Macdonald processes and universal limits in discrete random matrix theory

Roger Van Peski (KTH)

Leipzig integrable probability seminar
April 24, 2023
Based on https://arxiv.org/abs/2310.12275,
https://arxiv.org/abs/2312.11702, and work in preparation

Question 1: Let $A_1, \ldots, A_\tau$ be iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. What does the distribution of

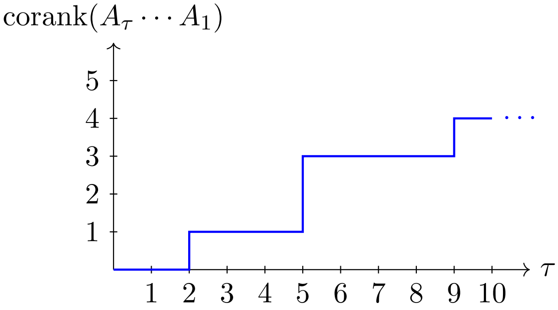
$$\mathrm{rank}(A_\tau \cdots A_2 A_1)$$

look like for large N and τ?

Question 1: Let $A_1, \ldots, A_\tau$ be iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. What does the distribution of

$$\mathrm{rank}(A_\tau \cdots A_2 A_1)$$

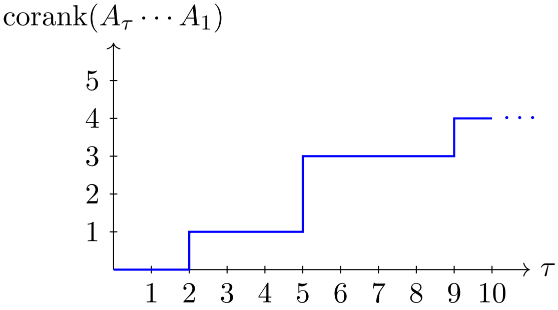
look like for large N and τ?



Question 1: Let $A_1, \ldots, A_\tau$ be iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. What does the distribution of

$$\mathrm{rank}(A_\tau \cdots A_2 A_1)$$

look like for large N and τ?



Fact: $\mathrm{corank}(A_\tau \cdots A_1) \approx \log_p \tau$, finite limit fluctuations.

An intriguing random integer

Theorem (VP '23, special case)

For each $N \geq 1$ take $A_1^{(N)}, A_2^{(N)}, \ldots$ iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. Then as $N \to \infty$,

$$\mathrm{corank}(A_{\tau_N}^{(N)} \cdots A_1^{(N)}) - [\log_p \tau_N + \zeta] \to \mathcal{L}_{1,p^{-1},p^{-\zeta}/(p-1)}$$

(an explicit $\mathbb{Z}$-valued random variable), for any sequence $\tau_N, N \geq 1$ s.t. $1 \ll \tau_N \ll p^N$ and $-\log_p \tau_N$ converges in $\mathbb{R}/\mathbb{Z}$ to some $\zeta \in [0,1)$.

An intriguing random integer

Theorem (VP '23, special case)

For each $N \geq 1$ take $A_1^{(N)}, A_2^{(N)}, \ldots$ iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. Then as $N \to \infty$,

$$\mathrm{corank}(A_{\tau_N}^{(N)} \cdots A_1^{(N)}) - [\log_p \tau_N + \zeta] \to \mathcal{L}_{1,p^{-1},p^{-\zeta}/(p-1)}$$

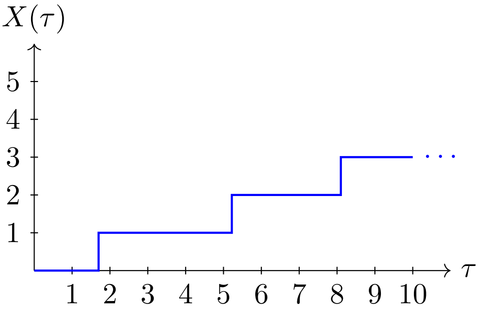
(an explicit $\mathbb{Z}$-valued random variable), for any sequence $\tau_N, N \geq 1$ s.t. $1 \ll \tau_N \ll p^N$ and $-\log_p \tau_N$ converges in $\mathbb{R}/\mathbb{Z}$ to some $\zeta \in [0,1)$.

