

# THE GAMMA-DISORDERED AZTEC DIAMOND

ROGER VAN PESKI (COLUMBIA UNIVERSITY)

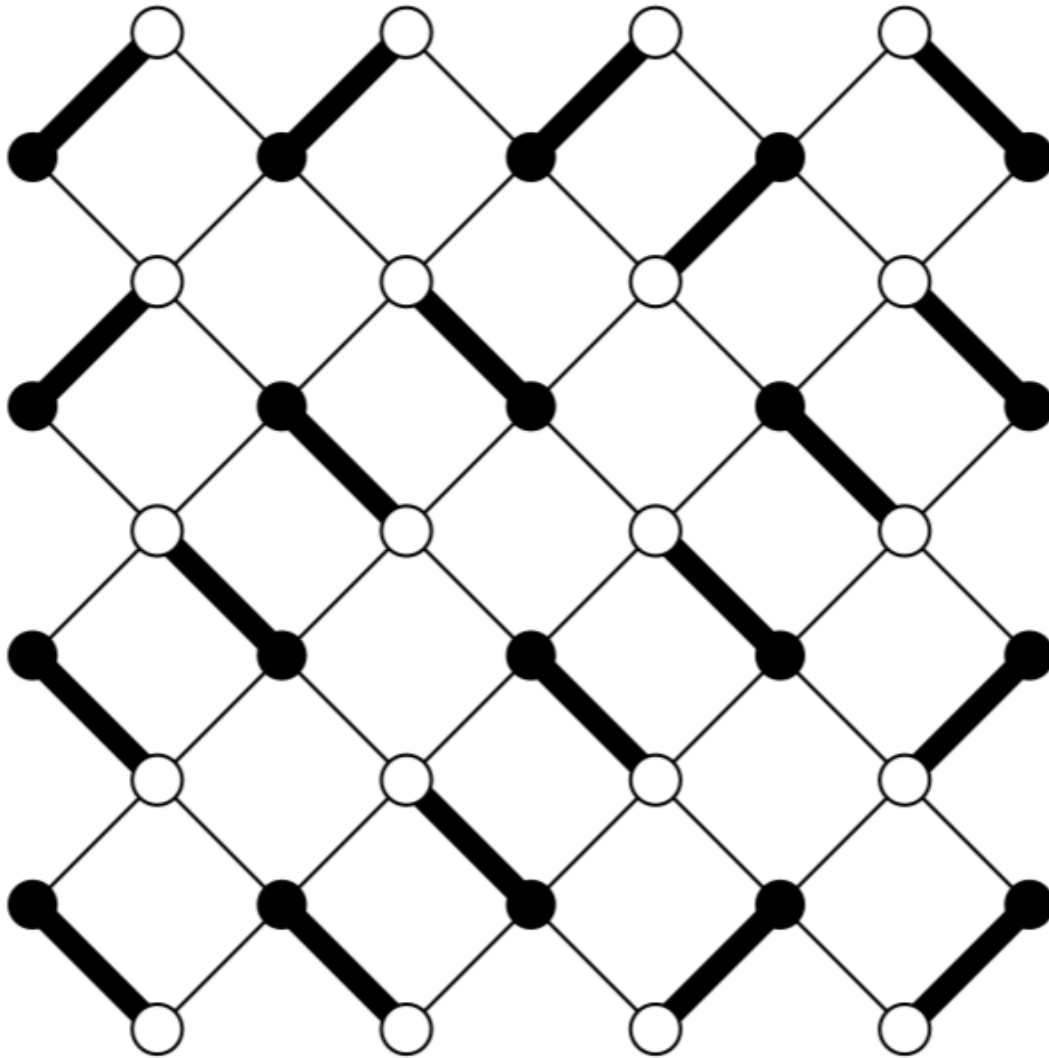
UNIVERSALITY IN THE KARDAR-PARISI-ZHANG CLASS AND  
RANDOM MATRIX THEORY

ICMS

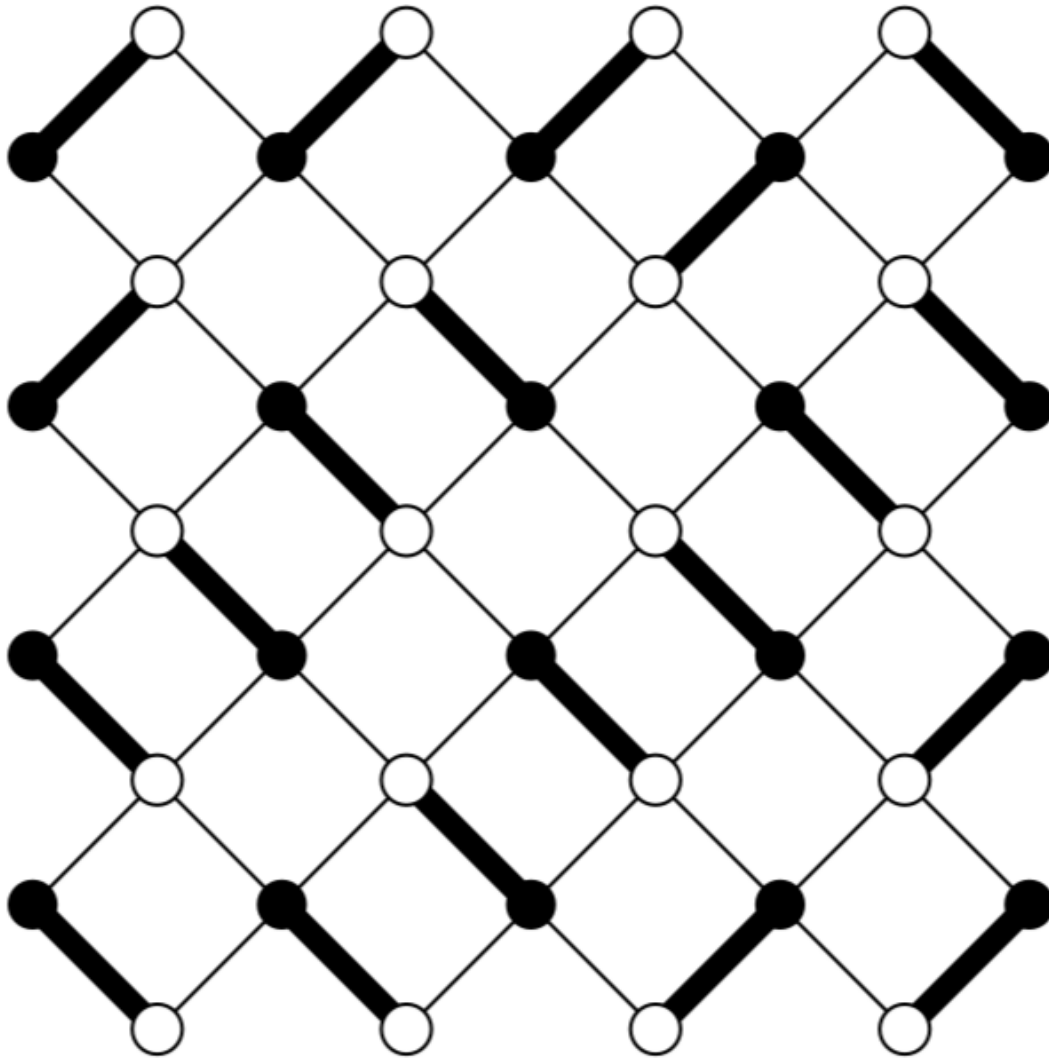
JUNE 3, 2026

*Joint work with Maurice Duits  
(KTH Royal Institute of Technology)*

Dimer model: random perfect matching of underlying bipartite graph



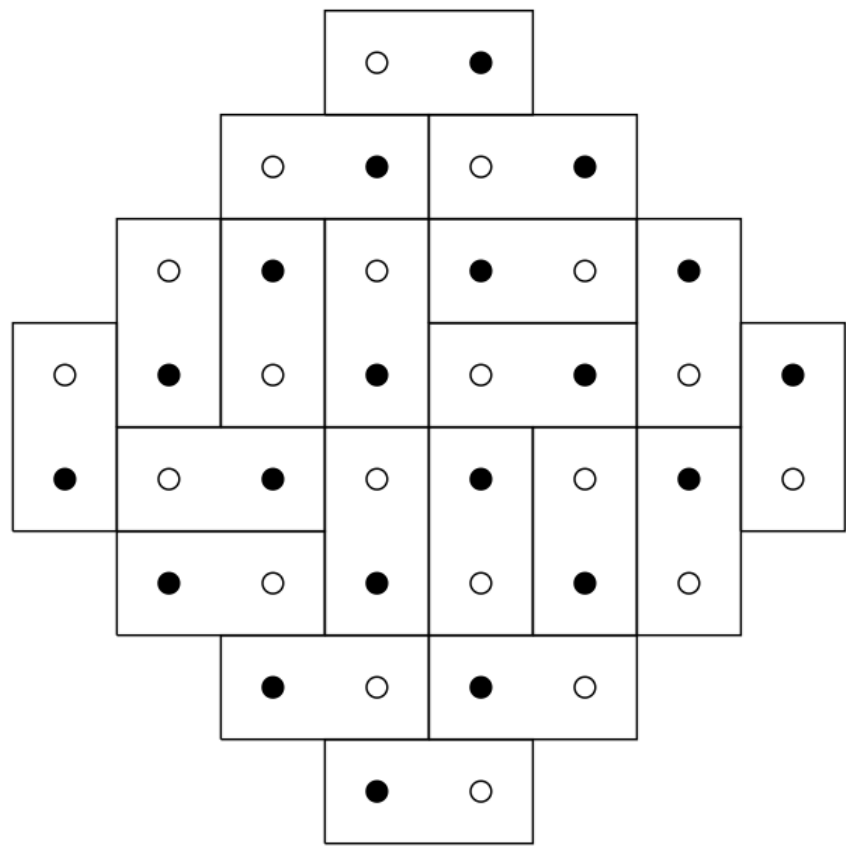
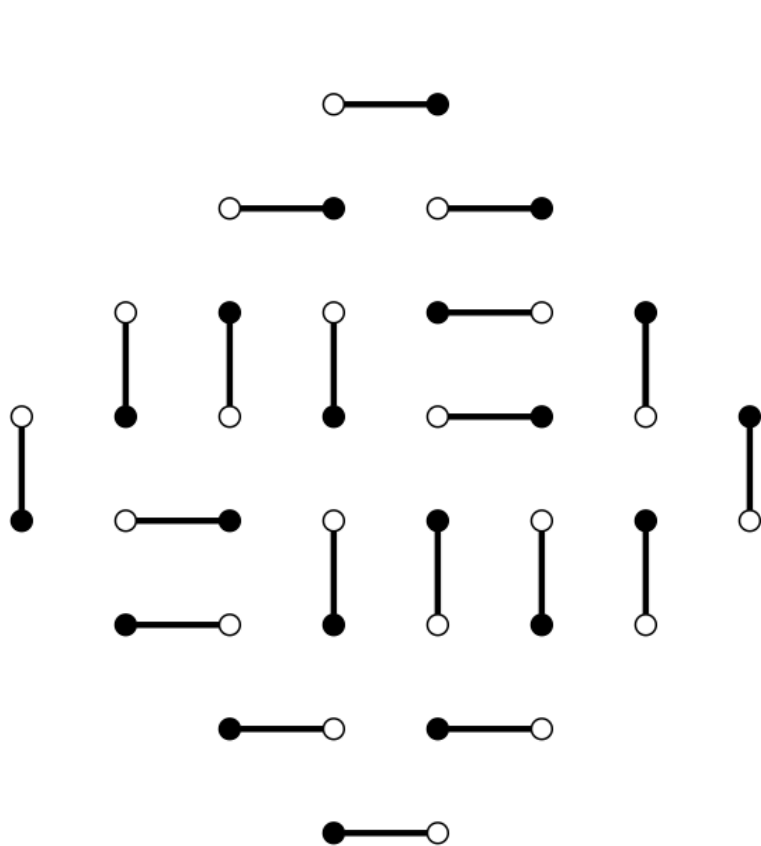
Dimer model: random perfect matching of underlying bipartite graph



If graph  $G$  is weighted, take

$$P(M) = \frac{1}{Z_G} \prod_{e \in M} \text{weight}(e)$$

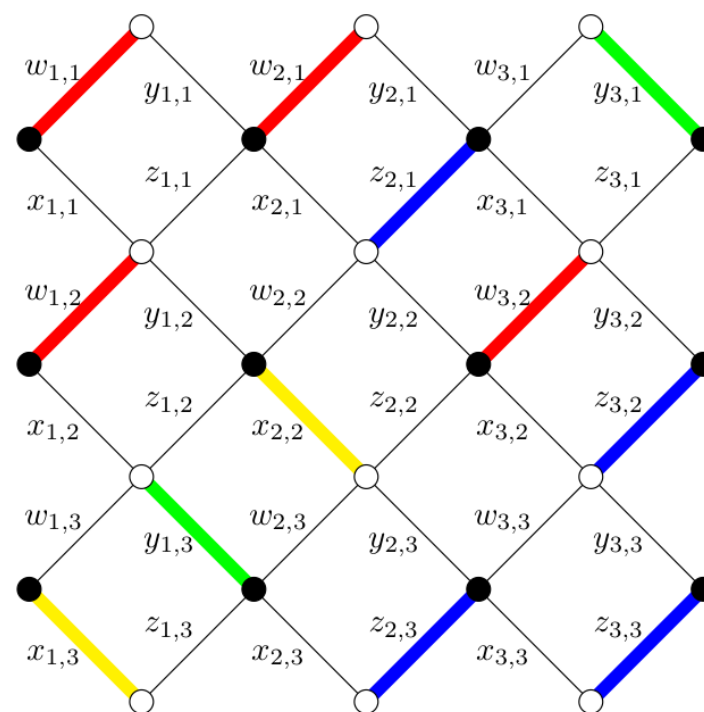
Higher-weight edges included more often in  $M$



## Shuffling algorithm [Elkies-Kuperberg-Larsen-Propp 1992], [Propp 2001]

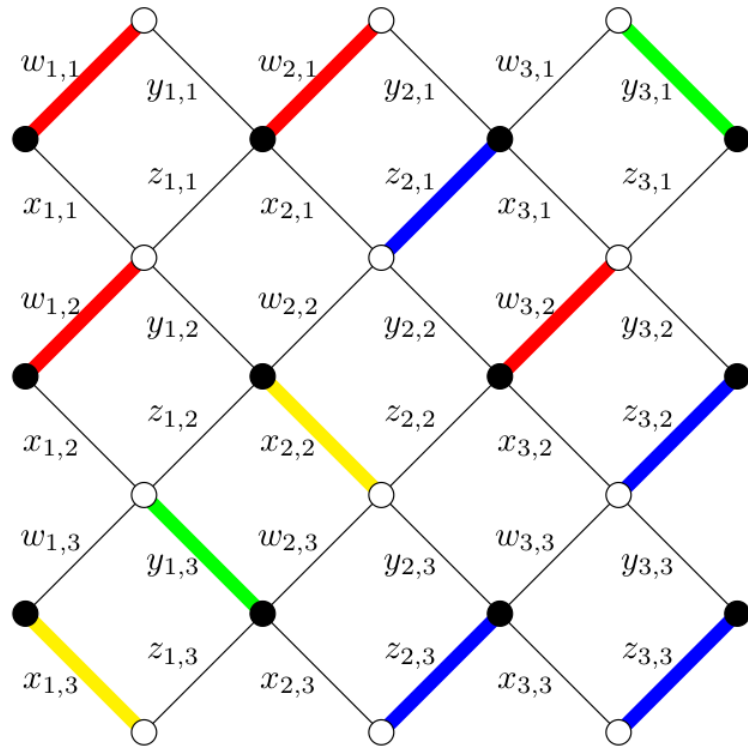
son Woolever, and Laurence Yogman. I thank Chen Zeng, who has made use of the generalized shuffling algorithm [23] in his own work on and thereby encouraged me to write up this algorithm for publication. Lastly, I thank the anonymous ref-

factors of  $\log n$ ). We have found that in practice, shuffling is extremely efficient. The physicist Zeng has applied the algorithm to the study of the dimer model in the presence of random edge-weights [23].



[HTTPS://HALCANARY.ORG/MATHAPPLETS/TOADSHUFFLE/#1](https://halcanary.org/mathappletoadshuffle/#1)

# Random edge-weights



Ex.

$$x_{ij} = e^{\frac{1}{T} x_{ij}}$$

$$w_{ij} = e^{\frac{1}{T} w_{ij}}$$

⋮

$T \rightarrow \infty$  : all weights 1  
(‘high-temperature’)

$T \rightarrow 0$  : dimer measure is a delta mass on  
best matching (‘freezing’)

# (incomplete) history

super-roughening, RG [Toner-DiVincenzo '90],...



conflicting predictions!

MCMC simulations [Cule-Shapir '95]



slow convergence!

Zero-T numerics via combinatorial optimization  
[Zeng-Middleton-Shapir '96], [Rieger-Blasum '97],  
[Middleton '99]



doesn't capture  $T > 0$ !

