p-adic random matrix products and particle systems

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Based on https://arxiv.org/abs/2011.09356,
https://arxiv.org/abs/2112.03725, and work in progress

Quick crash course in p-adic numbers

Fix a prime p. $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, concretely

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$$x = u \cdot p^n, u \in \mathbb{Z}_p^\times, n \geq 0.$$

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$(\mathbb{Z}_p, +)$ is compact, so has Haar probability measure $\mu_{Haar}^{\mathbb{Z}_p}$, explicitly given by taking $a_i \in \{0, \ldots, p-1\}$ iid uniformly random.

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$\mathrm{GL}_n(\mathbb{Z}_p)$ also compact, hence has Haar probability measure.

Two worlds

	RMT over $\mathbb{C}$	RMT over $\mathbb{Z}_p$
Matrices	$\mathrm{Mat}_n(\mathbb{C})$	$\mathrm{Mat}_n(\mathbb{Z}_p)$
Compact group	$U(n)$	$\mathrm{GL}_n(\mathbb{Z}_p)$
Structure theorem	SVD: $UAV = \mathrm{diag}(e^{-r_1}, \ldots, e^{-r_n})$ for $U, V \in U(n)$	Smith normal form: $UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})$ for $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$
We study	Singular values e^{-r_i}	Singular numbers λ_i

Some natural ensembles: $n \times m$ corner of Haar $U(N)$ or $\mathrm{GL}_N(\mathbb{Z}_p)$ ('Jacobi'), or iid Gaussian/$\mu_{Haar}^{\mathbb{Z}_p}$ ('Ginibre'). Both are invariant under left- and right- multiplication by $U(N)$ or $\mathrm{GL}_N(\mathbb{Z}_p)$.

Denote $\mathrm{SN}(A) := (\lambda_1, \ldots, \lambda_n) \in \mathbb{Y}_n$, an integer partition $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$.

Motivation: random abelian p-groups

If $A_n \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$, get abelian p-group

$$\mathrm{coker}(A_n) := \mathbb{Z}_p^n/\mathsf{Im}(A_n) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$$

with $\lambda = \mathrm{SN}(A_n)$.

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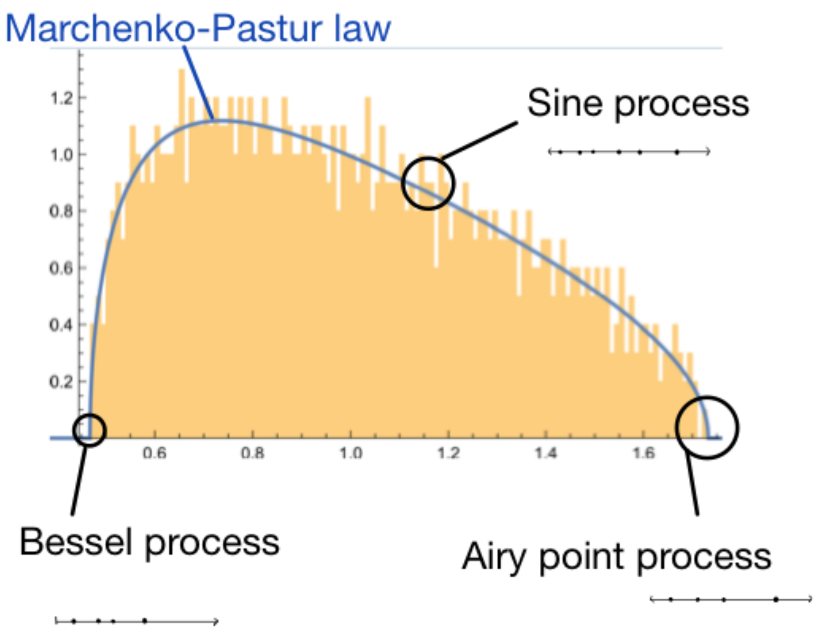
For A_n with iid additive Haar entries ('Ginibre'), [Friedman-Washington '87] showed

