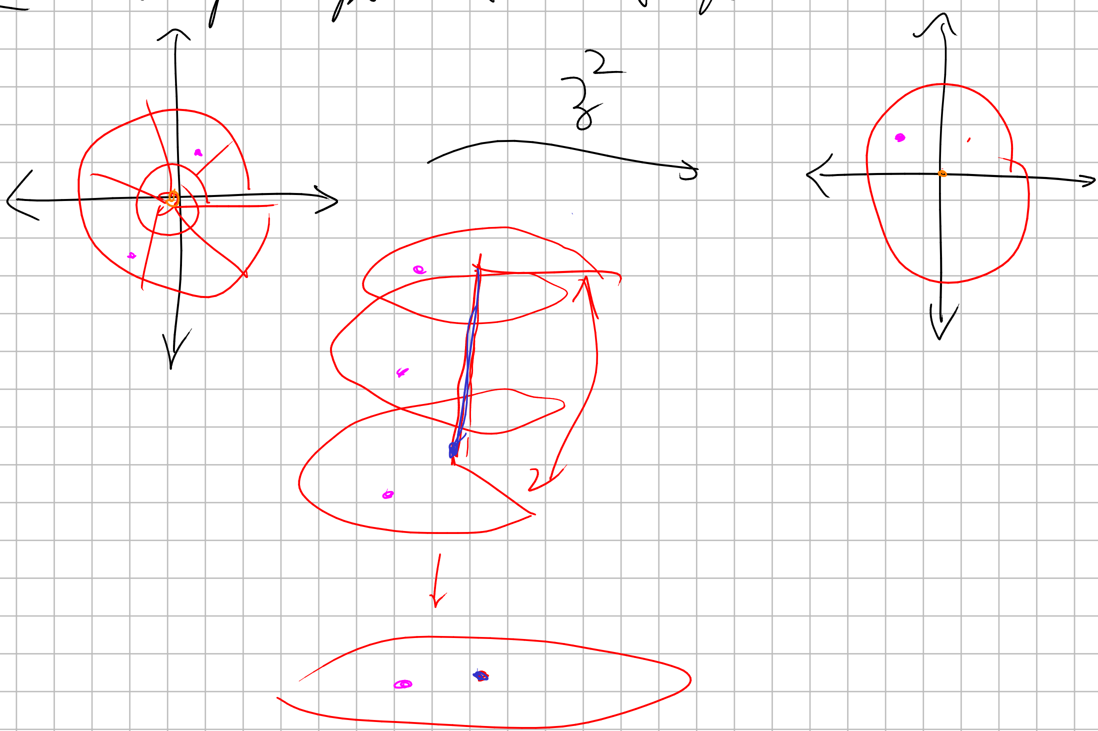


Eg: Where $\mathcal{U}$ appears unbounded




this region is simply connected & bounded on Riemann sphere $= \mathbb{C} \cup \{\infty\}$

Eg: Holomorphic maps with critical points

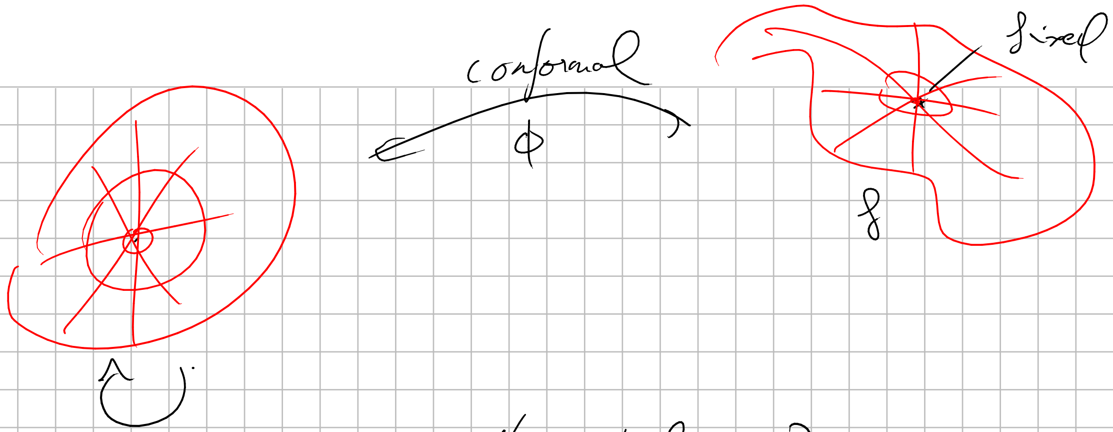



This is what all critical points look like to 1st non-zero order in a Taylor expansion @ the critical point

As an application of this idea:
Study the local behavior of $f_c$ (or any $f$) near a fixed point

Idea periodic for $f_c$ = fixed for $f_c^n$

~ neighborhood looks like unit disk




$\phi^{-1} \circ f \circ \phi$ is a map of the disk fixing $0$.

$$f(0)=0 \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and}$$

$$f(0) = a_0 + a_1 (0)^1 + \dots$$
$$= a_0 = 0$$

$$\Rightarrow f(z) = a_1 z + a_2 z^2 + \dots \qquad a_1 = \lambda$$

$$= \lambda z(1 + a_2' z + \dots) \quad (\text{or } \lambda = 0 \text{ and } f(z) = z^k(a_k + \dots))$$

This analysis tells us the structure of Julia sets, $f_c$, etc. near fixed or periodic points

**Prop:** The behavior of periodic orbits of $f_c$ falls into 3 classes based on $\lambda$

- $|\lambda| < 1$ – periodic orbit is attracting
- $|\lambda| > 1$ – periodic orbit is repelling
- $|\lambda| = 1$ – neutral.