Here for any $\chi \in \mathbb{R}_{>0}, t \in (0,1)$, $\mathcal{L}_{1,t,\chi}$ is the $\mathbb{Z}$-valued r.v. defined by

$$\Pr(\mathcal{L}_{1,t,\chi} = x) = \frac{1}{\prod_{i \geq 1}(1-t^i)} \sum_{j \geq 0} e^{-\chi t^{x-j}} \frac{(-1)^j t^{\binom{j}{2}}}{\prod_{i=1}^{j}(1-t^i)}$$

for any $x \in \mathbb{Z}$.

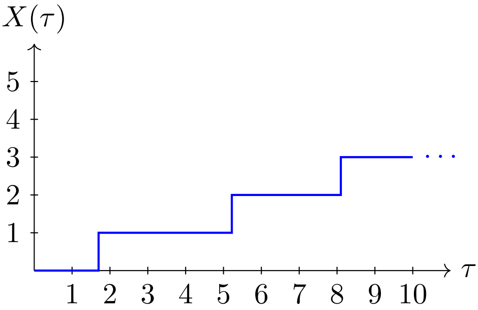
Let $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ be the $\mathbb{Z}_{\geq 0}$-valued process which jumps by 1, and waits at state $x \in \mathbb{Z}_{\geq 0}$ for an $\mathrm{Exp}(t^x)$-distributed time[1].



Question 2: What are the limiting fluctuations of $X(\tau)$?

[1] i.e. $\Pr((\text{time spent at } x) > T) = e^{-t^{-x}T}$.

Let $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ be the $\mathbb{Z}_{\geq 0}$-valued process which jumps by 1, and waits at state $x \in \mathbb{Z}_{\geq 0}$ for an $\mathrm{Exp}(t^x)$-distributed time[1].



Question 2: What are the limiting fluctuations of $X(\tau)$?

Theorem (VP '23, special case)

$$X(\tau) - (\log_{t^{-1}} \tau + \zeta) \to \mathcal{L}_{1,t,t^{-\zeta+1}/(1-t)}$$

in distribution as $\tau \to \infty$ along the sequence $\tau \in t^{-\mathbb{N}+\zeta}$.

[1] i.e. $\Pr((\text{time spent at } x) > T) = e^{-t^{-x}T}$.

Question 3: Let M_n be a uniformly random strictly upper-triangular matrix over $\mathbb{F}_q$. What are the limiting fluctuations of $\operatorname{corank}(M_n)$?

$$M_n \in \begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \ddots & * \\ 0 & 0 & 0 & \ddots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \subset \operatorname{Mat}_n(\mathbb{F}_q)$$

Question 3: Let M_n be a uniformly random strictly upper-triangular matrix over $\mathbb{F}_q$. What are the limiting fluctuations of $\operatorname{corank}(M_n)$?

$$M_n \in \begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \ddots & * \\ 0 & 0 & 0 & \ddots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \subset \operatorname{Mat}_n(\mathbb{F}_q)$$

Theorem (VP '24+, special case)

Let $n_j, j = 1, 2, \ldots$ be an increasing sequence with $-\log_q \tau_N$ converging in $\mathbb{R}/\mathbb{Z}$ to some $\zeta \in [0, 1)$. Then as $j \to \infty$,

$$\operatorname{corank}(M_{n_j}) - [\log_q n_j + \zeta] \to \mathcal{L}_{1,q^{-1},q^{-\zeta}}.$$

p-adic random matrices

Fix a prime p. $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^k\mathbb{Z}$, concretely

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \ldots : a_i \in \{0, \ldots, p-1\}\}.$$

Have ring maps $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^d\mathbb{Z}$.

Fix a prime p. $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^k\mathbb{Z}$, concretely

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \ldots : a_i \in \{0, \ldots, p-1\}\}.$$

Have ring maps $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^d\mathbb{Z}$.

$(\mathbb{Z}_p, +)$ has Haar probability measure $\mu_{Haar}^{\mathbb{Z}_p}$, explicitly given by taking $a_i \in \{0, \ldots, p-1\}$ iid uniformly random, projects to uniform on $\mathbb{Z}/p^k\mathbb{Z}$ for any k.