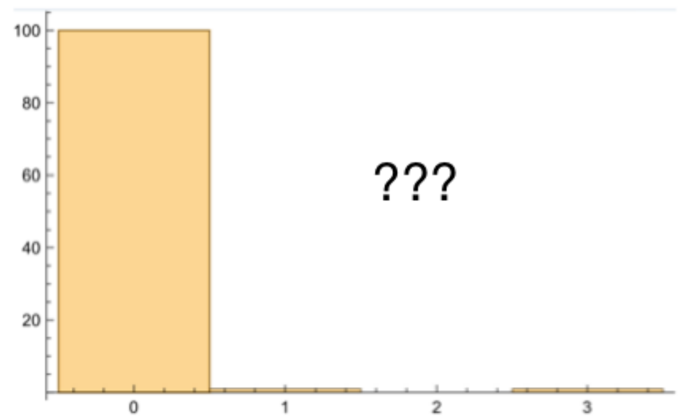
$$\lim_{n\to\infty} \Pr\left(\mathrm{coker}(A_n) \cong \bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}\right) = \frac{\text{const}}{|\mathrm{Aut}\left(\bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}\right)|}$$

matching numerically observed distribution of p-torsion part of class groups of quadratic imaginary number fields. Newer results: **[Bhargava et al. '15], many works by Wood '10-'20, [Clancy-Kaplan-Leake-Payne-Wood '15], [Fulman '16], [Nguyen-Wood '18],...**

Ginibre singular values vs. iid $\mu_{Haar}^{\mathbb{Z}_p}$ singular numbers



Histogram of singular values of a single $10^3 \times 10^4$ Ginibre (iid Gaussian) matrix



Histogram of singular numbers of a single 100×100 iid additive Haar matrix

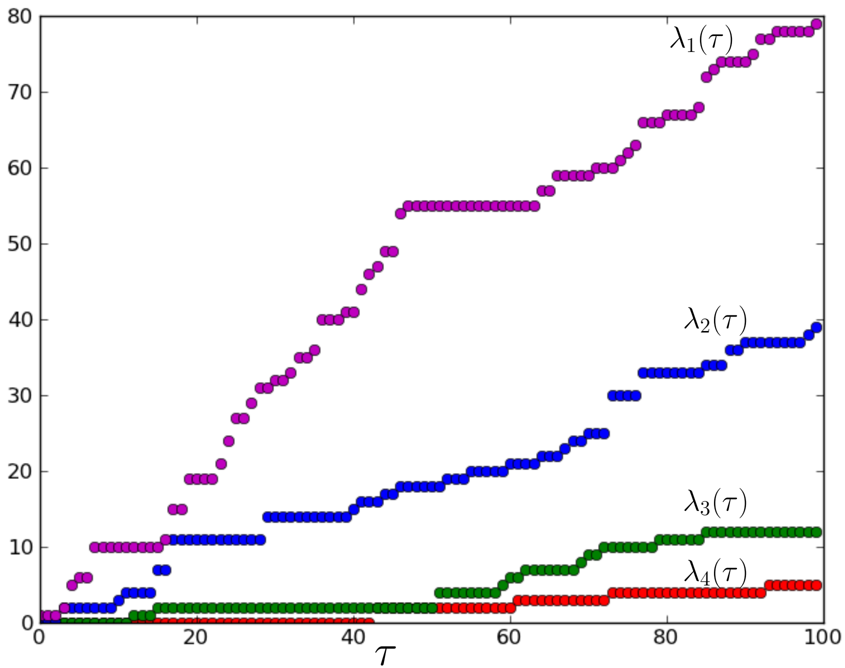
The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i random real/complex matrices, $\tau = 1, 2, \ldots$.

