A LEFSCHETZ THEOREM FOR CRYSTALLINE REPRESENTATIONS

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Abstract. As a corollary of nonabelian Hodge theory, Simpson proved a strong Lefschetz theorem for complex polarized variations of Hodge structure. We show an arithmetic analog. Our primary technique is p-adic nonabelian Hodge theory.

1. Introduction

An easy corollary of Simpson's nonabelian Hodge theorem is the following.

Theorem 1.1. [Sim92, Corollary 4.3] *Let X and Y be smooth projective complex varieties, and let $f\colon Y \to X$ be a morphism that induces a surjection:*

$$f_*\colon \pi_1(Y) \twoheadrightarrow \pi_1(X).$$

Let $\mathbb{L}$ be a $\mathbb{C}$-local system on X such that $f^*\mathbb{L}$ underlies a complex polarized variation of Hodge structures on Y. Then $\mathbb{L}$ under lies a complex polarized variation of Hodge structures on X.

As Simpson notes, this is especially useful when Y is the smooth complete intersection of smooth hyperplane sections of X. This article is concerned with an arithmetic analog of Theorem 1.1. In this setting, the condition that $\mathbb{L}$ underlies a complex polarized variation of Hodge structures is replaced by the condition that $\mathbb{L}$ is a (log) crystalline representation. To state this precisely, we need the following notation.

Setup 1.2. Let $k \cong \mathbb{F}_q$ be a finite field of odd characteristic, let $W := W(k)$ be the ring of Witt vectors, and let $K := \mathrm{Frac}(W)$ be the field of fractions. Let X/W be a smooth projective scheme of relative dimension at least 2. Let $S \subset X$ be a strict normal crossings divisor, flat over W. Let $j\colon D \hookrightarrow X$ be a relative smooth ample divisor, flat over W that intersects S transversely, so that $S \cap D \subset D$ is a strict normal crossings divisor. Let $X_K^o = X_K - S_K$ and let $D_K^o = D_K - (D_K \cap S_K)$. There is a natural continuous surjective homomorphism

$$j_{K*}\colon \pi_1(D_K^o) \twoheadrightarrow \pi_1(X_K^o).$$

Question 1.3. In the context of Setup 1.2, let $\rho_X\colon \pi_1(X_K^o) \to \mathrm{GL}_N(\mathbb{Z}_{p^f})$ be a continuous p-adic representation. Restricting ρ_X via j_K, one gets a representation $\rho_D\colon \pi_1(D_K^o) \to \mathrm{GL}_N(\mathbb{Z}_{p^f})$. Suppose ρ_D is log crystalline. Is ρ_X also log crystalline?

In this article, we answer Question 1.3 under several additional assumptions.

Theorem 1.4. *In the context of Question 1.3, suppose further that:*

(1) *$N^2 < p - \dim X$;*
(2) *ρ_D is residually irreducible; and*
(3) *the line bundle $\mathcal{O}_X(D - S)$ is ample on X.*

Then ρ_X is log crystalline.

We do not yet know how to relax these assumptions.

Remark 1.5. Note that the assumptions imply that we may Tate twist ρ_D so that it has Hodge-Tate weights in the interval $[0, \sqrt{p-\dim(X)}]$. As we have assumed that $p \geq 3$ and $\dim(X) \geq 2$, this means that $\sqrt{p-\dim(X)} \leq p-2$. Tate twists do not change the property of "being crystalline", so we may assume without loss of generality that ρ_D has Hodge-Tate weights in the interval $[0, p-2]$. This is necessary to apply the theory of Lan-Sheng-Zuo.

We briefly explain the structure of the proof.

Step 1 In Section 3, we transform the question into a problem about extending a periodic Higgs-de Rham flow. The theory of Higgs-de Rham flows has its origins in the seminal work of Ogus-Vologodsky on nonabelian Hodge theory in characteristic p [OV07]. This theory has recently been enchanced to a p-adic theory by Lan-Sheng-Zuo. According to the theory of Lan-Sheng-Zuo, there is an equivalence between the category of certain crystalline representations (with bounds on the Hodge-Tate weights) and periodic Higgs-de Rham flows. Let $\mathrm{HDF}_{\mathcal{D}}$ be the logarithmic Higgs-de Rham flow over $(\mathcal{D}, \mathcal{S} \cap \mathcal{D})$ associated to the representation ρ_D:

$$\begin{tikzcd}[column sep=small] & (V_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathrm{Fil}_{\mathcal{D}})_0 \arrow[dr] & & (V_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathrm{Fil}_{\mathcal{D}})_1 \arrow[dr] & & \cdots \\ (E_{\mathcal{D}}, \theta_{\mathcal{D}})_0 \arrow[ur] & & (E_{\mathcal{D}}, \theta_{\mathcal{D}})_1 \arrow[ur] & & (E_{\mathcal{D}}, \theta_{\mathcal{D}})_2 \arrow[ur] & \end{tikzcd} \tag{1}$$

Then we need to extend $\mathrm{HDF}_{\mathcal{D}}$ to some periodic Higgs-de Rham flow over $(\mathcal{X}, \mathcal{S})$.