Proposition (Smith normal form)

For nonsingular $A \in \mathrm{Mat}_N(\mathbb{Q}_p)$*, there are* $U, V \in \mathrm{GL}_N(\mathbb{Z}_p)$ *for which*

$$UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_N})$$

for singular numbers $\lambda_i \in \mathbb{Z}$ *(unique).*

Proposition (Smith normal form)

For nonsingular $A \in \mathrm{Mat}_N(\mathbb{Q}_p)$, *there are* $U, V \in \mathrm{GL}_N(\mathbb{Z}_p)$ *for which*

$$UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_N})$$

for singular numbers $\lambda_i \in \mathbb{Z}$ *(unique).*

We study $\mathrm{SN}(A) = (\mathrm{SN}(A)_1, \ldots, \mathrm{SN}(A)_N) := (\lambda_1, \ldots, \lambda_N)$ above (a random integer signature of length N).

Two worlds: RMT over $\mathbb{C}$ and $\mathbb{Q}_p$

	RMT over $\mathbb{C}$	**RMT over $\mathbb{Q}_p$**
Group	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{Q}_p)$
Maximal compact subgroup	$U(n)$	$\mathrm{GL}_n(\mathbb{Z}_p)$
Structure theorem	SVD: $UAV = \mathrm{diag}(e^{-r_1}, \dots, e^{-r_n})$ for $U, V \in U(n)$	Smith normal form: $UAV = \mathrm{diag}(p^{\lambda_1}, \dots, p^{\lambda_n})$ for $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$
We study	Singular values e^{-r_i}	Singular numbers λ_i

Two worlds: RMT over $\mathbb{C}$ and $\mathbb{Q}_p$

	RMT over $\mathbb{C}$	**RMT over $\mathbb{Q}_p$**
Group	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{Q}_p)$
Maximal compact subgroup	$U(n)$	$\mathrm{GL}_n(\mathbb{Z}_p)$
Structure theorem	SVD: $UAV = \mathrm{diag}(e^{-r_1}, \ldots, e^{-r_n})$ for $U, V \in U(n)$	Smith normal form: $UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})$ for $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$
We study	Singular values e^{-r_i}	Singular numbers λ_i

A matrix $A \in \mathrm{Mat}_N(\mathbb{Z}_p)$ yields a map $\mathbb{Z}_p^N \to \mathbb{Z}_p^N$,

$$\mathrm{cok}(A) := \mathbb{Z}_p^N / A\mathbb{Z}_p^N .$$

Two worlds: RMT over $\mathbb{C}$ and $\mathbb{Q}_p$

	RMT over $\mathbb{C}$	**RMT over $\mathbb{Q}_p$**
Group	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{Q}_p)$
Maximal compact subgroup	$U(n)$	$\mathrm{GL}_n(\mathbb{Z}_p)$
Structure theorem	SVD: $UAV = \mathrm{diag}(e^{-r_1}, \ldots, e^{-r_n})$ for $U, V \in U(n)$	Smith normal form: $UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})$ for $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$
We study	Singular values e^{-r_i}	Singular numbers λ_i

A matrix $A \in \mathrm{Mat}_N(\mathbb{Z}_p)$ yields a map $\mathbb{Z}_p^N \to \mathbb{Z}_p^N$,

$$\mathrm{cok}(A) := \mathbb{Z}_p^N / A\mathbb{Z}_p^N.$$

For $\lambda = \mathrm{SN}(A)$, get finite abelian p-group

$$\mathrm{cok}(A) \cong \bigoplus_{i=1}^{N} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}.$$

Motivation: random groups

In many contexts such as

- Arithmetic statistics—distributions of class groups of number fields, Tate-Shafarevich groups—(Bhargava, Cohen, Ellenberg, Kane, Lenstra Jr., Nguyen, Poonen, Rains, Sawin, Venkatesh, Westerland, Wood... '83-present),
- Sandpile groups of random graphs—(Clancy, Fulman, Kaplan, Koplewitz, Leake, Nguyen, Payne, Wood... '14-present),
- (co)homology groups of random chain complexes—(Kahle, Lutz, Meszaros, Newman, Parsons,...)

we want asymptotics of a (pseudo-)random finite abelian group G.