- (Furstenberg-Kesten 1960) Gaussian fluctuations for $\log($largest singular value of $A_\tau \cdots A_1)$ via ergodic theory.
- Works by Akemann, Burda, Forrester, Ipsen, Kieburg, Liu, Wang, Wei, and others, 2010s onward. Connections to statistical physics, dynamical systems, other areas...
- Works by Ahn, Gorin, Strahov, Sun analyze via degenerations of Macdonald processes, also 2010s onward.

Simulating p-adic matrix products

Set $\lambda(\tau) = (\lambda_1(\tau), \lambda_2(\tau), \ldots, \lambda_n(\tau)) := \mathrm{SN}(A_\tau \cdots A_1)$.



Example: $\lambda(\tau)$ when $A_i \in M_4(\mathbb{Z}_2)$ are iid with iid additive Haar entries.

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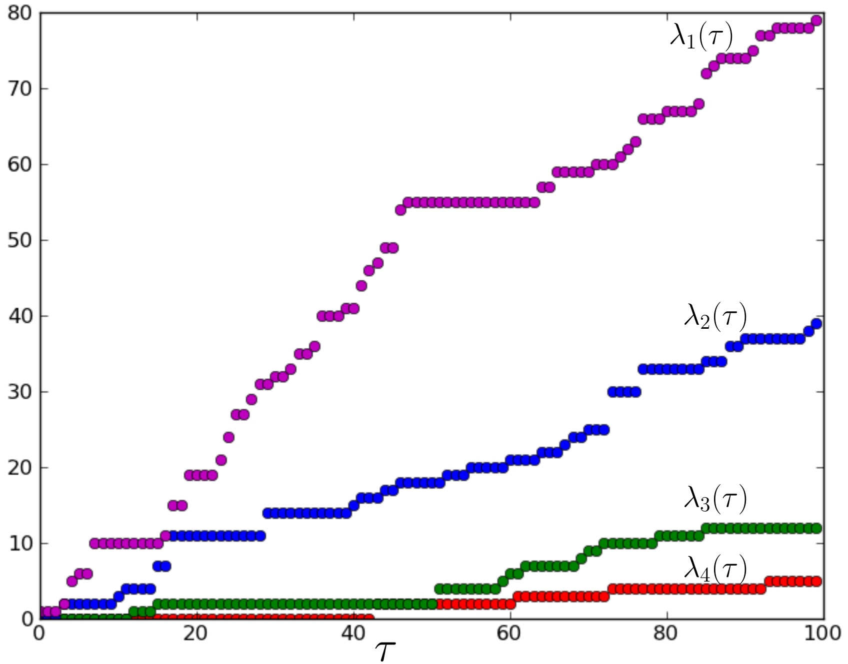


Example: $\lambda(\tau)$ when $A_i \in M_4(\mathbb{Z}_2)$ are iid with iid additive Haar entries.

View $(\lambda_1(\tau), \lambda_2(\tau), \ldots, \lambda_n(\tau))$ as n particles on $\mathbb{Z}$ evolving in discrete time.

LLN and dynamical CLT

Theorem (VP 2020)

Let $A_1, A_2, \ldots \in M_n(\mathbb{Z}_p)$ *have iid additive Haar entries, and recall* $\lambda(\tau) := \mathrm{SN}(A_\tau \cdots A_1)$. *Then as* $\tau \to \infty$,

$$\frac{\lambda_i(\tau)}{\tau} \xrightarrow{a.s.} \frac{1}{p^i - 1}. \qquad \text{(LLN)}$$

