3-6 SHP Chaos and Fractals

Goal: Understand theoretical properties of a class of functions $f: \mathbb{C} \to \mathbb{C}$ (eg. $z \mapsto z^2 + c$) called <u>holomorphic functions</u>, and how these properties give structure to Julia sets.

Def / Theorem Given $f: \mathbb{C} \to \mathbb{C}$ (or $f: \underset{\mathbb{C}}{\overset{}{\mathcal{U}}} \to \mathbb{C}$)

$\mathcal{U} \subset \mathbb{C}$, $\mathcal{U} \subset \{x+iy \mid x, y \text{ are real}\}$ is a subset, eg. $\mathcal{U} =$ ○, $\mathcal{U} = \{x+iy \mid x^2+y^2 < 1\}$

it is called <u>holomorphic</u> if any one of the following is true:

1) There is a continuous $f'(z) = \frac{df}{dz}$ s.t. $\frac{d}{dz}(i f(z)) = i \frac{d}{dz} f(z)$

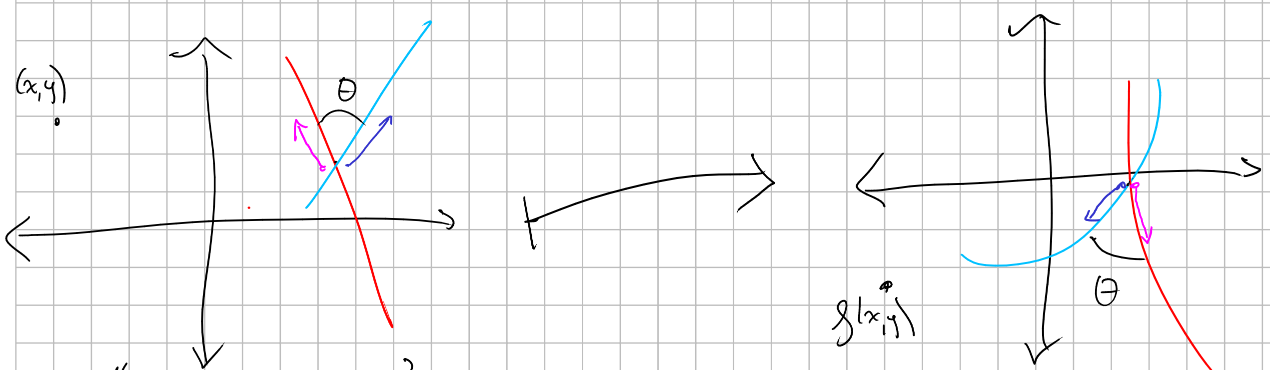
$z^2 + c \mapsto 2z$
$f(z) \mapsto f'(z)$

2) f is the same as it's Taylor series near any point z_0 i.e.

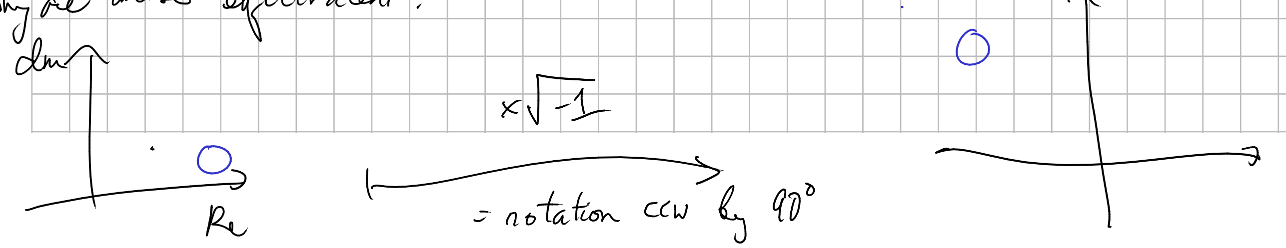
$$f(z - z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \text{near } z_0$$

3) f is <u>conformal</u> where $f'(z) \neq 0$

Def: $f: \mathbb{R}^2 \to \mathbb{R}^2$ is <u>conformal</u> if it preserves angles



Why are these equivalent?



Examples of holomorphic functions

1) Polynomials

$$z \longmapsto a_n z^n + \cdots + a_1 z + a_0$$

2) Trig functions

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

etc.

Eg: $\exp : \mathbb{C} \longrightarrow \mathbb{C}$

$$z \longmapsto e^{iz}$$