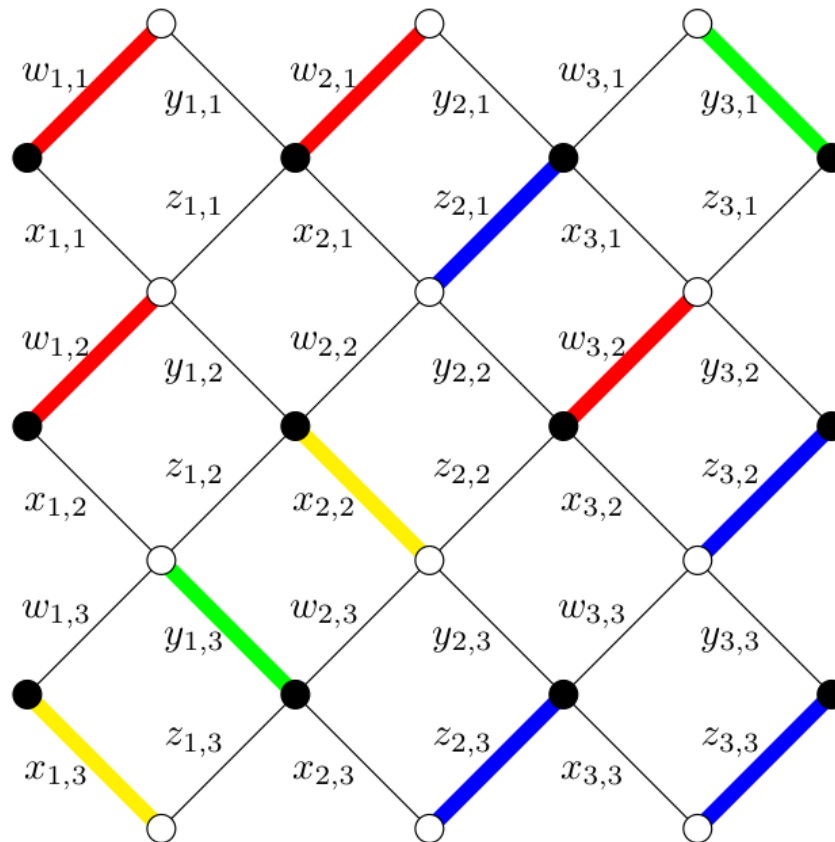
Better numerics on Aztec diamond via shuffling  
algorithm

[Zeng-Leath-Hwa '99]

[Emig-Bogner-Taha-Zeng '04]

[Perret-Ristivojevic-Le Doussal-Schehr '12]

No mathematical work on independent random weights ([Bufetov-Petrov-Zografos 2025], [Mouillard-Toninelli 2025], [Zografos 2025] nonlocal, different universality class)



Basic questions:

1. Can we find some observable with different  $n \rightarrow \infty$  asymptotics for deterministic vs. random weights?
2. If so, is there a phase transition in  $T$ ?

## Is There a Glass Transition in Planar Vortex Systems?

Thorsten Emig and Simon Bogner

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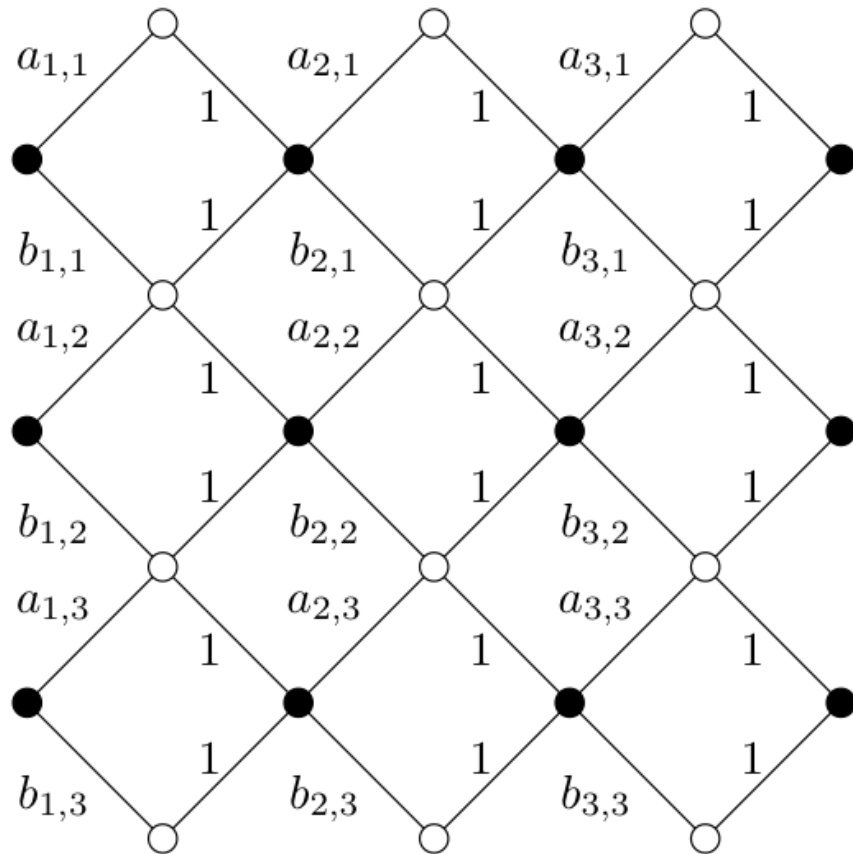
## Is There a Glass Transition in Planar Vortex Systems?

Thorsten Emig and Simon Bogner

Answers [Duits-VP 2025]:

1. Yes
2. No

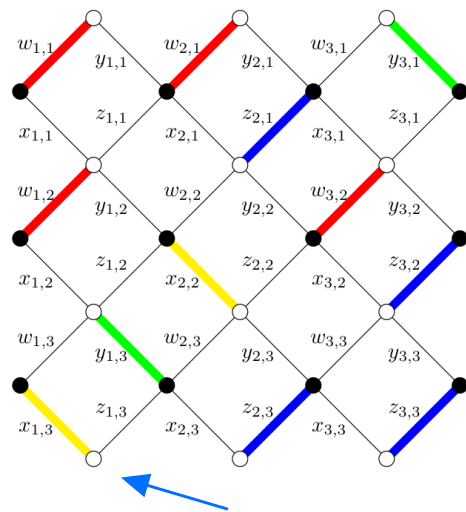
# Gamma-disordered Aztec diamond [Duits-VP 2025]



Fix  $T > 0$ .

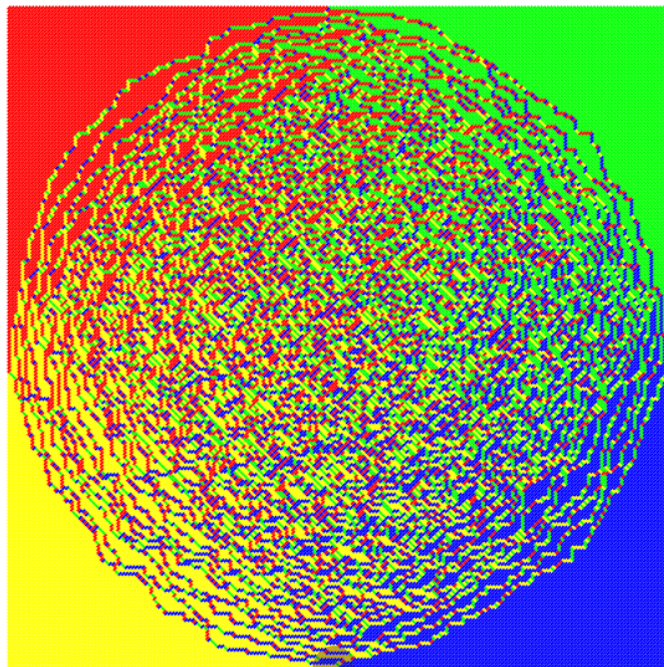
$a_{ij}, b_{ij} \sim \mathcal{P}(T, 1)$  iid

$$\text{Var}\left(\frac{a_{ij}}{\mathbb{E}[a_{ij}]}\right) = \frac{1}{T}$$



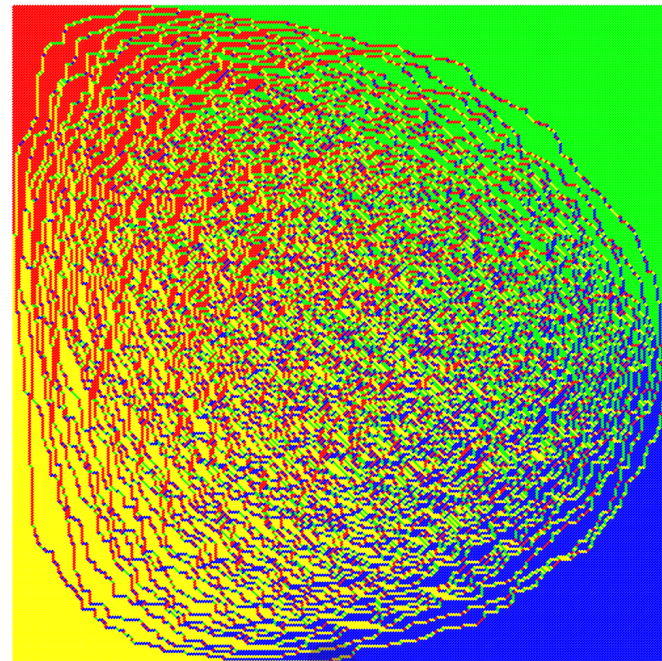
South turning point =  $\#(\text{yellow edges on bottom}) = 1$

all weights 1 [Jokusch-Propp-Shor 1992]



$n^{1/2}$

gamma-disordered [Duits-VP 2025]



$n^{2/3}$

### THEOREM [Duits-VP 2025]

Let  $X_n$  be the South turning point of the Gamma-disordered Aztec diamond with temperature  $T$  and size  $n$ . Then

$$\frac{X_n}{n} \rightarrow \frac{\Psi_1(2T)}{\Psi_1(T)} \quad \text{in probability as } n \rightarrow \infty.$$

Furthermore, there exists  $C > 0$  such that for all  $b$  large enough and all  $n$ ,

$$\mathbb{P}\left(\left|X_n - \frac{\Psi_1(2T)}{\Psi_1(T)} \cdot n\right| > b n^{2/3}\right) < \frac{C}{b^3} \quad (\text{tail bound})$$

and there exist  $C_0, C_1 > 0$  such that for all  $n$  large enough,

$$\mathbb{P}\left(\left|X_n - \frac{\Psi_1(2T)}{\Psi_1(T)} \cdot n\right| > C_1 n^{2/3}\right) > C_0 \quad (\text{anticoncentration})$$

trigamma function

$$\Psi_1(z) = \frac{d^2}{dz^2} \log \zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$$

# Free energy

Free energy of size  $n$   
Gamma-disordered  
Aztec

$$F_n = T \log Z_n \quad (\text{random})$$

**THEOREM** [Duits-VP 2025]

$$\mathbb{P}\left(\frac{F_n - \mathbb{E}[F_n]}{\sqrt{2n(n+1)\Psi_1(2T)T^2}} < s\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-x^2/2} dx + O\left(\frac{1}{n}\right)$$

with uniform error term over  $T \in (0, \infty)$ .