Furthermore

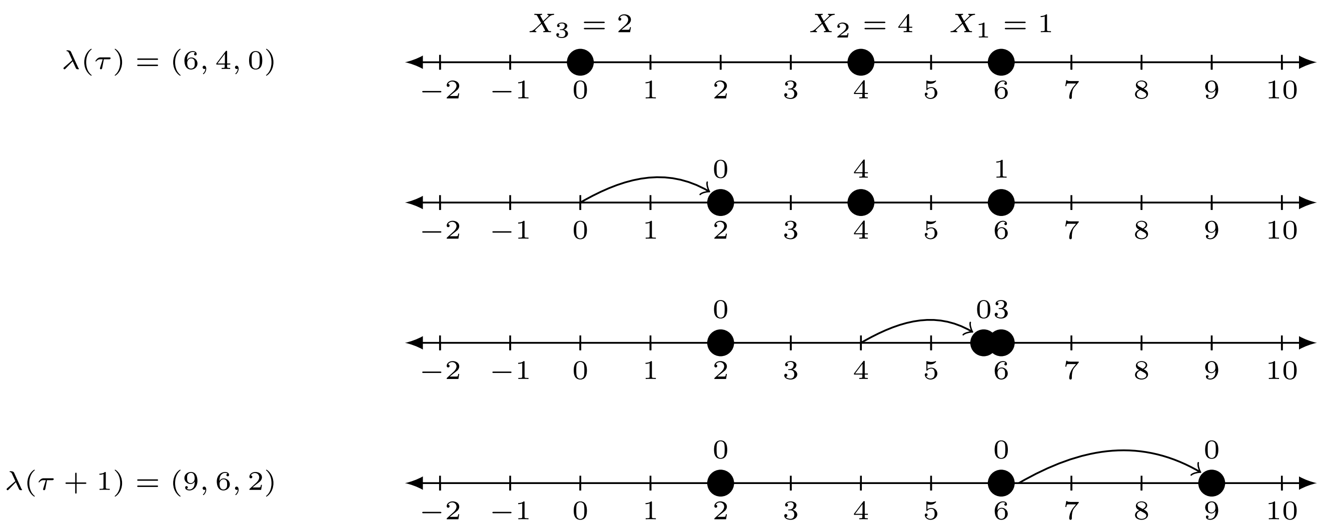
$$\frac{\lambda_i(\lfloor ks \rfloor) - \frac{ks}{p^i - 1}}{\sqrt{k} C_i} \xrightarrow{k \to \infty} B_s^{(i)} \qquad \text{(CLT)}$$

for $B_s^{(1)}, \ldots, B_s^{(n)}$ *independent Brownian motions,* C_i *explicit constants.*

Also holds when A_i are $n \times n$ corners of Haar-distributed elements of $\mathrm{GL}_D(\mathbb{Z}_p)$, not necessarily identical (universal!).

Proof idea

Proof uses new explicit random-walk descriptions of $\lambda(\tau)$. Each $\lambda_i(\tau)$ *independently* samples a ‘desired jump’ $X_i \geq 0$ and donates if necessary to preserve interlacing.



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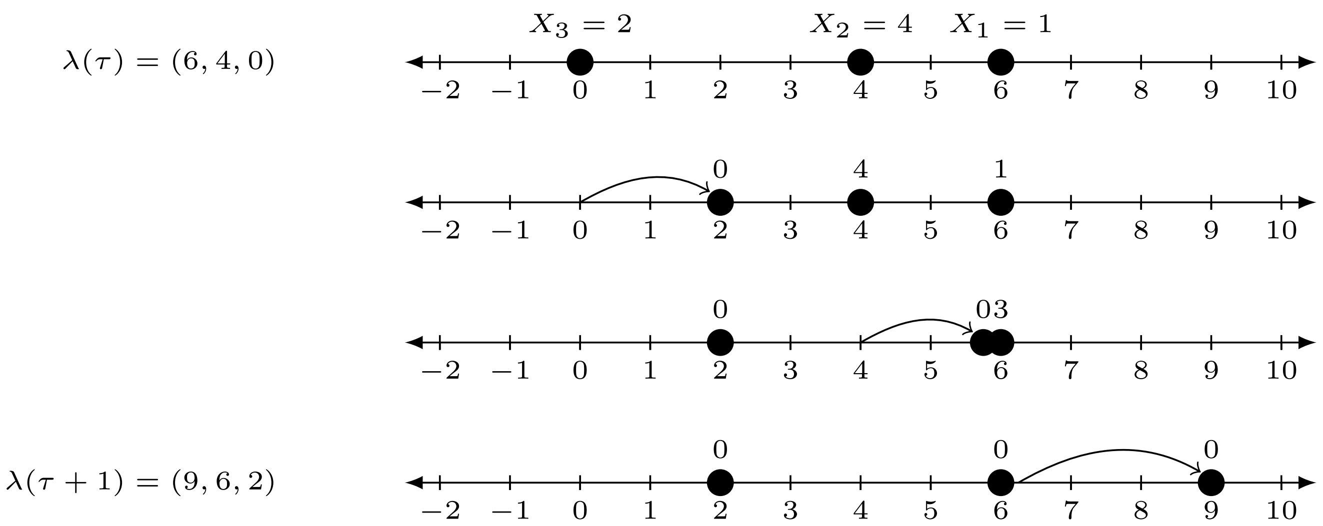