Step 2 In Section 4, we extend $(E_{\mathcal{D}}, \theta_{\mathcal{D}})_i$ to a graded logarithmic semistable Higgs bundles $(E_{\mathcal{X}}, \theta_{\mathcal{X}})_i$ over $(\mathcal{X}, \mathcal{S})$. Using Scholze's notion of *de Rham local systems* together with a rigidity theorem due to Liu-Zhu (and Diao-Lan-Liu-Zhu), we construct a graded logarithmic Higgs bundle $(E_{\mathcal{X}_K}, \theta_{\mathcal{X}_K})$ over $(\mathcal{X}_K, \mathcal{S}_K)$ such that

$$(E_{\mathcal{X}_K}, \theta_{\mathcal{X}_K}) \mid_{\mathcal{D}_K} = (E_{\mathcal{D}}, \theta_{\mathcal{D}}) \mid_{\mathcal{D}_K} .$$

One gets graded Higgs bundles $(E_{\mathcal{X}_K}, \theta_{\mathcal{X}_K})_i$ extending $(E_{\mathcal{D}}, \theta_{\mathcal{D}})_i \mid_{\mathcal{D}_K}$. In Section 5, using a result of Langer (extending work of Langton), we extend $(E_{\mathcal{X}_K}, \theta_{\mathcal{X}_K}) = (E_{\mathcal{X}_K}, \theta_{\mathcal{X}_K})_0$ to a semistable Higgs torsion free sheaf $(E_{\mathcal{X}}, \theta_{\mathcal{X}})$ on $(\mathcal{X}, \mathcal{S})$. We show that this extension $(E_{\mathcal{X}}, \theta_{\mathcal{X}})$ is unique up to an isomorphism and has trivial Chern classes in Section 6, which implies that $(E_{\mathcal{X}}, \theta_{\mathcal{X}})$ is locally free using work of Langer.

Step 3 We show that $(E_{\mathcal{X}}, \theta_{\mathcal{X}})$ constructed in Step 2 has stable reduction modulo p and is graded. To do this, we prove a Lefschetz theorem for semistable Higgs bundles with vanishing Chern classes using a vanishing theorem of Arapura in Section 7. It is here that our assumptions transform from $N < p$ to $N^2 < p - \dim(X)$. The argument that $(E_{\mathcal{X}}, \theta_{\mathcal{X}})$ is graded is contained in Section 8.

Step 4 In Section 9, we extend $\mathrm{HDF}_{\mathcal{D}} \mid_{D_1}$ to a Higgs-de Rham flow HDF_{X_1} over X_1 (here the subscript n denotes reduction modulo p^n). By the stability of the Higgs bundle, this flow extends $\mathrm{HDF}_{\mathcal{D}} \mid_{D_1}$.

Step 5 Finally, in Section 10, we deform HDF_{X_1} to a (p-adic, periodic) Higgs-de Rham flow $\mathrm{HDF}_{\mathcal{X}}$ over $\mathcal{X}$, which extends $\mathrm{HDF}_{\mathcal{D}}$. To do this, we use results of Krishnamoorthy-Yang-Zuo [KYZ20] that explicitly calculate the obstruction class of deforming each piece of the Higgs-de Rham flow together with a Lefschetz theorem relating this obstruction class to the obstruction class over D.

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2. Notation

- p is an odd prime number.
- $k \cong \mathbb{F}_q$ be a finite field of characteristic p.
  - $W := W(k)$,
  - $K := \operatorname{Frac} W$.
- (X, S): X is a smooth projective scheme over $\operatorname{Spec}(W)$ and $S \subset X$ is a relative (strict) normal crossings divisor, flat over W.
  - (X_n, S_n): the reduction of (X, S) modulo p^n for any $n \geq 1$;
  - (X_K, S_K): the generic fiber of (X, S);
  - $X^o = X - S$
  - $\mathcal{X}$: the p-adic formal completion of X along the special fiber X_1;
  - $\mathcal{X}_K$: the rigid analytic space associated to $\mathcal{X}$.
- $D \subset X$: a relative smooth ample divisor, flat over W, that intersects S transversely.
  - Same notation for D_1, D_n, D_K, $\mathcal{D}$, $\mathcal{D}_K$, D^o, etc.

3. The theory of Lan-Sheng-Zuo

The following fundamental theorem is a combination of work of Lan-Sheng-Zuo ([LSZ19, Theorem 1.4] for non-logarithmic case and joint with Y. Yang in [LSYZ19, Theorem 1.1] for the logarithmic setting) together with the work of Faltings [Fal89, Theorem 2.6(*), p. 43] relating logarithmic Fontaine-Faltings modules to crystalline (lisse) p-adic sheaves.

Theorem 3.1. *(Lan-Sheng-Zuo) Let X/W be a smooth projective scheme and let $S \subset X$ be a relative simple normal crossings divisor, and let $X_K^o = X_K \backslash S_K$. Then for each natural number $f \in \mathbb{N}$, there is an equivalence between the category of logarithmic crystalline representations $\pi_1(X_K^o) \to \mathrm{GL}_N(\mathbb{Z}_{p^f})$ with Hodge-Tate weights in the interval $[0, p-2]$ and the category of f-periodic Higgs-de Rham flow over $(\mathcal{X}, \mathcal{S})$ where the exponents of nilpotency are less than or equal to $p-2$.*

Let us use this theorem to investigate Question 1.3. Since ρ_D is crystalline with Hodge-Tate weights in $[0, p-2]$, there is a periodic Higgs-de Rham flow $\mathrm{HDF}_{\mathcal{D}}$ on $\mathcal{D}$ associated to this crystalline representation under Lan-sheng-Zuo's equivalent functors.

Lemma 3.2. *Setup as in Question 1.3 and suppose $N \leq p-2$. Then Question 1.3 has an affirmative answer if and only if there exists a periodic Higgs-de Rham flow over $(\mathcal{X}, \mathcal{S})$ extending the Higgs-de Rham flow $\mathrm{HDF}_{\mathcal{D}}$ on $(\mathcal{D}, \mathcal{D} \cap \mathcal{S})$.*

Proof. If ρ_X is crystalline, we choose the Higgs-de Rham flow $\mathrm{HDF}_{\mathcal{X}}$ associated to ρ_X.