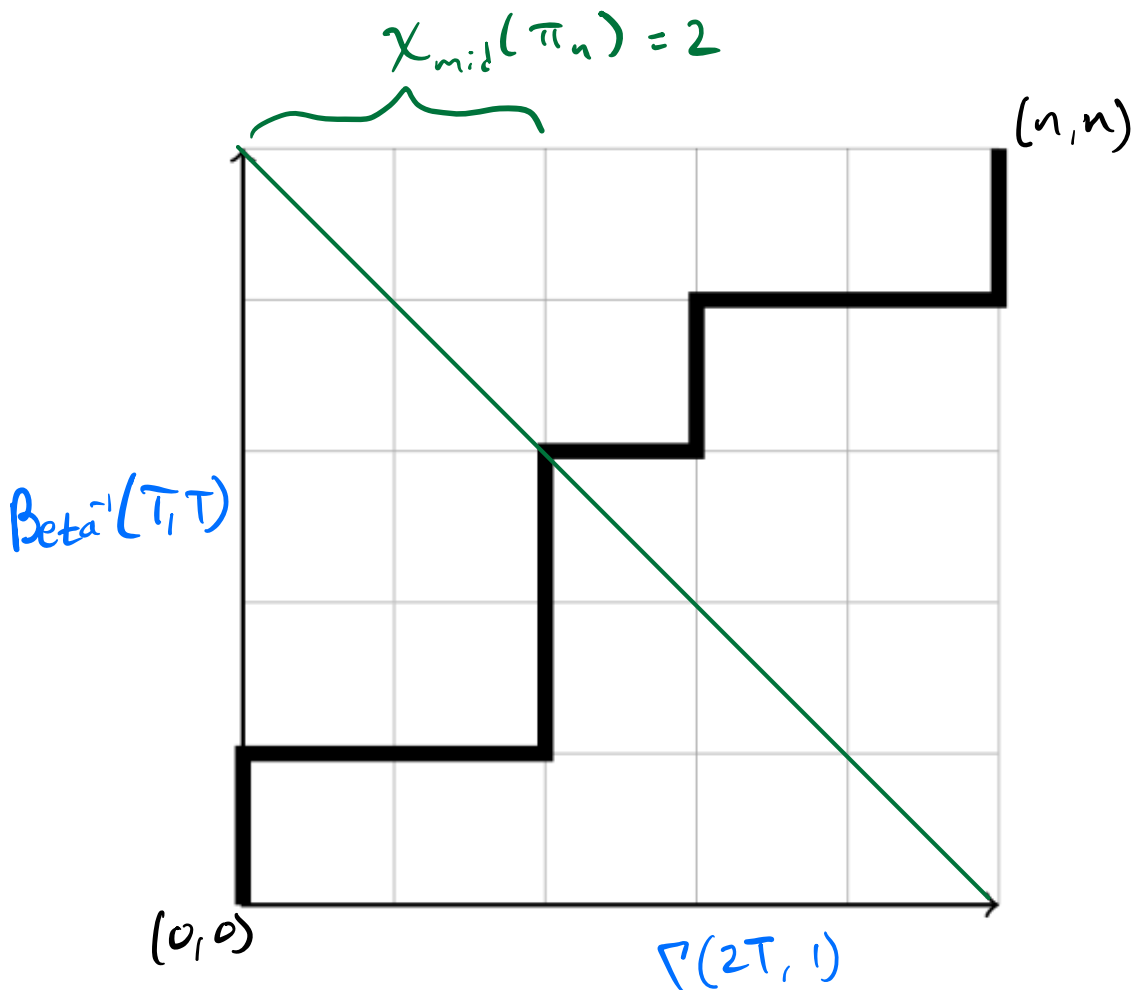
This is a consequence of....

$$Z_n = \prod_{i=1}^{\binom{n+1}{2}} X_i$$

$$X_i \sim \Gamma(2T, 1) \quad \underline{\text{iid}}$$

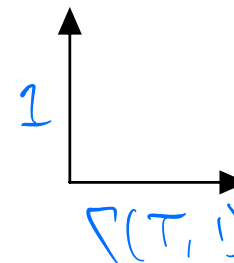
**THEOREM [Duits-VP 2025]**

The South turning point  $\chi_n$  is equal in distribution to  $\chi_{mid}(\pi_n)$  for a stationary strict-weak polymer path  $\pi_n$ .



Stationary strict-weak polymer  
[Corwin-Seppäläinen-Shen 2014]

iid random weights

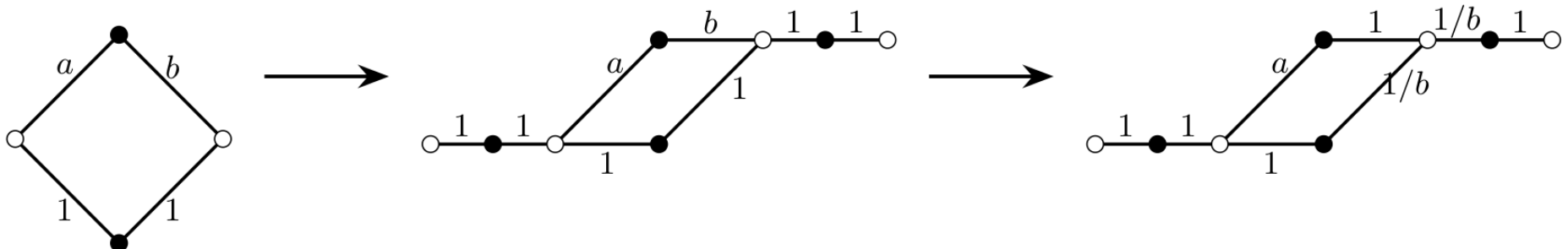
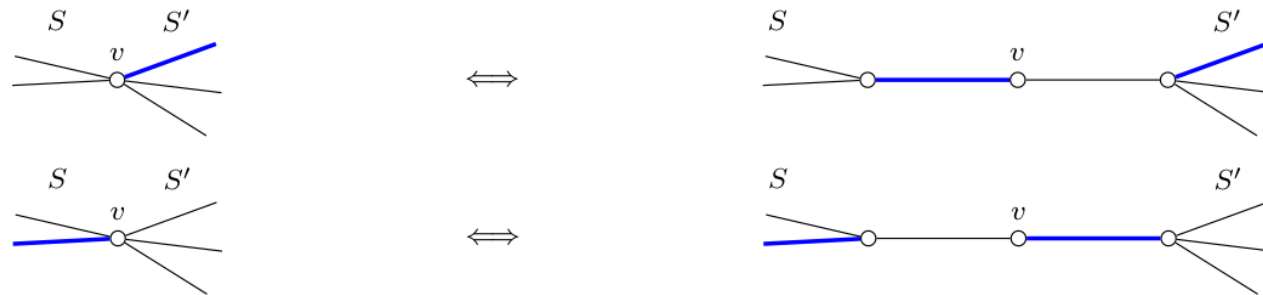
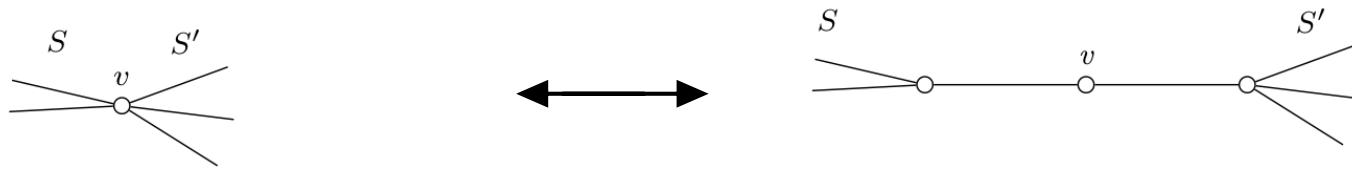