Motivation: random groups

In many contexts such as

- Arithmetic statistics—distributions of class groups of number fields, Tate-Shafarevich groups—(Bhargava, Cohen, Ellenberg, Kane, Lenstra Jr., Nguyen, Poonen, Rains, Sawin, Venkatesh, Westerland, Wood... ‘83-present),
- Sandpile groups of random graphs—(Clancy, Fulman, Kaplan, Koplewitz, Leake, Nguyen, Payne, Wood... ‘14-present),
- (co)homology groups of random chain complexes—(Kahle, Lutz, Meszaros, Newman, Parsons,...)

we want asymptotics of a (pseudo-)random finite abelian group G.

$$G = \bigoplus_{p \text{ prime}} G_p$$

$$G_p = \bigoplus_i \mathbb{Z}/p^{\lambda_i^{(p)}}\mathbb{Z} \rightsquigarrow \text{random partition } \lambda.$$

Random matrices produce random groups

The first such distribution (1983) was the *Cohen-Lenstra distribution* on abelian p-groups $G = \bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ given by

$$\Pr(G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\operatorname{Aut}(G)|},$$

observed empirically for p-parts of class groups of fields $\mathbb{Q}(\sqrt{-d})$.

Random matrices produce random groups

The first such distribution (1983) was the *Cohen-Lenstra distribution* on abelian p-groups $G = \bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ given by

$$\Pr(G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\operatorname{Aut}(G)|},$$

observed empirically for p-parts of class groups of fields $\mathbb{Q}(\sqrt{-d})$.

Theorem (Friedman-Washington 1987)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p)$ have iid additive Haar entries. Then as $N \to \infty$, $\mathrm{cok}(A^{(N)})$ limits to the Cohen-Lenstra distribution $\Pr(G) = \prod_{i \geq 1}(1 - p^{-i})/|\operatorname{Aut}(G)|$.

Theorem (Wood 2015)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ have iid entries from any distribution which is nonconstant modulo p. Then $\mathrm{cok}(A^{(N)})_p$ converges to the Cohen-Lenstra distribution.

At a probabilistic level things look quite different

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

For fixed N, the logarithms of singular values have Gaussian fluctuations as $\tau \to \infty$ ([Furstenberg-Kesten 1960], [Akemann-Burda-Kieburg 2014], [Liu-Wang-Wang 2018]).

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

For fixed N, the logarithms of singular values have Gaussian fluctuations as $\tau \to \infty$ ([Furstenberg-Kesten 1960], [Akemann-Burda-Kieburg 2014], [Liu-Wang-Wang 2018]).

When $N, \tau \to \infty, N/\tau \to c \in (0, \infty)$, the bulk (resp. soft edge) statistics are c-parametrized deformations of sine (resp. Airy) kernel ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018]).

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

For fixed N, the logarithms of singular values have Gaussian fluctuations as $\tau \to \infty$ ([Furstenberg-Kesten 1960], [Akemann-Burda-Kieburg 2014], [Liu-Wang-Wang 2018]).

When $N, \tau \to \infty, N/\tau \to c \in (0, \infty)$, the bulk (resp. soft edge) statistics are c-parametrized deformations of sine (resp. Airy) kernel ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018]).

Question

How do singular numbers of p-adic matrix products behave?

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

For fixed N, the logarithms of singular values have Gaussian fluctuations as $\tau \to \infty$ ([Furstenberg-Kesten 1960], [Akemann-Burda-Kieburg 2014], [Liu-Wang-Wang 2018]).

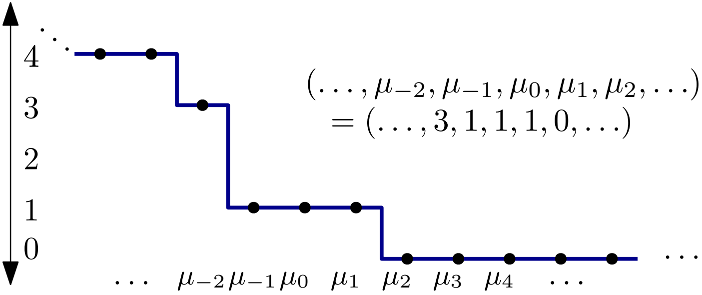
When $N, \tau \to \infty, N/\tau \to c \in (0, \infty)$, the bulk (resp. soft edge) statistics are c-parametrized deformations of sine (resp. Airy) kernel ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018]).