Particles separate since $\mathbb{E}X_1 > \mathbb{E}X_2 > \ldots$, and Donsker applies.

Local limits

Question

For $r \in (0,1)$, does joint evolution of bulk singular numbers $\lambda_{\lfloor rN \rfloor + i}(\tau), i \in \{\dots, -1, 0, 1, \dots\}$ converge as matrix size $N \to \infty$ to some Markov process on

$$\mathbb{Y}_{2\infty} := \{\mu = (\dots, \mu_{-1}, \mu_0, \mu_1, \dots) \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}} : \mu_{i+1} \leq \mu_i\}?$$

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Definition

The bi-infinite reflected Poisson walk (BRPW) $\mu(T) = (\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots), T \geq 0$ *is the continuous-time stochastic process with* $\mu_i(T)$ *jumping by rate-*p^{-i} *Poisson clock, donating move if blocked.*

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In previous examples, $X_i \approx \text{Ber}(p^{-i})$ for large i, so Poisson limit theorem implies BRPW bulk limit. How universal?

Universal bulk limit

Theorem (VP 2022+, to appear)

Let $r \in (0,1)$, *and for each* $N \geq 1$, *let*

- $A_i^{(N)}, i \geq 1$ *iid* $N \times N$ *matrices with distribution invariant under left- and right-* $\mathrm{GL}_N(\mathbb{Z}_p)$ *multiplication,*
- $\lambda^{(N)}(\tau) = \mathrm{SN}(A_\tau^{(N)} \cdots A_1^{(N)})$.

Then the joint evolution of $L_i(T) := \lambda_{\lfloor rN \rfloor + i}(\lfloor c^{-1} T \rfloor), i \in \mathbb{Z}$ *converges to the BRPW* $\mu(T)$ *for explicit* $c = c(r, \textit{Law}(\mathrm{SN}(A_i^{(N)})))$, *provided that the random partitions* $\mathrm{SN}(A_N^{(i)}), N \geq 1$ *satisfy*

1. $\mathrm{SN}(A_i^{(N)})$ *is not identically* $(0, \ldots, 0)$, *and*
2. $rN - \#\{\textit{nonzero SNs of } A_i^{(N)}\} \xrightarrow{N \to \infty} \infty$ *in probability.*