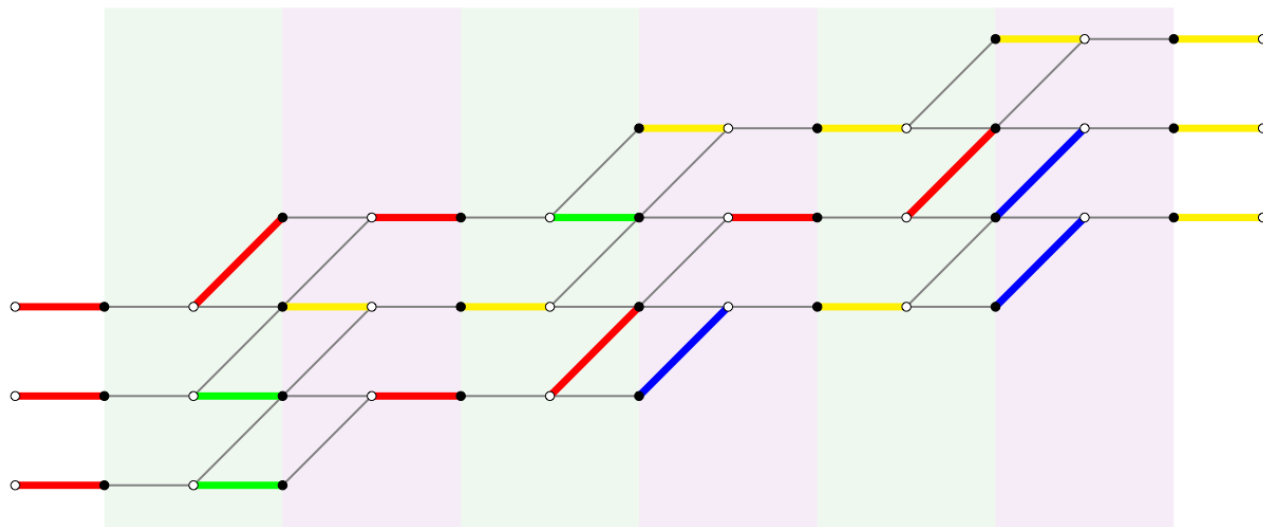
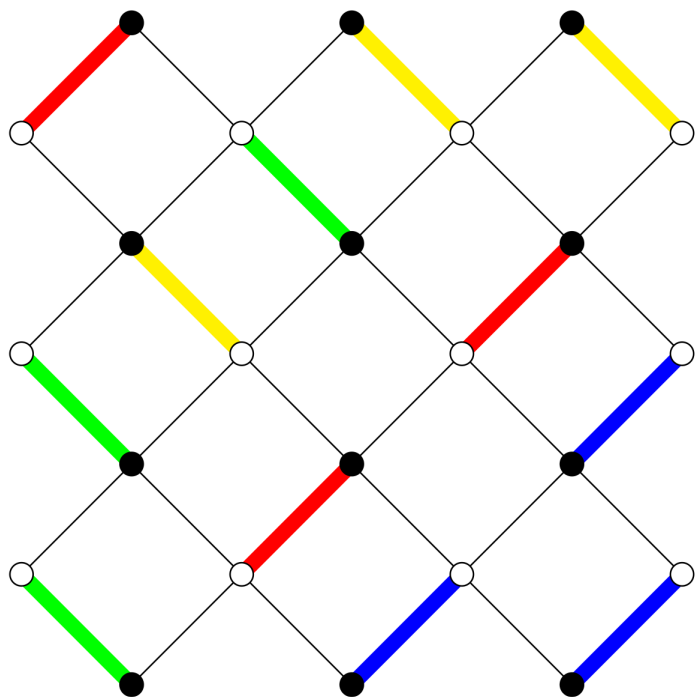
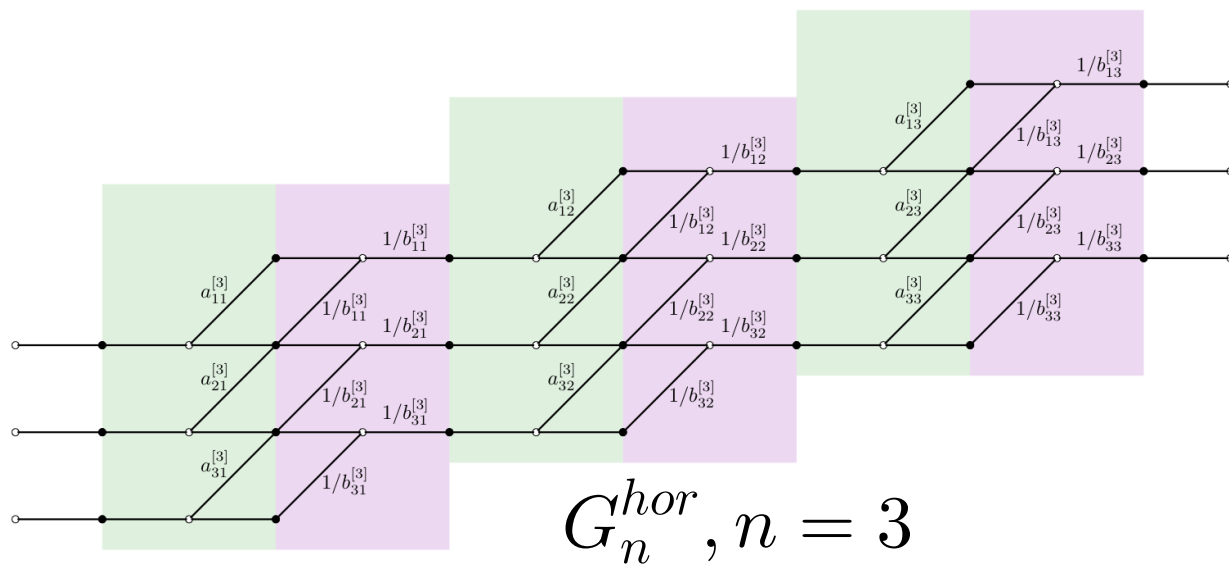
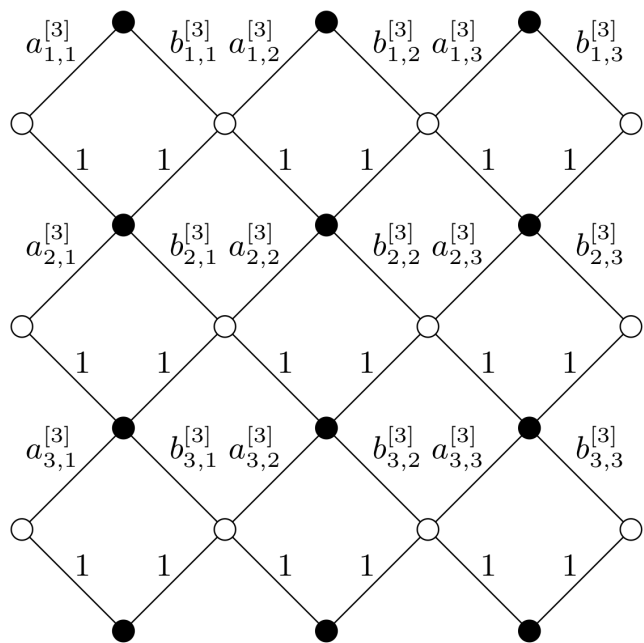
Random probability measure on paths

$$P(\pi_n) = \frac{1}{Z_n} \prod_{e \in \pi_n} w_t(e)$$

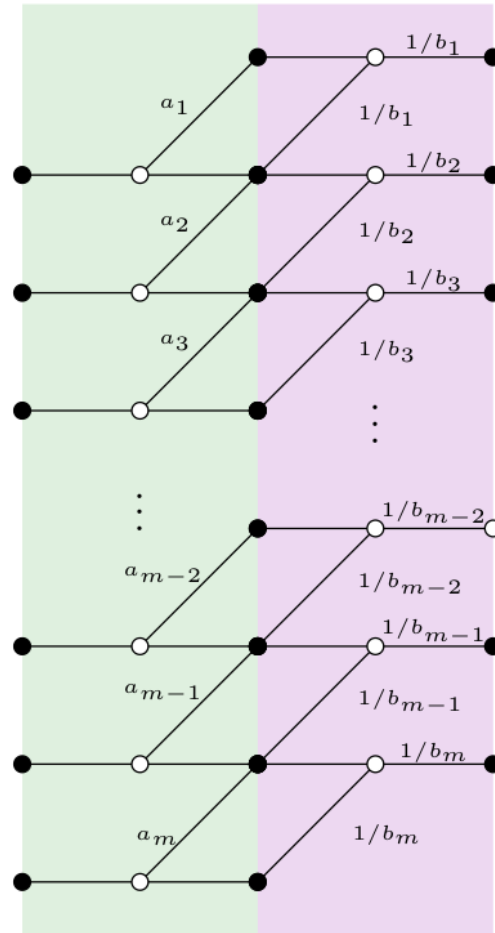
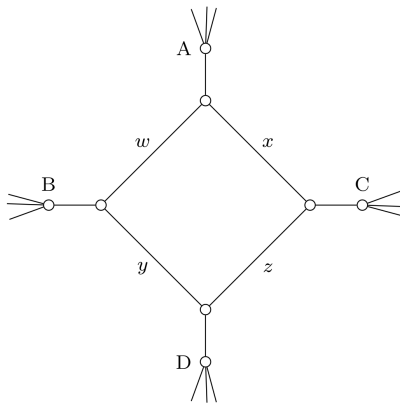
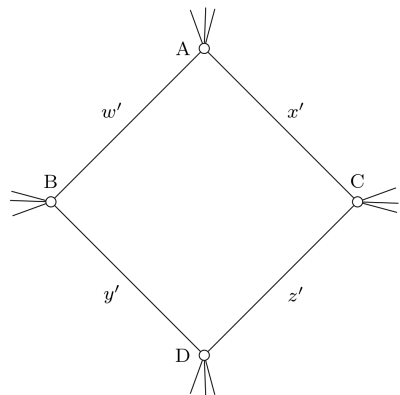
# PROOF IDEA