Conversely, suppose there exists a Higgs-de Rham flow $\mathrm{HDF}'_{\mathcal{X}}$ on $\mathcal{X}$ extending the Higgs-de Rham flow $\mathrm{HDF}_{\mathcal{D}}$ on $(\mathcal{D}, \mathcal{D} \cap \mathcal{S})$. As $N \leq p-2$, the exponents of nilpotency are all $\leq p-2$. Then by Theorem 3.1, one obtains a crystalline representation ρ'_X extending ρ_D. As the map

$$\pi_1(D_K^o) \to \pi_1(X_K^o)$$

is surjective (by e.g. [EK16, Theorem 1.1(a)]), we see that ρ_X is isomorphic to ρ'_X; hence ρ_X is crystalline as desired. $\square$

4. De Rham local systems and graded Higgs bundles.

In this section, we construct a logarithmic Higgs bundle $(E_{\mathcal{X}_K}, \theta_{\mathcal{X}_K})$ over $\mathcal{X}_K$ which is associated to the representation ρ_X.

For clarity, we first discuss the non-logarithmic case, i.e. $S = \emptyset$. For the reader's convenience, we recall Faltings' definition of crystalline local systems and Scholze's definition of de Rham local systems. A $\mathbb{Z}_p$-local system $\mathbb{L}$ on X_K is called crystalline, if it has an associated Fontaine-Faltings module $(M, \nabla, \mathrm{Fil}, \varphi)$ over $\mathcal{X}$. Using Faltings' $\mathbb{D}$-functor [Fal89, p. 36], locally on a small affine open set $U = \mathrm{Spec}(R)$, one can reconstruct the local system $\mathbb{L}$

$$\mathbb{L}(U_K) = \mathbb{D}(M)(U) = \varprojlim_n \mathrm{Hom}\Big(M(U) \otimes B^+(\hat{R})/p^n B^+(\hat{R}), B^+(\hat{R})[\frac{1}{p}]/B^+(\hat{R})\Big),$$

where homomorphisms on the RHS are $B^+(\hat{R})$-linear and respect the filtrations and the φ's. In particular, one has

$$M_D^\vee(U) \otimes_{\hat{R}} B^+(\hat{R}) = \mathbb{L}_D(U) \otimes_{\mathbb{Z}_p} B^+(\hat{R}) \tag{2}$$

A *filtered de Rham bundle* is denoted to be vector bundle together with a separated and exhaustive decreasing filtration by locally direct summands, and an integrable connection satisfying Griffiths transversality. Following Scholze [Sch13, Definition 7.5 and Definition 8.3] a local system $\mathbb{L}$ on X_K is said to be *de Rham* if there exists a filtered de Rham bundle $(\mathcal{E}, \nabla, \mathrm{Fil})_{X_K}$ over X_K such that

$$\mathcal{E}_{\mathcal{X}_K} \otimes_{\mathcal{O}_{\mathcal{X}_K}} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \simeq \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}.[1]$$

Following [TT19, Definition 3.10], one can also use the crystalline period sheaf $\mathcal{O}\mathbb{B}_{cris}$ to redefine Faltings' notion of a crystalline representation. A local system $\mathbb{L}$ on X_K is said to be *crystalline* if there exists a filtered F-isocrystal on X_1 with realization $(\mathcal{E}, \nabla, \mathrm{Fil})_{\mathcal{X}_K}$ over $\mathcal{X}_K$ such that

$$\mathcal{E}_{\mathcal{X}_K} \otimes_{\mathcal{O}_{\mathcal{X}_K}} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}.$$

Let $\mathbb{L}_X$ (respectively $\mathbb{L}_D$) be the $\mathbb{Z}_p$-local system on X_K (respectively D_K) associated to the representation ρ_X (respectively ρ_D). As ρ_D is crystalline, it is de Rham: this is well-known and essentially follows from the natural inclusion $\mathcal{O}\mathbb{B}_{\mathrm{cris}} \hookrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}$ [TT19, Corollary 2.25(1)].

Proposition 4.1 (Tan-Tong)**.** *The local system $\mathbb{L}_D$ is de Rham on D_K.*

Remark 4.2. As we assume that ρ_D is crystalline, there is an associated Fontaine-Faltings module $(M, \nabla, \mathrm{Fil}, \varphi)_{\mathcal{D}}$. As crystalline representations are de Rham, there is also a filtered de Rham bundle $(\mathcal{E}, \nabla, \mathrm{Fil})_{D_K}$ associated to ρ_D. One then has

$$(M, \nabla, \mathrm{Fil})_{\mathcal{D}} \mid_{D_K} = (\mathcal{E}, \nabla, \mathrm{Fil})_{D_K}^\vee.$$

The appearance of a dual is simply because Faltings' original functor is contravariant.

[1]Here, this isomorphism is of sheaves on the pro-étale site of $\mathcal{X}_K$, the rigid analytic generic fiber.

Corollary 4.3. *Let $X/W(k)$ be a smooth projective scheme of relative dimension at least 2 and let $D \subset X$ be a relative smooth ample divisor. Suppose $\rho_D \colon \pi_1(D_K) \to GL_N(\mathbb{Z}_{p^f})$ is a crystalline representation. It follows Remark 4.2 that there is an associated filtered de Rham bundle $(M, \nabla, Fil)_{\mathcal{D}}$ over $\mathcal{D}$ associated to ρ_D. By taking the associated graded, we obtain a Higgs bundle $(E, \theta)_{\mathcal{D}}$. Then we have the following.*

(1) *There exists a filtered de Rham bundle $(M, \nabla, \mathrm{Fil})_{X_K}$ over X_K such that*

$$(M, \nabla, \mathrm{Fil})_{\mathcal{D}} \mid_{D_K} = (M, \nabla, \mathrm{Fil})_{X_K} \mid_{D_K} .$$

In other words, the filtered de Rham bundle $(M, \nabla, \mathrm{Fil})_{\mathcal{D}} \mid_{D_K}$ extends to a filtered de Rham bundle on X_K.

