Liouville Quantum Gravity

Guillaume Remy

École Normale Supérieure

July 6, 2016

What is Liouville Quantum Gravity?

First introduced by Polyakov in 1981:

"Quantum Geometry of Bosonic Strings"

Brownian motion :

- Canonical random path
- Scaling limit of random walks

Liouville Quantum Gravity :

- Canonical two-dimensional geometry
- Conjectured limit of planar maps

Conformal Field Theory

Goal of field theory: compute correlations of certain observables called fields: $\langle \prod_{i \in I} \phi_i(z_i) \rangle$

Conformal Field Theory: conformal invariance in 2D
$\Rightarrow$ Imposes constraints on correlation functions

Examples of CFT's:

- Continuum limit of critical Ising model
- Liouville Quantum Gravity

Another point of view: SLE curves (see Miller/Sheffield course)

Brownian motion seen as a path integral

Space of paths: $\Sigma = \{\sigma : [0,1] \to \mathbb{R}, \sigma(0) = 0\}$

Action functional: $S_{BM}(\sigma) = \frac{1}{2}\int_0^1 |\sigma'(r)|^2 dr$

$$\mathbb{E}[F((B_s)_{0 \leq s \leq 1})] = \frac{1}{Z}\int_\Sigma D\sigma F(\sigma) e^{-S_{BM}(\sigma)}$$

$D\sigma$: formal uniform measure on Σ

Classical Theory / Quantum Theory

Minimum of S_{BM} $\to$ straight line $=$ classical solution
Path integral $\to$ Brownian motion $=$ quantum correction

Some definitions

Let M be a two-dimensional surface (sphere, torus,..).

Metric tensor g : $M \to S_2^+(\mathbb{R})$

Simple case: $g(x) = \begin{pmatrix} e^{f(x)} & 0 \\ 0 & e^{f(x)} \end{pmatrix}$

- Area of A: $\int_A e^{f(x)} dx^2 = \int_A \lambda_g(dx)$
- Gradient squared: $|\partial^g X|^2 = e^{-f} |\partial X|^2$
- Scalar curvature $R_g = -e^{-f} \Delta f$.

Spherical metric on $\mathbb{R}^2$: $g(x) = \frac{4}{(1+|x|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R_g = 2$.

Classical Liouville Theory

For all maps $X : M \to \mathbb{R}$, we define:

$$S_L(X,g) = \frac{1}{4\pi}\int_M (|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X})\lambda_g$$

$Q, \gamma, \mu > 0$ positive constants

Uniformization of (M, g)

Assume X_{min} to be the minimum of S_L and define $g' = e^{\gamma X_{min}} g$. Then $R_{g'} = -2\pi\mu\gamma^2$ if we choose $Q = \frac{2}{\gamma}$.
$\implies$ *The minimum of S_L provides a metric of constant negative curvature.*

Defining Liouville Quantum Gravity

Formal definition

Random metric $e^{\gamma\phi}g$ where the law of ϕ is given by:
$\mathbb{E}[F(\phi)] = \frac{1}{Z}\int F(X)e^{-S_L(X,g)}DX$

First goal: give a meaning to ϕ for different M.

- M = Riemann sphere: David-Kupiainen-Rhodes-Vargas
- M = Torus or higher genus: David-Guillarmou-Rhodes-Vargas
- M = Unit disk: Huang-Rhodes-Vargas
- M = Annulus: Remy

ϕ = Liouville field

Why the Liouville action?

- $|\partial^g X|^2$: analogue of the $|\sigma'|^2$ for Brownian motion

  $\frac{1}{Z}\int F(X) e^{-\frac{1}{4\pi}\int_M |\partial^g X|^2 \lambda_g} DX$: formally defines the law of the Gaussian Free Field (GFF)

- $QR_g X$: curvature term

- $\int_M e^{\gamma X}\lambda_g =$ area of M in the metric $g' = e^{\gamma X} g$
  $\Rightarrow$ penalizes large areas
  $\Rightarrow$ required to have a well defined Liouville field

Insertion points

- For $M = \mathbb{S}^2$, Gauss-Bonnet: $\int_{\mathbb{S}^2} R_g \lambda_g = 8\pi > 0$
- No metric of constant negative curvature
  $\Rightarrow S_L$ has no minimum
  $\Rightarrow \frac{1}{Z} \int F(X) e^{-S_L(X,g)} DX$ not defined
- Instead we consider:
  $\frac{1}{Z} \int F(X) e^{\sum_{i=1}^{n} \alpha_i X(z_i)} e^{-S_L(X,g)} DX = \langle \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \rangle$
  $=$ correlation function of the fields $e^{\alpha_i \phi(z_i)}$
- (z_i, α_i): insertion points $=$ singularities of the metric
- For $\mathbb{S}^2$: at least 3 insertions required

Computing the partition function

Consider $M = \mathbb{S}^2$

Main objective: give a mathematical meaning to

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F,g) = \int DX \prod_i e^{\alpha_i X(z_i)} e^{-S_L(X,g)} F(X)$$

where again:

$$S_L(X,g) = \frac{1}{4\pi}\int_M (|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X})\lambda_g$$

Remark: $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$

Step 1 : the squared gradient term

Goal: give meaning to $\frac{1}{Z}\int \widetilde{F}(X) e^{-\frac{1}{4\pi}\int_M |\partial^g X|^2 \lambda_g} DX$

$$e^{-\frac{1}{4\pi}\int_M |\partial^g X|^2 d\lambda_g} = e^{-\frac{1}{2}\int_M X(-\frac{\Delta_g}{2\pi})X d\lambda_g}$$

Density of an infinite dimensional Gaussian vector of covariance function $(-\frac{\Delta}{2\pi})^{-1}$ = Green function

$\Rightarrow$ defines a GFF

Gradient term: defines X up to a constant c

c = average value of the field

We integrate over c with the Lebesgue measure.