vertex-dilation gives weight-preserving bijection of dimer covers





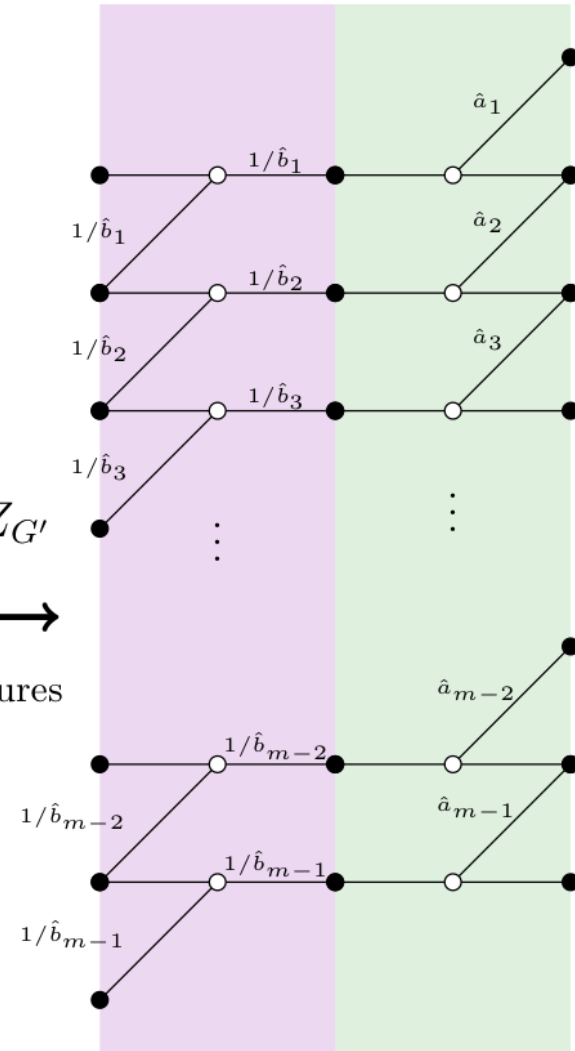
# KEY LOCAL MOVE



$G$

$$Z_G = (1 + a_1/b_1)Z_{G'}$$

coupling of dimer measures



$G'$

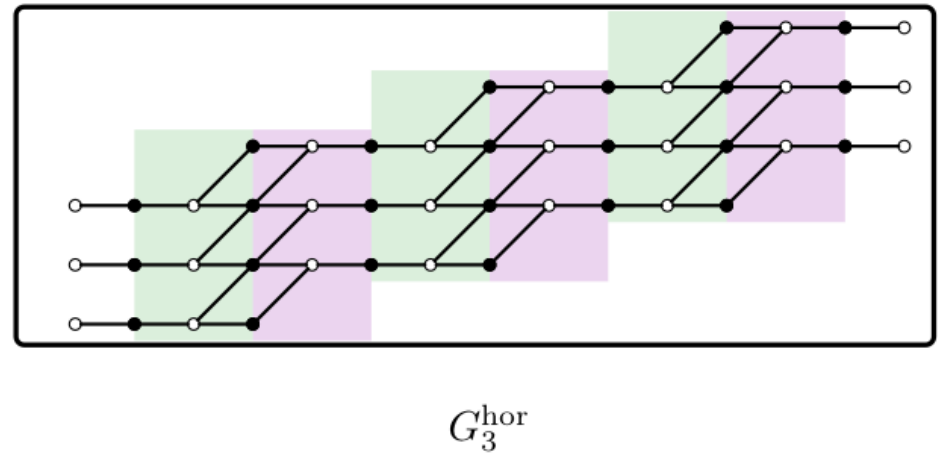
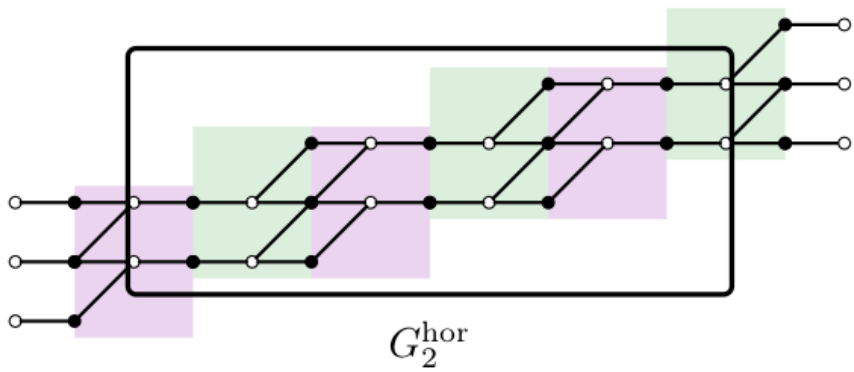
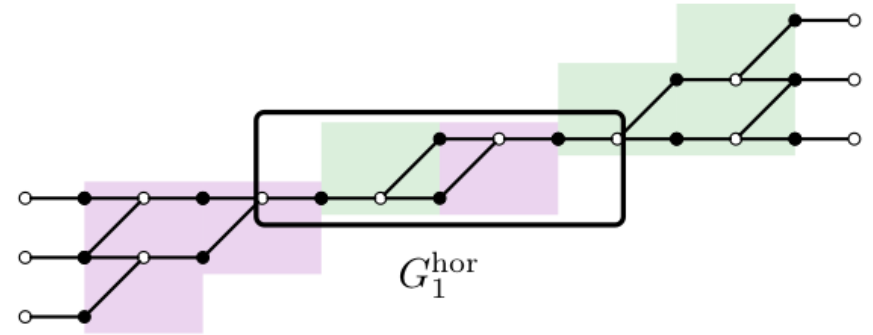
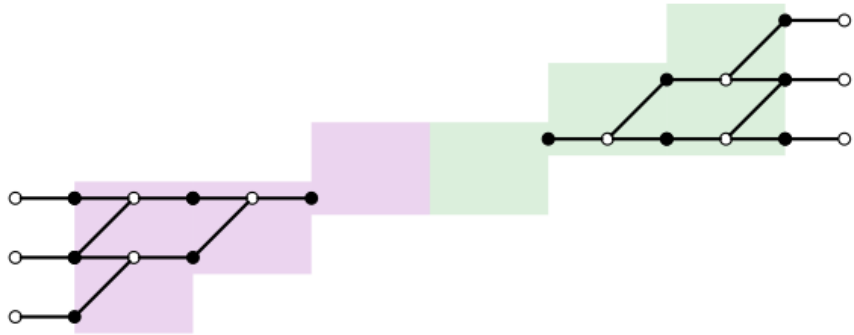
$$\hat{a}_i = \frac{a_i}{a_i + b_i}(a_{i+1} + b_{i+1})$$