(2) *There exists a graded Higgs bundle $(E, \theta)_{X_K}$ over X_K extending $(E, \theta)_{\mathcal{D}} \mid_{D_K}$.*

Proof. Taking the associated graded, one derives (2) from (1) trivially. Since $\mathbb{L}_X \mid_{X_D} = \mathbb{L}_D$ is de Rham, $\mathbb{L}_X$ is also de Rham by [LZ17, Theorem 1.5(iii)]. Taking $(M, \nabla, \mathrm{Fil})_{X_K}$ to be the dual of the filtered de Rham bundle associated to $\mathbb{L}_X$, then (1) follows the functor property. □

For the general case $S \neq \emptyset$, we still have the following result, which crucially uses a recent Riemann-Hilbert correspondence [DLLZ18].

Corollary 4.4. *Let $X/W(k)$ be a smooth projective scheme of relative dimension at least 2, and let $S \subset X$ be a relative simple normal crossings divisor. Let $D \subset X$ be a relative smooth ample divisor that meets S transversally. Suppose $\rho_D \colon \pi_1(D_K^o) \to GL_N(\mathbb{Z}_{p^f})$ is a logarithmic crystalline representation. It follows from the Remark 4.2*[2] *that there is an associated logarithmic filtered de Rham bundle $(M, \nabla, Fil)_{\mathcal{D}}$ over $(\mathcal{D}, \mathcal{D} \cap \mathcal{S})$ associated to ρ_D. By taking the associated graded, we obtain a logarithmic Higgs bundle $(E, \theta)_{\mathcal{D}}$. Then we have the following.*

(1). There exists a logarithmic filtered de Rham bundle $(M, \nabla, \mathrm{Fil})_{X_K}$ over (X_K, S_K) such that

$$(M, \nabla, \mathrm{Fil})_{\mathcal{D}} \mid_{D_K} = (M, \nabla, \mathrm{Fil})_{X_K} \mid_{D_K} .$$

Furthermore, the connection has nilpotent residues around S_K.

(2). There exists a logarithmic graded Higgs bundle $(E, \theta)_{X_K}$ over (X_K, S_K) extending $(E, \theta)_{\mathcal{D}} \mid_{D_K}$.

Proof. First of all, knowing that ρ_D is log crystalline means $\mathbb{L}_{D_K^o}$ is crystalline. Then $\mathbb{L}_{D_K^o}$ is de Rham by Remark 4.2. By [LZ17, Theorem 1.5(iii)], it follows that $\mathbb{L}_{X_K^o}$ is de Rham. Using [DLLZ18, Theorem 1.1], we see that there is an (algebraic) filtered vector bundle with integrable connection that has *regular singularities* on X_K^o that is associated to $\mathbb{L}_{X_K^o}$: $(M, \nabla, \mathrm{Fil})_{X_K}$.

Note that D_K meets every component of S_K as $D_K \subset X_K$ is ample. As ρ_D is log crystalline, the pair (E_{D_K}, ∇) has nilpotent residues around $S_K \cap D_K$. On the other hand, the residues are constant along each component of S_K; therefore ∇ has nilpotent residues around S_K.[3] □

The fact that M_{X_K} admits an integrable connection with logarithmic poles and nilpotent residues implies that its de Rham Chern classes all vanish [EV86, Appendix B]. By

[2]in its logarithmic form

[3]Another argument for the nilpotence of the residues; the local monodromy of ρ_D along $D_K \cap S_K$ is unipotent. This implies that the local monodromy of ρ_X along S_K is also unipotent because D_K intersects each component of S_K non-trivially and transversally. Then simply apply [DLLZ18, Theorem 3.2.12].

comparison between de Rham and l-adic cohomology, this implies that the $\mathbb{Q}_l$ Chern classes also vanish.

Consider the associated graded Higgs bundle $(E,\theta)_{X_K} = \mathrm{Gr}((M,\nabla,\mathrm{Fil})_{X_K})$ over X_K, which extends the Higgs bundle $(E,\theta)_{\mathcal{D}} \mid_{D_K}$. Since $(E,\theta)_{X_K} \mid_{D_K}$ is semistable, it follows that $(E,\theta)_{X_K}$ is also semistable. As the Chern classes of M_X all vanish, so do the Chern classes of E.

In summary, we have constructed a logarithmic Higgs bundle $(E,\theta)_{X_K}$ extending the Higgs bundle $(E_{\mathcal{D}},\theta_{\mathcal{D}}) \mid_{D_K}$ that is semistable, and has (rationally) trivial Chern classes. In the next sections, we will extend $(E,\theta)_{X_K}$ to a Higgs *bundle* on X whose special fiber is also semistable.

5. A theorem in the style of Langton

In this section, we apply results due to Langer [Lan14, Lan19] in the vein of Langton to graded semistable logarithmic Higgs bundles.

Theorem 5.1. *Let (E_{X_K},θ_{X_K}) be a graded semistable logarithmic Higgs bundle over X_K such that the underlying vector bundle has rank $r \le p$ and trivial $\mathbb{Q}_\ell$ Chern classes. Then there exists a semi-stable logarithmic Higgs bundle (E_X,θ_X) over X, which satisfies*

- $(E_X,\theta_X) \mid_{X_K} \simeq (E_{X_K},\theta_{X_K})$;
- $(E_X,\theta_X) \mid_{X_1}$ *is semistable over X_1.*

Proof. The proof works in several steps.