Question

How do singular numbers of p-adic matrix products behave?

[VP 2020]: For A_i 'nice' $N \times N$ p-adic random matrices, each $\mathrm{SN}(A_\tau \cdots A_1)_i$ has Gaussian fluctuations as $\tau \to \infty$.

Dynamical local limits?



Question

For $1 \ll r_N \ll N$, *does joint evolution of bulk singular numbers* $\mathrm{SN}(A_\tau \cdots A_1)_{r_N+i}, i \in \{\dots, -1, 0, 1, \dots\}$ *converge as matrix size* $N \to \infty$ *to some Markov process on*

$$\mathrm{Sig}_{2\infty}^{\geq 0} := \{\mu = (\dots, \mu_{-1}, \mu_0, \mu_1, \dots) \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}} : \mu_{i+1} \leq \mu_i\}?$$

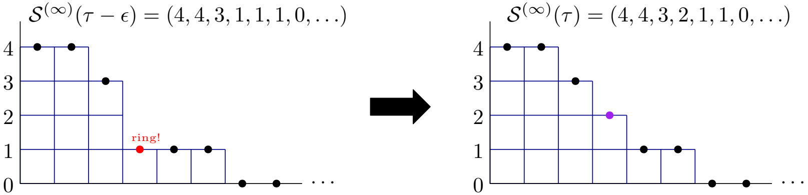
The reflecting Poisson sea and limit theorems

Extending $X(\tau)$ to a growth process on integer partitions

$\mathcal{S}^{(\infty)}(\tau) = (\mathcal{S}^{(\infty)}(\tau)_1, \mathcal{S}^{(\infty)}(\tau)_2, \ldots), \tau \in \mathbb{R}_{\geq 0}$.

Starts at $\mathcal{S}^{(\infty)}(0) = (0, 0, \ldots)$.

Indep. exp. clocks at $1, 2, \ldots$ of rates $t, t^2, \ldots$ control jumps.

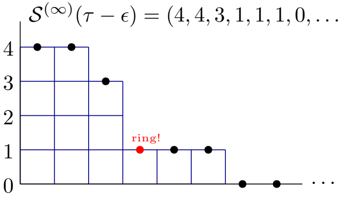


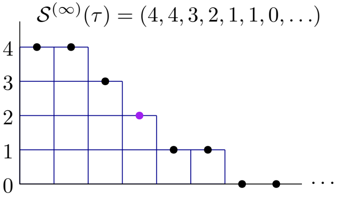
Extending $X(\tau)$ to a growth process on integer partitions

$$\mathcal{S}^{(\infty)}(\tau) = (\mathcal{S}^{(\infty)}(\tau)_1, \mathcal{S}^{(\infty)}(\tau)_2, \ldots), \tau \in \mathbb{R}_{\geq 0}.$$

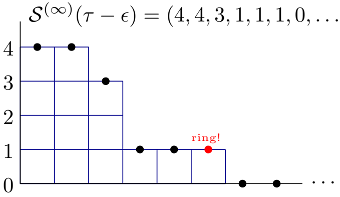
Starts at $\mathcal{S}^{(\infty)}(0) = (0, 0, \ldots)$.

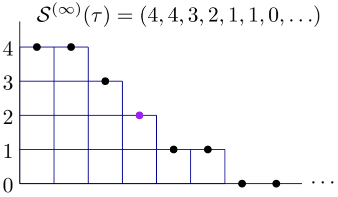
Indep. exp. clocks at $1, 2, \ldots$ of rates $t, t^2, \ldots$ control jumps.