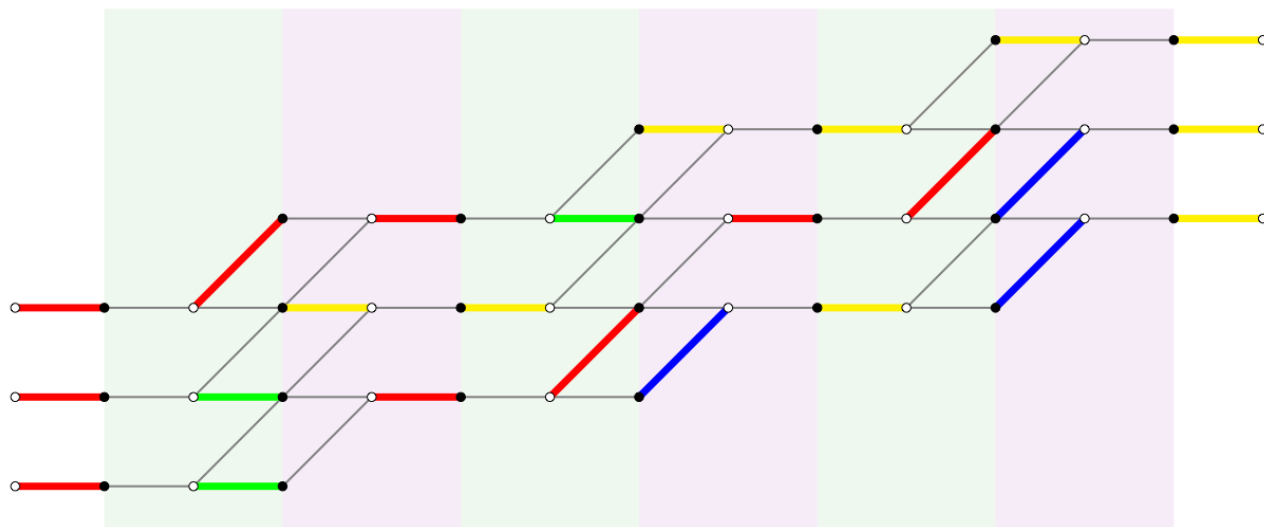
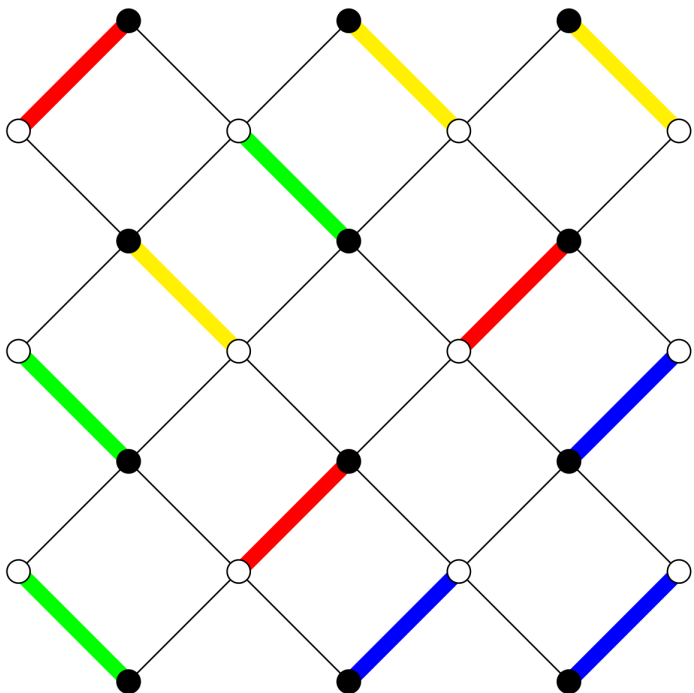
$$\hat{b}_i = \frac{b_i}{a_i + b_i}(a_{i+1} + b_{i+1})$$

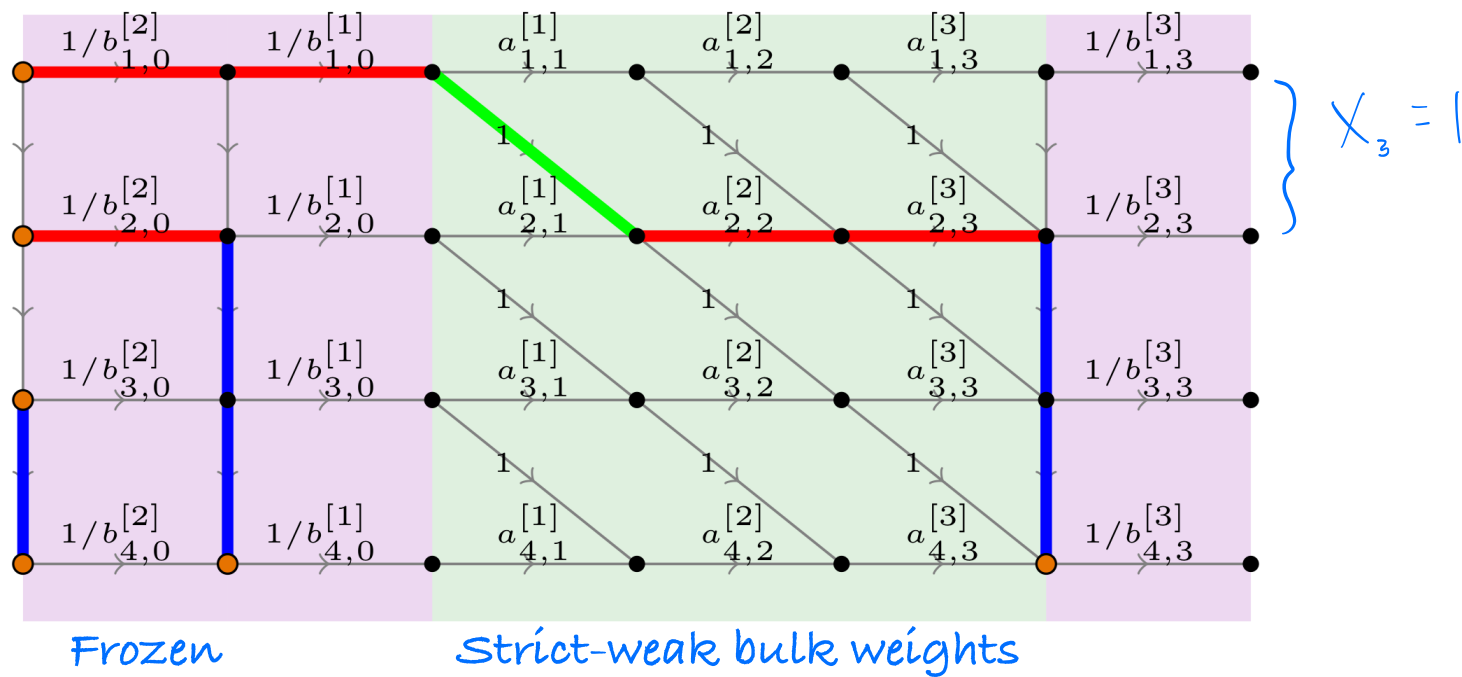
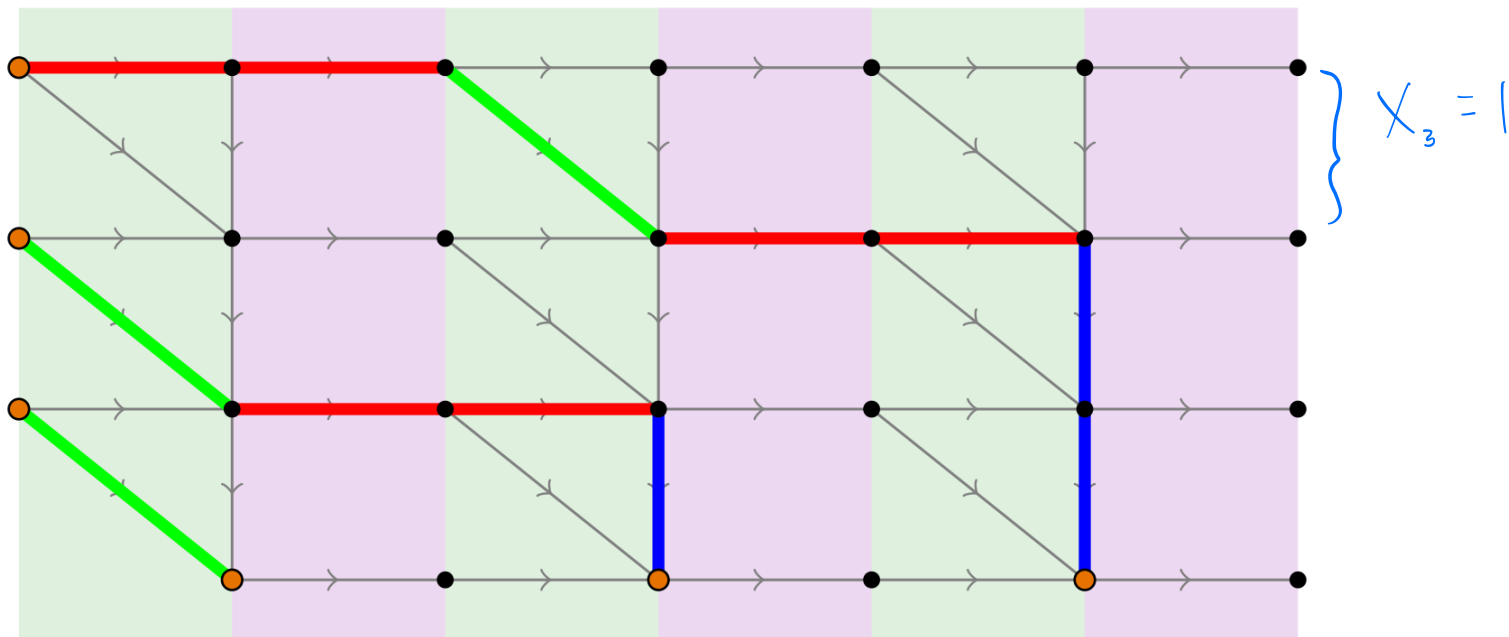
Independent iff gamma!