Step 1. We first construct an auxiliary logarithmic coherent Higgs sheaf (F_X,Θ_X) on X such that
 (a) F_X is reflexive (and hence torsion-free);
 (b) there is an isomorphism $(F_X,\Theta_X)|_{X_K} \cong (E_{X_K},\theta_{X_K})$.

First of all, E_K admits a coherent extension F_X to X. Replacing F_X with F_X^{**}, this is reflexive.

We must construct a logarithmic Higgs field. We know that $F_X \subset F_{X_K} \cong E_{X_K}$. Work locally, over open $W(k)$-affines $\mathcal{U}_\alpha$, and pick a finite set of generators f_i for $F_{\mathcal{U}_\alpha}$. Then, for each α, there exists an $r_\alpha \ge 0$ with

$$\theta_{X_k}(f_i) \subset p^{-r_\alpha} F_{\mathcal{U}_\alpha} \otimes_{\mathcal{O}_{\mathcal{U}_\alpha}} \Omega^1_{\mathcal{U}_\alpha/W}(\log S \cap \mathcal{U}_\alpha)$$

for each i. As X is noetherian, there are only finitely many α and hence there is a *uniform* r such that

$$p^r\theta_{X_k} \colon F_X \to F_X \otimes_{\mathcal{O}_X} \Omega^1_{X/W}(\log S). \tag{3}$$

Set $\Theta_X := p^r\theta_{X_K}$; this makes sense by the above formula 3. Then (F_X,Θ_X) is a Higgs sheaf on X. We claim that $(F_X,\Theta_X)_{X_K} \cong (E_{X_K},\theta_{X_K})$. Indeed, as (E_{X_K},θ_{X_K}) is a graded Higgs bundle, there exists an isomorphism

$$(E_{X_K}, p^r\theta_{X_K}) \cong (E_{X_K},\theta_{X_K}).$$

Finally, as F_X is torsion-free, it is automatically $W(k)$-flat.

Step 2. We now claim that there exists a logarithmic Higgs sheaf (E_X,θ_X) extending (E_{X_K},θ_{X_K}) such that the logarithmic Higgs sheaf (E_{X_1},θ_{X_1}) on the special fiber X_1 is semistable. This is a direct consequence of [Lan14, Theorem 5.1].[4] We

[4]To apply Langer's theorem directly, set L to be the smooth Lie algebroid whose underlying coherent sheaf is the logarithmic tangent sheaf, and whose bracket and anchor maps are trivial.

emphasize that (E_X, θ_X) is merely a Higgs sheaf; we don't yet know it is a vector bundle.

The (E_X, θ_X) constructed by Langer's theorem is torsion-free by design, hence also $W(k)$-flat. We claim that the Chern classes of (E_{X_1}, θ_{X_1}) vanish because the Chern classes of (E_{X_K}, θ_{X_K}) do; this is what we do in Section 6.

Step 3. The special fiber (E_{X_1}, θ_{X_1}) is a semistable logarithmic Higgs sheaf with rank $r \leq p$. Moreover, the Chern classes all vanish. Then it directly follows from [Lan19, Lemmas 2.5 and 2.6] that (E_{X_1}, θ_{X_1}) is locally free.[5][6]

Step 4. Finally, it follows from the easy argument below that (E_X) is a vector bundle.

□

Lemma 5.2. *Let $\mathcal{F}_X$ be a torsion free and coherent sheaf over X. If $\mathcal{F}_X \mid_{X_1}$ is locally free, then $\mathcal{F}_X$ is locally free.*

Proof. Since $\mathcal{F}_X$ is torsion free and coherent sheaf, its singular locus Z is a closed sub scheme of X. And $\mathcal{F}_X$ is locally free if and only if Z is empty. Since $\mathcal{F}_X \mid_{X_1}$ is locally free, its singular locus $Z \cap X_1$ is empty. On the other hand Z is empty if and only if $Z \cap X_1$ is empty, because X is proper over W. Thus Z is also empty. □

We emphasize that we do not know the Higgs sheaf (E_{X_1}, θ_{X_1}) is graded in general. We will deduce this in our situation using a Lefschetz-style theorem.

6. Local constancy of Chern classes

To explain the local constancy of this section, we first need some preliminaries. Let S be a separated scheme and let $f\colon X \to S$ be smooth projective morphism. Let l be invertible on S. Then the lisse l-adic sheaf $R^i f_* \mathbb{Z}_l(j)$ is a constructible, locally constant sheaf by the smooth base change theorem. Moreover, for any point $s \in S$, the proper base change theorem implies that the natural morphism $R^i f_* \mathbb{Z}_l(j)_s \to R^i (f_s)_* \mathbb{Z}_l(j)$ is an isomorphism. In particular, if $\xi \in H^0(S, R^i(f_s)_*\mathbb{Z}_l(j)$ then it is automatically "locally constant".

Proposition 6.1. *Let S be a separated irreducible scheme. Let $f\colon X \to S$ be a smooth proper morphism. Let $\mathcal{E}$ be a vector bundle on X. If $\bar{s}$ is a geometric point such that the l-adic Chern classes of $\mathcal{E}_{\bar{s}}$ vanish, then the Chern classes of $\mathcal{E}$ vanish at every geometric point.*

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[5]See Section 1.6 and Lemma 1.7 of *loc. cit.* for why all Chern classes vanishing implies that $\Delta_i = 0$ for all $i > 1$ and hence why the hypotheses Lemma 2.5 of *loc. cit.* are met.