When particle is blocked, donates jump:





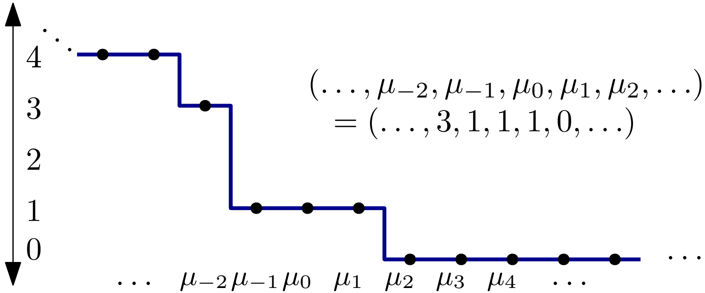
Have $\mathbb{Z}^d$-valued random variables $\mathcal{L}_{d,p^{-\zeta}/(p-1)}$ such that:

Theorem (VP '23)

Let $\tau_N = p^{N-\zeta}$. *Then*

$$(\mathcal{S}^{(\infty)}(\tau_N)'_i - \log_p(\tau_N) - \zeta)_{1\leq i\leq d} \xrightarrow{N\to\infty} \mathcal{L}_{d,p^{-\zeta}/(p-1)}.$$

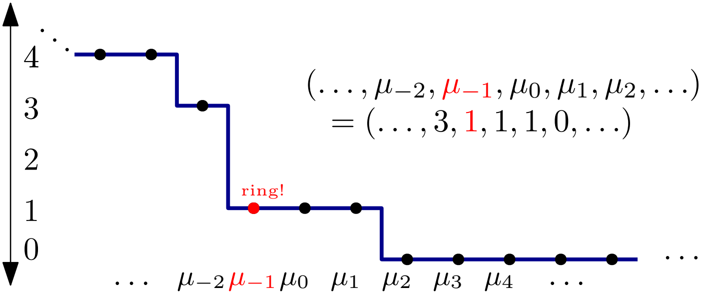
The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

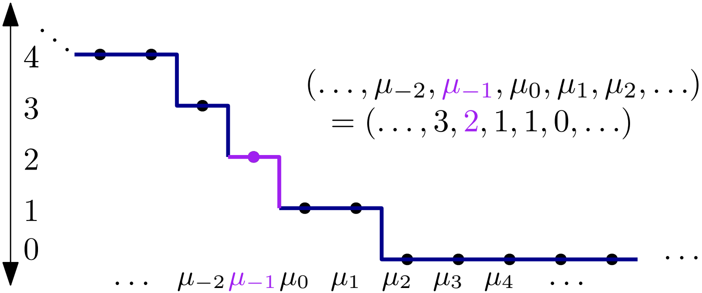
The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\dots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \dots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

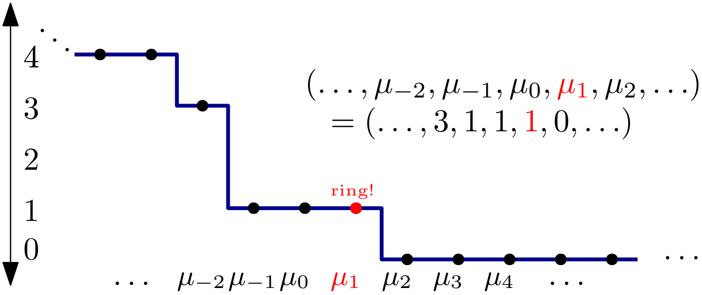
The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots)$, $T \geq 0$ is the continuous-time stochastic process with each $\mu_i(T)$ increasing by 1 according to rate-t^i exponential clock (independent of each other), donating move if blocked.

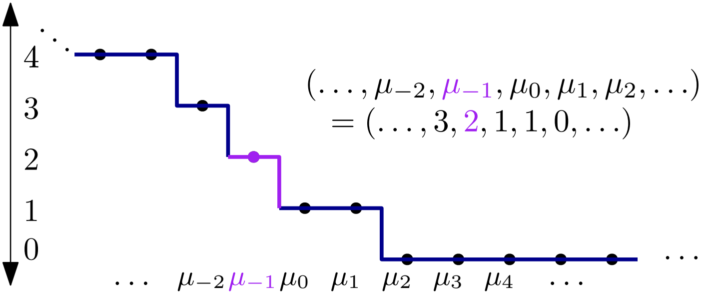
The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\dots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \dots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