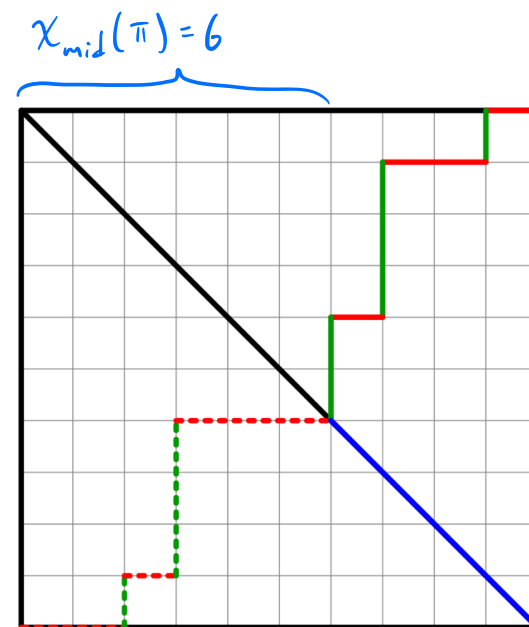
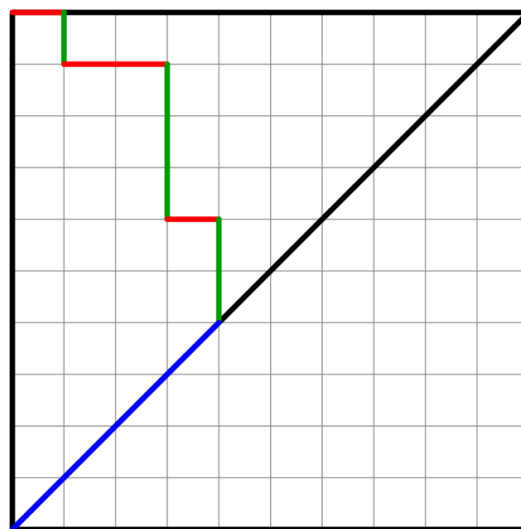
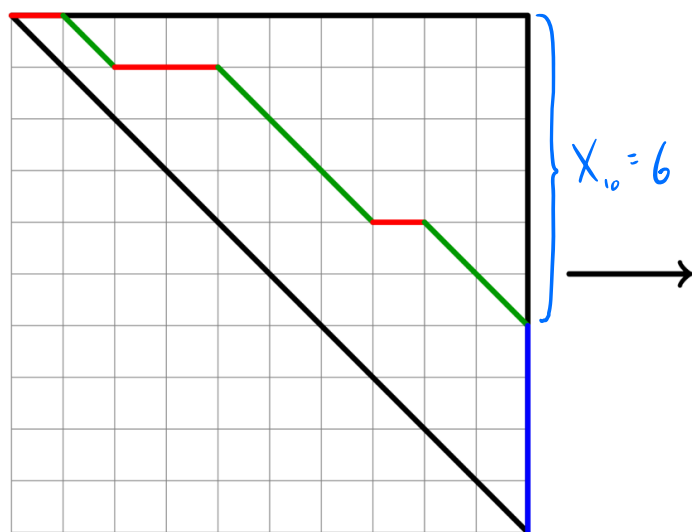
Kuperberg,  
[Propp 2001]

# Shuffling via column-swaps







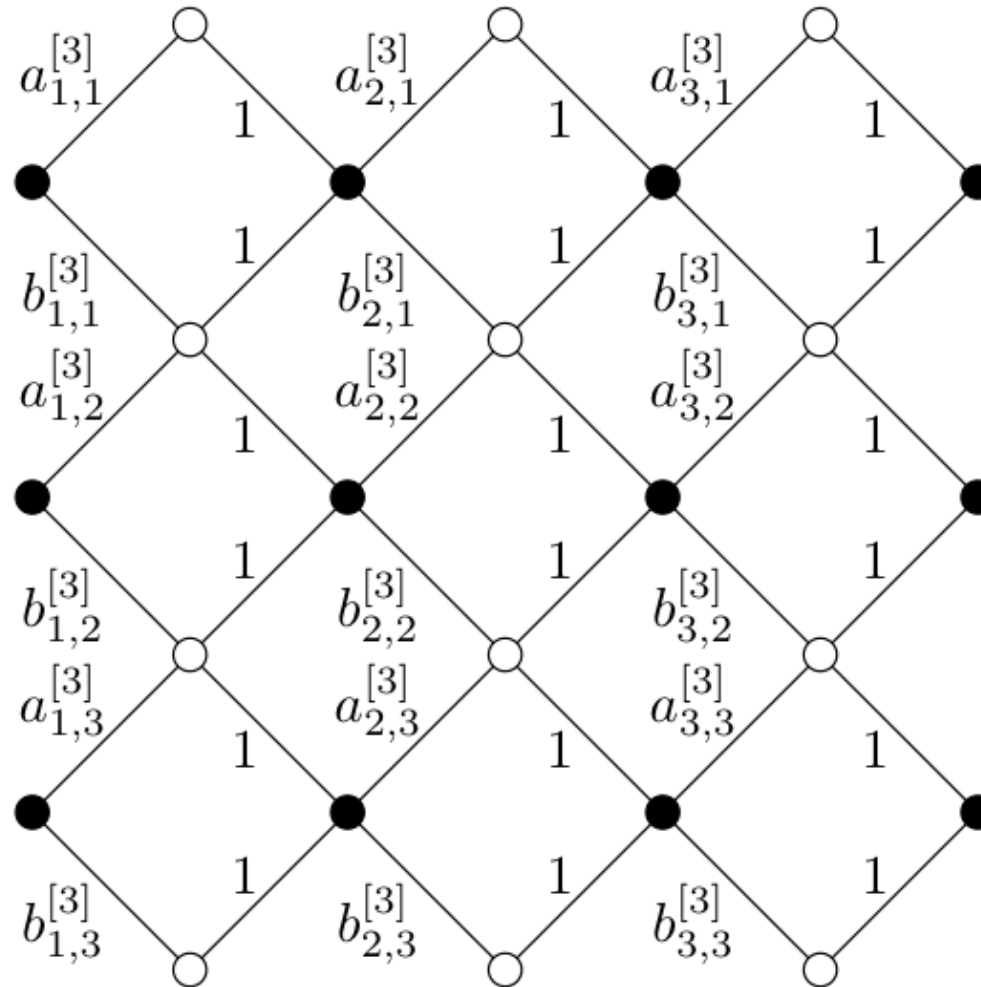


**N**

Stationary strict-weak polymer  
 [Corwin-Seppäläinen-Shen 2014]

**W**

Stationary log-  
gamma polymer  
 [Seppäläinen 2009]

**E**

Deformed Beta  
random walk in  
random  
environment  
 [Barraquand-  
Corwin 2015],  
 [Korotkiĭkh 2021]

**S**

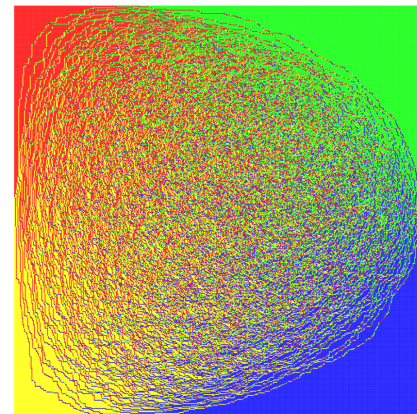
Stationary strict-weak polymer  
 [Corwin-Seppäläinen-Shen 2014]

Dimers with random weights



Natural dimer property  
(independence after shuffling)

Gamma-disordered Aztec diamond



Dimers with random weights

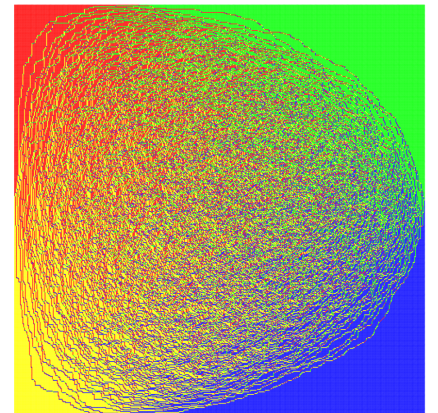
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Gamma-disordered Aztec diamond

Known exactly-solvable models in Kardar-Parisi-Zhang (KPZ) universality class (log-gamma, ...)

with

many known probabilistic results



New questions:

- Distribution of turning points?
- QUE corners analogue at edge?
- Height function? (in progress)

Dimers with random weights

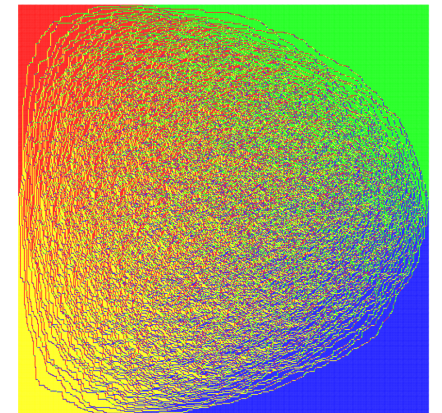
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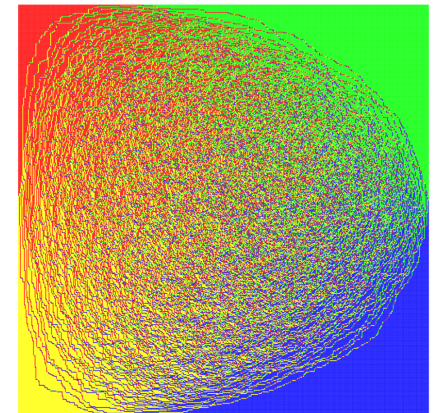
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**THANKS!**