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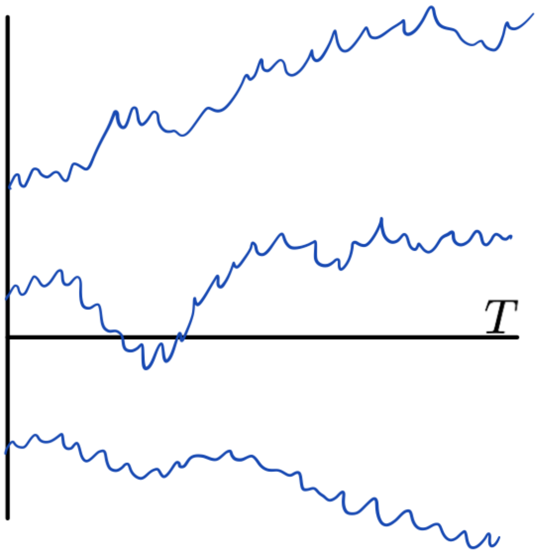
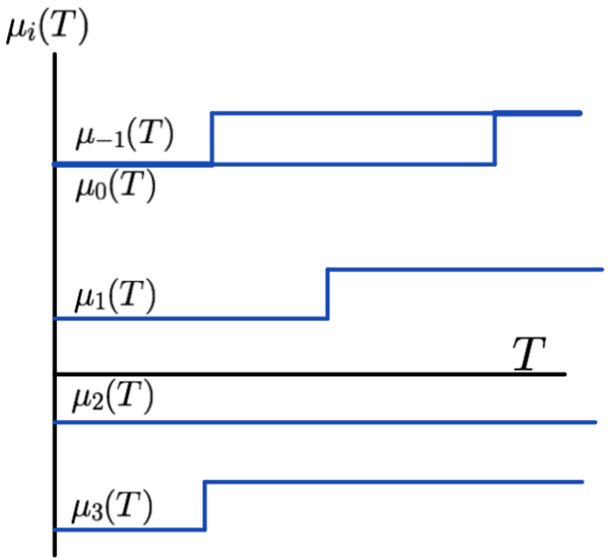
Taking $r = 1$, same hypotheses, get **edge limit** to an *infinite reflected Poisson walk* $(\ldots, \mu_{-1}(T), \mu_0(T))$.

Comparison to dynamical local limits over $\mathbb{C}$

Bulk local limits of log singular values of complex matrix products: Brownian motions with drift conditioned never to intersect (highly nonlocal) [Akemann-Burda-Kieburg '20], [Ahn '22]



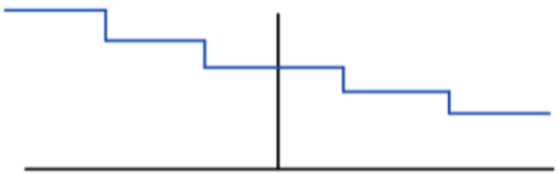
...while p-adic local limits feature only local interactions at collisions



Limits of height function?

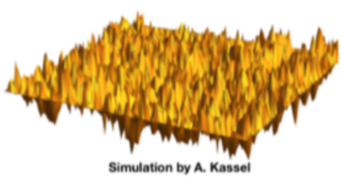
Random $A \in \mathrm{Mat}_n(\mathbb{C})$

$$f_A(x) = |\{i : \lambda_i(A) \geq x\}|$$



Bulk scaling limit

1D slices of 2D *Gaussian free field* [Borodin '10] (logarithmic correlations, not a function!)