Local limit picture

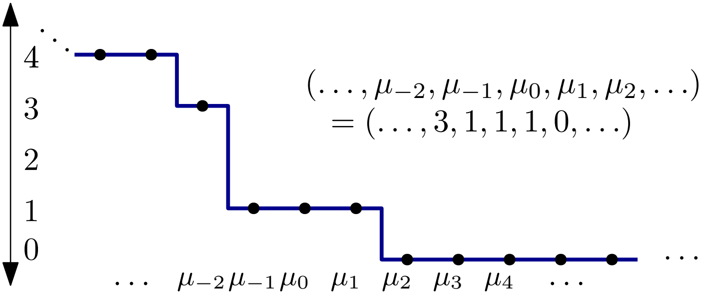
Theorem (VP '23)

For each $N \geq 1$ *let* $A_i^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p), i \geq 1$ *be iid additive Haar matrices and* $\Pi^{(N)}(\tau) := \mathrm{SN}(A_\tau^{(N)} \cdots A_1^{(N)})$.
Let $(r_N)_{N \geq 1}$ *be such that (1)* $r_N \to \infty$ *and (2)* $N - r_N \to \infty$. *Then* $(\ldots, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N - 1}, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N}, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N + 1}, \ldots)$ *converges to* $\mu(T)$ *(with* $t = 1/p$*).*

Comparison to dynamical local limits over $\mathbb{C}$

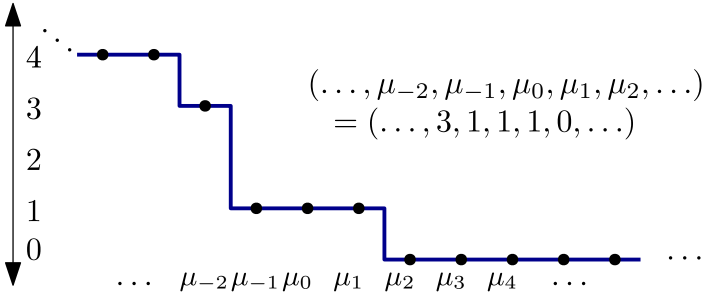
Fixed-time limits: analogues of deformed sine/Airy process

Let $\mu_i'(T) := \max(\{j : \mu_j(T) \geq i\})$.
Here $(\mu_1', \mu_2', \mu_3', \mu_4', \ldots) = (1, -2, -2, -3, \ldots)$.



Fixed-time limits: analogues of deformed sine/Airy process

Let $\mu_i'(T) := \max(\{j : \mu_j(T) \geq i\})$.
Here $(\mu_1', \mu_2', \mu_3', \mu_4', \ldots) = (1, -2, -2, -3, \ldots)$.



Theorem (VP '23)

The joint distribution of $(\mu_1'(T), \mu_2'(T), \ldots, \mu_k'(T))$ *is an explicit random variable* $\mathcal{L}_{k,t,tT/(1-t)}$.

What is $\mathcal{L}_{k,t,\chi}$?

Example

When $k = 2$ (taking $t = 1/p$), $\chi \in \mathbb{R}_{>0}$, $(L + x, L) \in \mathrm{Sig}_2$,

$$\Pr(\mathcal{L}_{2,t,\chi} = (L + x, L)) = \frac{t^{\binom{x}{2}}}{(t;t)_\infty} \sum_{m \geq 0} e^{-t^{L-m}\chi}$$
$$\times (-1)^m t^{m^2+(x-1)m} \sum_{i=0}^{x} \frac{(-1)^{x-i}}{(t;t)_{x-i}} \begin{bmatrix} m+i \\ i \end{bmatrix}_t$$
$$\times \left(\frac{(t^{L-m}\chi)^{i+m}}{(i+m)!} + \frac{(t^{L-m}\chi)^{i+m-1}\mathbb{1}(i+m \geq 1)}{(i+m-1)!} \right)$$

where

$$(a;t)_n := (1-a)(1-ta)\cdots(1-t^{n-1}a) \text{ and } \begin{bmatrix} a \\ b \end{bmatrix}_t := \frac{(t;t)_a}{(t;t)_b (t;t)_{a-b}}.$$

A word on proofs

Given convergence of fixed-time marginals, explicit linear-algebraic arguments show multi-time convergence to $\mu(T)$ (robust, universal for generic $\mathrm{GL}_N(\mathbb{Z}_p)$-invariant distributions).