[6]This argument actually proves the following: let (X_1, S_1) be a smooth projective variety and simple normal crossings divisor defined over $\bar{\mathbb{F}}_p$. Suppose the pair (X_1, S_1) lifts to W_2. Then any semistable logarithmic Higgs bundle of rank $r \leq p$ with vanishing Chern classes on (X_1, S_1) is preperiodic. The inverse Cartier C^{-1} sends slope semistable Higgs bundles to slope semistable flat connections. Any semistable logarithmic flat connection has a distinguished gr-semistable Griffiths tranverse filtration, the *Simpson filtration*, whose field of definition is the same as the field of definition of the flat connection [Lan19, Theorem 5.5]. Let $\mathcal{M}$ be the moduli space of semistable logarithmic Higgs bundles of rank r on (X_1, S_1) with trivial $\mathbb{Q}_l$-Chern classes. This is a finite dimensional moduli space by a boundedness result of Langer [Lan19, Theorem 1.2].Consider the constructible map $\mathcal{M} \to \mathcal{M}$ induced by the $\mathrm{Gr} \circ C^{-1}$, where Gr is the associated graded of the Simpson filtration. (Compare with [Lan19, Theorem 1.6].) There exists a finite field $\mathbb{F}_q$ such that X_1 and this constructible map are both defined. For any $m \geq 1$, the set of all $\mathbb{F}_{q^m}$-points of $\mathcal{M}$ is finite and preserved by this self map. Thus every $\bar{\mathbb{F}}_p$-point in this moduli space is preperiodic. This preperiodicity result is also obtained as [Ara19, Theorem 8(2)].

Proof. As discussed above, the result directly follows if we show that the Chern classes live in $H^0(S, R^if_*\mathbb{Z}_l(j))$. This amounts to *defining* Chern classes in this level of generality. We indicate how to do this. The key is the *splitting principle*; see, e.g., [Sta20, 02UK] for a reference (in the context of Chow groups). Let $f\colon \mathbb{F}\mathcal{E} \to X$ be the associated full flag scheme; then the following two properties hold:

- The induced map $f^*\colon H^i_{\text{ét}}(X, \mathbb{Q}_l) \to H^i_{\text{ét}}(\mathbb{F}\mathcal{E}, \mathbb{Q}_l)$ is injective.
- The vector bundle $f^*\mathcal{E}$ has a filtration whose subquotients are line bundles.

By the splitting principle, it suffices to construct the *first Chern class* for line bundles in the appropriate cohomology group. Let L be a line bundle on X. Then the isomorphism class of L gives a class in $H^1_{\text{ét}}(X, \mathcal{O}_X^*)$. Consider the Leray spectral sequence:

$$E_2^{pq} := H^p_{\text{ét}}(S, R^qf_{*,\text{ét}}\mathcal{O}_X^*) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathcal{O}_X^*)$$

By the low-degree terms of the Leray spectral sequence, there is a natural exact sequence

$$0 \to H^1_{\text{ét}}(S, f_*\mathcal{O}_X^*) \to H^1_{\text{ét}}(X, \mathcal{O}_X^*) \to H^0(S, R^1f_*\mathcal{O}_X^*).$$

(See e.g. [FGI+05, Eqns. 9.2.11.3 and 9.2.11.4 on p. 256-257].) For a line bundle, we define the l-adic Chern class via the Kummer sequence, which induces a map

$$R^1f_*(\mathcal{O}_X^*) \to R^2f_{*,\text{ét}}\mathbb{Z}_l(-1).$$

This map is compatible with base change and hence agrees with the notion of l-adic fist Chern class for smooth projective varieties over fields. In particular, we obtain a class $c_1(L)$ in $H^0_{\text{ét}}(X, R^2f_*\mathbb{Z}_l(-1))$. The result follows. □

Corollary 6.2. *Let S be a separated irreducible scheme. Let $f\colon X \to S$ be a smooth proper morphism. Let $\mathcal{E}$ be a coherent sheave over X with a finite locally free resolution*

$$0 \to \mathcal{E}^m \to \mathcal{E}^{m-1} \to \cdots \to \mathcal{E}^0 \to \mathcal{E} \to 0.$$

Suppose $\mathcal{E}$ is flat over S, i.e. for all $x \in X$, the stalk of $\mathcal{E}$ at x is flat over the local ring $\mathcal{O}_{S,s}$ where $s = f(x) \in S$. Suppose there is a point $s \in S$ such that the $\mathbb{Q}_l$ Chern classes of $\mathcal{E}_s$ vanish. Then for every point $s \in S$, the $\mathbb{Q}_l$ Chern classes of $\mathcal{E}_s$ vanish.

Proof. We only need to show that the restriction on the fiber X_s for any point $s \in S$ induces the following exact sequence

$$0 \to \mathcal{E}^m|_{X_s} \to \mathcal{E}^{m-1}|_{X_s} \to \cdots \to \mathcal{E}^0|_{X_s} \to \mathcal{E}|_{X_s} \to 0. \tag{4}$$

where $\mathcal{E}^i|_{X_s}$ is the restriction of $\mathcal{E}^i$ on X_s. The Whitney sum formula implies that

$$c\left(\mathcal{E}|_{X_s}\right) = \prod_{i=0}^{m} c\left(\mathcal{E}^i|_{X_s}\right)^{(-1)^i},$$

where $c(\cdot)$ is the total Chern class. Then Proposition 6.1 implies the result.