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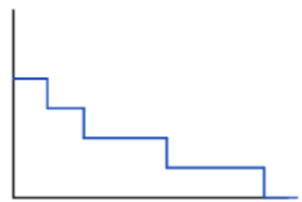
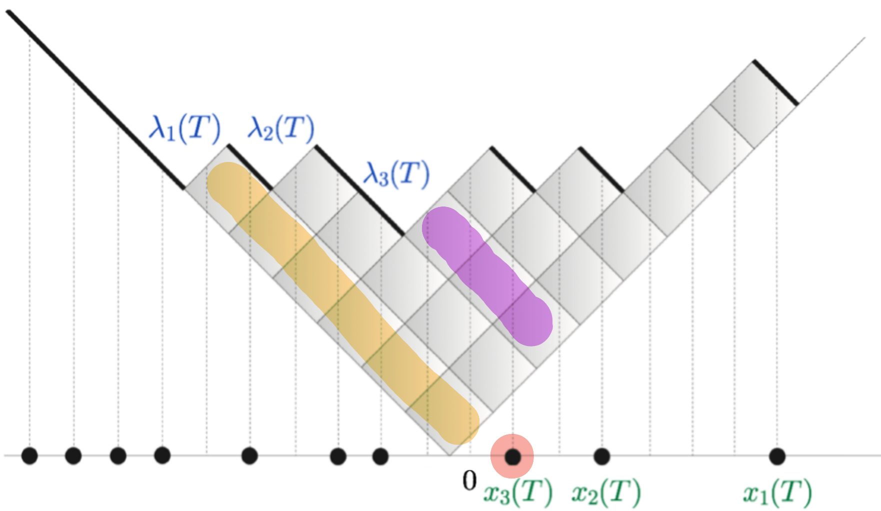


Bulk scaling limit

???

Mapping dynamics to particle system

Natural to consider randomly evolving integer partition $\lambda(T) = (\lambda_1(T), \lambda_2(T), \ldots), T \geq 0$ where $\lambda_i(T)$ jumps by rate $(1-p)p^{-i}$ Poisson clock, with reflection/move donation.



Map $x_k = |\{i : \lambda_i \geq k\}| - k$; above $\lambda = (6, 5, 3, 3, 2, 1, 1, 1)$, $\mathbf{x} = (7, 3, 1, -2, -3, -5, -7, -8, \ldots)$. Then

$$\text{rate}(x_k \mapsto x_k + 1) = p^{-(x_k+k)}(1 - p^{-(x_{k-1}-x_k-1)})$$

Slowed t-TASEP and 'p-adic β-ensembles'

Extrapolate to particle system at general $t = p^{-1} \in (0,1)$,

$$\text{rate}(x_k \mapsto x_k + 1) = t^{x_k+k}(1 - t^{\mathrm{gap}_k}), \quad \mathrm{gap}_k := x_{k-1} - x_k - 1$$

'Slowed t-TASEP': if instead rate$(x_k) = (1 - t^{\mathrm{gap}_k})$, get q-*totally asymmetric exclusion process (q-TASEP)* of [Borodin-Corwin '11].

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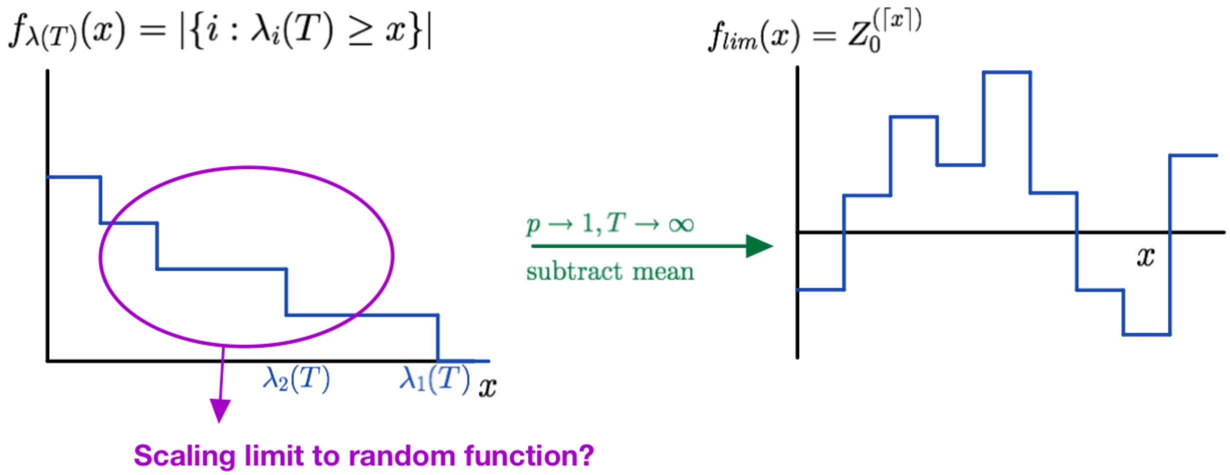
Theorem (VP 2021)