Step 1 : the squared gradient term

$$\frac{1}{Z}\int \widetilde{F}(X)e^{-\frac{1}{4\pi}\int_M |\partial^g X|^2 \lambda_g} DX = \int_{\mathbb{R}} dc\,\mathbb{E}[\widetilde{F}(X+c)]$$

- On the l.h.s: formal functional integral
- On the r.h.s: X has the law of a GFF

The partition function $\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F,g)$ becomes:

$$\int_{\mathbb{R}} dc\,\mathbb{E}[F(X+c)\prod_i e^{\alpha_i(X(z_i)+c)} e^{-\frac{1}{4\pi}\int_M (QR_g(X+c)+4\pi\mu e^{\gamma(X+c)})\lambda_g}]$$

Step 2 : Gaussian multiplicative chaos

X is a random distribution, $e^{\gamma X}$ is ill defined

Regularization procedure: circle average X_ϵ

$$X_\epsilon(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} X(z + \epsilon e^{i\theta}) d\theta$$

Gaussian multiplicative chaos

The following limit exists in probability in the sense of weak convergence of measures for $\gamma \in [0, 2)$:

$$\lim_{\epsilon \to 0} e^{\gamma X_\epsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(z)^2]} d\lambda_g(z) = e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} d\lambda_g(z)$$

Remark: $\mathbb{E}[X_\epsilon(z)^2] + \ln \epsilon$ remains bounded as $\epsilon \to 0$.

Step 3 : regularization of the partition function

Define $\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F, g, \epsilon) =$

$$\int_{\mathbb{R}} dc \mathbb{E}[F(X_\epsilon + c) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (X_\epsilon(z_i) + c)} e^{-\frac{1}{4\pi} \int_M (Q R_g (X_\epsilon + c) + 4\pi\mu \epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_\epsilon + c)}) \lambda_g}]$$

When does the limit $\lim_{\epsilon \to 0} \Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F, g, \epsilon)$ exist?

Main result

Consider $M = \mathbb{S}^2$

Non-triviality of the partition function

Assume $\gamma \in [0,2)$ and $\mu > 0$, then $\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F,g) = \lim_{\epsilon \to 0} \Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F,g,\epsilon)$ exists and is finite and non zero

$\iff \sum_i \alpha_i > 2Q$ and $\forall i$, $\alpha_i < Q$

Definition of the law of the Liouville field ϕ:

$$\mathbb{E}[F(\phi)] = \frac{\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F,g)}{\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(1,g)}$$

Liouville measure

- ϕ is a random distribution $\Rightarrow$ difficult to define $e^{\gamma\phi}$
- Well defined Liouville measure $Z(A) = \int_A e^{\gamma\phi}\lambda_g$

  Conjectured limit of uniform planar maps for $\gamma = \sqrt{\frac{8}{3}}$

  Conjectured limit of planar maps with an Ising model for $\gamma = \sqrt{3}$

Surfaces with boundary

Must add boundary terms to S_L:

$\tilde{S}_L(X,g) = S_L(X,g) + \frac{1}{2\pi}\int_{\partial M}(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X})\lambda_{\partial g}$

Example: M =unit disk

- Bulk insertion points (z_i, α_i)
- Boundary insertion points (s_j, β_j)

Non-triviality of the partition function

Assume $\gamma \in [0,2)$, $\mu_\partial > 0$, and $\mu > 0$, then $\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F,g) = \lim_{\epsilon\to 0}\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F,g,\epsilon)$ exists and is finite and non zero

$\iff \sum_i \alpha_i + \sum_j \frac{\beta_j}{2} > Q$, $\forall i$, $\alpha_i < Q$ and $\forall j$, $\beta_j < Q$

Surfaces of higher genus

Higher genus: non trivial moduli space

- Torus
- Annulus

$A(a,b)$ = annulus in the plane with radii $a < b$

Then $A(a,b) \sim A(a',b') \iff \frac{a}{b} = \frac{a'}{b'}$

$\tau = \frac{a}{b}$ = modular parameter, important in physics.

Thank you for listening!

References:

- Lecture notes on Gaussian multiplicative chaos and Liouville Quantum Gravity,
  R. Rhodes, V. Vargas (2016), arXiv.
- Liouville Quantum Gravity on the Riemann sphere,
  F. David, A. Kupiainen, R. Rhodes, V. Vargas (2014), to appear in Communications in Mathematical Physics.
- Liouville Quantum Gravity on the unit disk,
  Y. Huang, R. Rhodes, V. Vargas (2015), arXiv.
- Liouville Quantum Gravity on the complex tori,
  F. David, R. Rhodes, V. Vargas (2015), to appear in Journal of Mathematical Physics.