In the following we prove the exactness of (4). Recall that a complex of sheaves is exact if and only if the corresponding complexes on all stalks are exact. Thus we only need to show the exactness of the following complex of $\mathcal{O}_{X_s,x}$-modules

$$0 \to (\mathcal{E}^m|_{X_s})_x \to (\mathcal{E}^{m-1}|_{X_s})_x \to \cdots \to (\mathcal{E}^0|_{X_s})_x \to 0, \tag{5}$$

for all $x \in X_s$, where $(\mathcal{F}|_{X_s})_x$ is the stalk of $\mathcal{F}|_{X_s}$ at x. By assumption one has an exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{E}_x^m \to \mathcal{E}_x^{m-1} \to \cdots \to \mathcal{E}_x^0 \to \mathcal{E}_x \to 0, \tag{6}$$

for all $x \in X_s$, where $\mathcal{E}_x^i$ (resp. $\mathcal{E}_x$) is the stalk of $\mathcal{E}^i$ (resp. $\mathcal{E}$) at x. Since $\mathcal{E}_x$ is flat over $\mathcal{O}_{S,s}$ by assumption and $\mathcal{E}_x^i$ is flat over $\mathcal{O}_{S,s}$ by the local freeness of $\mathcal{E}^i$, the exactness of

(6) is preserved by tensoring any $\mathcal{O}_{S,s}$-module. In particular, the functor $-\otimes_{\mathcal{O}_{S,s}} k(s)$ preserves the exactness of (6), i.e. one has the following exact sequence of $k(s)$-modules

$$0 \to \mathcal{E}_x^m \otimes_{\mathcal{O}_{S,s}} k(s) \to \cdots \to \mathcal{E}_x^0 \otimes_{\mathcal{O}_{S,s}} k(s) \to \mathcal{E}_x \otimes_{\mathcal{O}_{S,s}} k(s) \to 0, \tag{7}$$

According the following Cartesian squares

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{O}_{X_s,x}) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[d] \\
X_s \arrow[r] \arrow[d, "f_s"'] & X \arrow[d, "f"] \\
s \arrow[r] & S
\end{tikzcd}$$

one has $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} k(s)$. Hence $-\otimes_{\mathcal{O}_{S,s}} k(s) = -\otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_s,x}$ for all $\mathcal{O}_{X,x}$-modules. In particular for any $\mathcal{O}_X$-module $\mathcal{F}$, one has canonical isomorphisms of $k(s)$-modules

$$\mathcal{F}_x \otimes_{\mathcal{O}_{S,s}} k(s) \xrightarrow{=} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_s,x} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s,x} = \mathcal{F}\mid_{X_s} \otimes_{\mathcal{O}_{X_s}} \mathcal{O}_{X_s,x} = (\mathcal{F}\mid_{X_s})_x.$$

Thus the exactness of (7) implies the exactness of (5). □

7. A Lefschetz-style theorem for morphisms of Higgs bundle, after Arapura

In this section, we temporarily change the notation. Let Y/k be a d-dimensional smooth projective variety defined over an algebraically closed field. Let S be a normal crossing divisor. We first review a vanishing theorem of Arapura.

Recall that a logarithmic Higgs bundle over (Y, S) is a vector bundle E over Y together with an $\mathcal{O}_Y$-linear map

$$\theta\colon E \to E \otimes \Omega^1_Y(\log S)$$

such that $\theta \wedge \theta = 0$. This integrability condition induces a *de Rham complex*

$$DR(E, \theta) = (0 \to E \xrightarrow{\theta} E \otimes \Omega^1_Y(\log S) \xrightarrow{\theta} E \otimes \Omega^2_Y(\log S) \xrightarrow{\theta} \cdots).$$

We set the *Higgs cohomology* to be:

$$H^*_{\mathrm{Hig}}(Y, (E, \theta)) := \mathbb{H}(Y, DR(E, \theta))$$

The following is a fundamental vanishing theorem due to Arapura.

Theorem 7.1. [Ara19, Theorem 1 on p.297] *Let (E, θ) be a nilpotent semistable Higgs bundle on (Y, S) with vanishing Chern classes in $H^*(Y_{et}, \mathbb{Q}_\ell)$. Let L be an ample line bundle on Y. Suppose that either*

(a) $\mathrm{char}(k) = 0$, *or*
(b) $\mathrm{char}(k) = p$, *(Y, S) is liftable modulo p^2*, $d + \mathrm{rank}\, E < p$.

Then

$$\mathbb{H}^i(Y, DR(E, \theta) \otimes L) = 0$$

for $i > d$.

As Arapura notes, all one really needs to assume is that $c_1(E) = 0$ and $c_2(E).L^{d-2} = 0$ in $H^*(Y_{et}, \mathbb{Q}_\ell)$.

Note that if (E, θ) is semistable with vanishing Chern classes, then so is the dual. By Grothendieck-Serre duality, one immediately deduces [Ara19, Lemma 4.3]

$$H^i\Big(Y, DR(E,\theta) \otimes L^\vee\big(-S\big)\Big) = 0$$

for any $i < d$.

Remark 7.2. We claim the nilpotence condition in Theorem 7.1 may be dropped. For any semistable Higgs bundle (E, θ) with trivial Chern classes, the limit $\lim_{t \to 0}(E, t\theta)$ is a graded semistable Higgs bundle with trivial Chern classes. As the limit is graded, the Higgs field is nilpotent and hence Theorem 7.1 directly applies to this limit. By upper semicontinuity of cohomology, Theorem 7.1 in fact holds for (E, θ). Arapura's result also clearly holds over non algebraically closed fields.

We use Theorem 7.1 to prove a Lefschetz theorem.

Setup 7.3. Let Y/k be a smooth projective variety over a perfect field of characteristic p and of dimension d. Let $S \subset Y$ be a simple normal crossings divisor (possibly empty). Let $D \subset Y$ be a smooth ample divisor that meets S transversely and such that $\mathcal{O}(D-S)$ is also ample.

We suppose that (Y, S) has a lifting $(\widetilde{Y}, \widetilde{S})$ over $W_2(k)$. We may define $\widetilde{D} \subset \widetilde{Y}$ to have the same topological space as D and the structure sheaf induced from $\mathcal{O}_{\widetilde{Y}}$.