In suitable joint $t \to 1$, time $\to \infty$ limit, fluctuations of x_k converge jointly to Gaussians $Z_0^{(k)}$, unique stationary distributions of

$$dZ_T^{(k)} = ((k-1)Z_T^{(k-1)} - kZ_T^{(k)})dT + dW_T^{(k)}, \qquad k \geq 1,$$

for $W_T^{(k)}, k \geq 1$ independent standard Brownian motions.

Convergence to random function?



Theorem (VP 2021)

The process $R_s^{(u)} := u^{1/4} f_{lim}(u + s\sqrt{u})$ *converges as* $u \to \infty$ *to the unique stationary Gaussian process* $R_s, s \in \mathbb{R}$ *with covariances*

$$\mathrm{Cov}(R_a, R_b) = \int_0^\infty y^2 e^{-y^2 - |b-a| y} dy.$$

$\mathbb{Z}_p$ vs. $\mathbb{C}$: correlations of random height function

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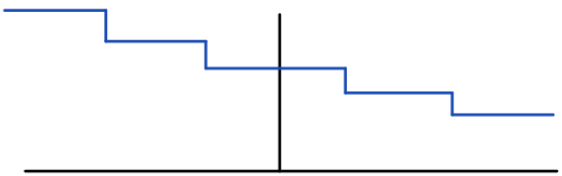

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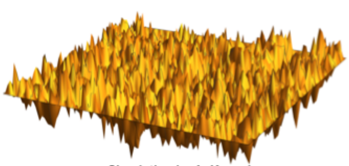

Simulation by A. Kassel

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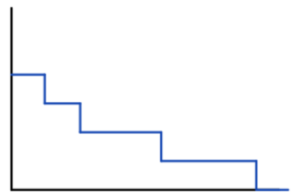
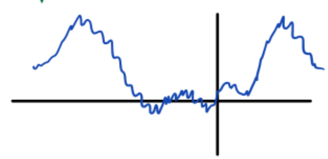


$p \to 1$ bulk scaling limit

1D Gaussian process [VP '21]



Integrable probability and $\mathbb{C}$ vs. $\mathbb{Z}_p$ RMT analogies

Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ indexed by $\lambda \in \mathbb{Y}_n$ are symmetric polynomials in $x_1, \ldots, x_n$ with two parameters q, t.

Macdonald measure
[Borodin-Corwin '11]

$$\Pr(\lambda) = \frac{P_\lambda(1, \ldots, t^{n-1}; q, t) Q_\lambda(t^{m-n+1}, \ldots, t^{D-n}; q, t)}{\Pi_{q,t}(1, \ldots, t^{n-1}; t^{m-n+1}, \ldots, t^{D-n})}$$

$q \to 0$
$t = 1/p$

$\beta \in \{1, 2, 4\}$
$t = q^{\beta/2}$
$q \to 1$
λ rescaled

Hall-Littlewood measure: singular numbers of $n \times m$ corners of Haar $GL_D(\mathbb{Z}_p)$ matrices [VP '20]

Heckman-Opdam measure: singular values of $n \times m$ corners of Haar $O(D), U(D), Sp^*(D)$ matrices [Forrester-Rains '05]

Conclusion

We showed:

- Structural parallels to RMT over $\mathbb{C}$ mediated by special functions, bringing Macdonald process tools and ideas
- Brownian fluctuations for each singular number of $A_\tau \cdots A_1$
- Bulk limit to bi-infinite reflected Poisson walk
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- Applications of p-adic matrix products to groups arising in NT, random graphs,...?

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Thanks for listening!