A word on proofs

Given convergence of fixed-time marginals, explicit linear-algebraic arguments show multi-time convergence to $\mu(T)$ (robust, universal for generic $\mathrm{GL}_N(\mathbb{Z}_p)$-invariant distributions).

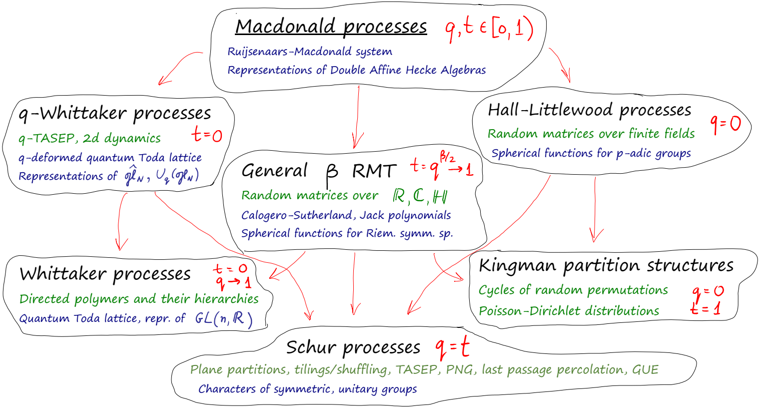
Convergence of fixed-time marginals to $\mu(T)$ uses symmetric function theory.

Macdonald processes [Borodin-Corwin '11]

Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ indexed by *integer partitions* $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ are symmetric polynomials in $x_1, \ldots, x_n$ with two parameters q, t.

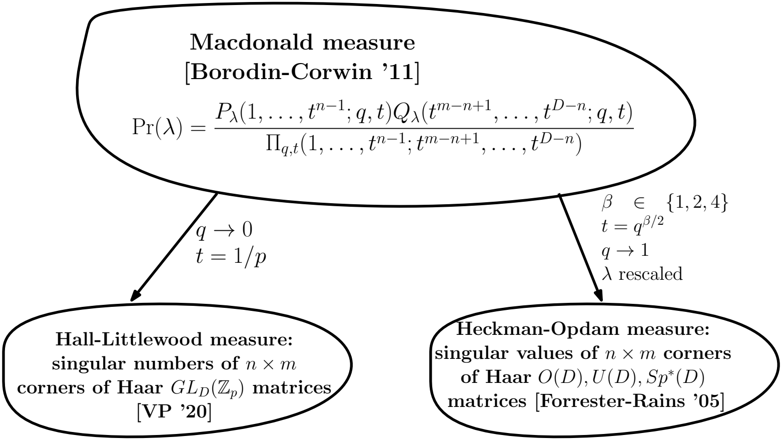
Macdonald processes [Borodin-Corwin '11]

Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ indexed by *integer partitions* $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ are symmetric polynomials in $x_1, \ldots, x_n$ with two parameters q, t.



(Figure credits: A. Borodin, ICM 2014 slides)

The $\mathbb{Z}_p \leftrightarrow \mathbb{C}$ analogy is actually extremely close



The $\mathbb{Z}_p \leftrightarrow \mathbb{C}$ analogy is actually extremely close

Macdonald measure
[Borodin-Corwin '11]

$$\Pr(\lambda) = \frac{P_\lambda(1, \ldots, t^{n-1}; q, t) Q_\lambda(t^{m-n+1}, \ldots, t^{D-n}; q, t)}{\Pi_{q,t}(1, \ldots, t^{n-1}; t^{m-n+1}, \ldots, t^{D-n})}$$

$q \to 0$
$t = 1/p$

Hall-Littlewood measure: singular numbers of $n \times m$ corners of Haar $GL_D(\mathbb{Z}_p)$ matrices [VP '20]

$\beta \in \{1, 2, 4\}$
$t = q^{\beta/2}$
$q \to 1$
λ rescaled

Heckman-Opdam measure: singular values of $n \times m$ corners of Haar $O(D), U(D), Sp^*(D)$ matrices [Forrester-Rains '05]

Macdonald processes used for real/complex matrix products (Ahn, Borodin, Gorin, Strahov, Sun 2015+), are also a key tool for us.