Let (E, θ) be a logarithmic Higgs bundle over Y The Higgs field $\theta\mid_D$ on $E\mid_D$ is defined as the composite map as in the following diagram:

$$\begin{tikzcd} E\mid_D \arrow[r, "j^*\theta"] \arrow[dr, "\theta|_D"'] & E\mid_D \otimes j^*\Omega^1_Y \arrow[d, "\mathrm{id}\otimes \mathrm{d}j"] \\ & E\mid_D \otimes \Omega^1_D \end{tikzcd}$$

Then one has the following result.

Lemma 7.4. *Setup as in 7.3. Let (E, θ) be a semistable logarithmic Higgs bundle on Y of rank r with trivial Chern classes and semistable restriction on D. Suppose further that $d + r \le p$. Then the restriction functor induces isomorphisms*

$$res\colon H^i_{\mathrm{Hig}}(Y, (E,\theta)) \xrightarrow{\sim} H^i_{\mathrm{Hig}}(D, (E,\theta)\mid_D)$$

for all $i \le d-2$ and an injection

$$res\colon H^{d-1}_{\mathrm{Hig}}(Y, (E,\theta)) \hookrightarrow H^{d-1}_{\mathrm{Hig}}(D, (E,\theta)\mid_D)$$

Corollary 7.5. *Setup as in 7.3. Let (E, θ) and $(E, \theta)'$ be two semistable logarithmic Higgs bundles over Y of rank r and r' respectively, where $d + rr' \le p$. Suppose further that both Higgs bundles have trivial $\mathbb{Q}_l$ Chern classes and semistable restrictions to D. Then one has*

(1) *an isomorphism*

$$\mathrm{Hom}((E,\theta),(E,\theta)') \simeq \mathrm{Hom}((E,\theta)\mid_D, (E,\theta)'\mid_D)$$

(2) *an injection*

$$H^1_{\mathrm{Hig}}(Y, \mathcal{H}\mathrm{om}((E,\theta),(E,\theta)')) \hookrightarrow H^1_{\mathrm{Hig}}(D, \mathcal{H}\mathrm{om}((E,\theta)\mid_D,(E,\theta)'\mid_D)).$$

Proof. Denote $(\mathcal{E},\Theta) := \mathcal{H}\mathrm{om}((E,\theta),(E,\theta)')$. Then

$$(\mathcal{E},\Theta)\mid_D \simeq \mathcal{H}\mathrm{om}((E,\theta)\mid_D,(E,\theta)'\mid_D).$$

Note that (E,θ) and $(E,\theta)'$ are both strongly semistable as $r,r'\leq p$.[7] It follows that $(\mathcal{E},\Theta)$ is also strongly semistable and hence semistable. Similarly, $(\mathcal{E},\Theta)\mid_D$ is also semistable. Then the result follows Lemma 7.4. $\square$

Proof of Lemma 7.4. In the following, we need to show the restrict functor induces an isomorphism between this two Higgs cohomology groups. According Arapura's theorem 7.1, one has

$$\mathbb{H}^i(Y, DR(E,\theta)\otimes\mathcal{O}_X(-D)) = 0,$$

for all $i<d$. The following exact sequence of complexes

$$\begin{array}{ccccccccccc}
0 & & & & 0 & & 0 & & 0 & & \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR(E,\theta)\otimes\mathcal{O}_Y(-D) & = & 0 & \longrightarrow & E(-D) & \xrightarrow{\theta} & E(-D)\otimes\Omega^1(\log S) & \xrightarrow{\theta} & E(-D)\otimes\Omega^2(\log S) & \xrightarrow{\theta} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR(E,\theta) & = & 0 & \longrightarrow & E & \xrightarrow{\theta} & E\otimes\Omega^1(\log S) & \xrightarrow{\theta} & E\otimes\Omega^2(\log S) & \xrightarrow{\theta} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR(E,\theta)\otimes j_*\mathcal{O}_D & = & 0 & \longrightarrow & j_*(E\mid_D) & \xrightarrow{\theta} & j_*(E\mid_D)\otimes\Omega^1(\log S) & \xrightarrow{\theta} & j_*(E\mid_D)\otimes\Omega^2(\log S) & \xrightarrow{\theta} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & & & 0 & & 0 & & 0 & &
\end{array}$$

induces

$$H^i_{\mathrm{Hig}}(Y,(E,\theta)) \xrightarrow{\sim} \mathbb{H}^i(DR(E,\theta)\otimes j_*\mathcal{O}_D).$$

for all $i\leq d-2$ and an injection

$$H^{d-1}_{\mathrm{Hig}}(Y,(E,\theta)) \hookrightarrow \mathbb{H}^{d-1}(DR(E,\theta)\otimes j_*\mathcal{O}_D).$$

On the other hand, we have the

$$\mathbb{H}^i(DR((E,\theta)\mid_D)\otimes\mathcal{I}/\mathcal{I}^2[-1]) = \mathbb{H}^{i-1}(DR((E,\theta)\mid_D)\otimes\mathcal{I}/\mathcal{I}^2) = 0,$$

for all $i<d$. This is because that $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}(-D)\mid_D$ is more negative than $\mathcal{O}_D(-D\cap S)$. The long exact sequence of the following exact sequence of complexes

$$\begin{array}{ccccccccccc}
0 & & & & 0 & & 0 & & 0 & & \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR((E,\theta)\mid_D)\otimes\mathcal{I}/\mathcal{I}^2[-1] & = & 0 & \longrightarrow & 0 & \longrightarrow & E\mid_D\otimes\mathcal{I}/\mathcal{I}^2 & \xrightarrow{\theta\mid_D} & E\mid_D\otimes\Omega^1_D(\log S)\otimes\mathcal{I}/\mathcal{I}^2 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR(E,\theta)\mid_D & = & 0 & \longrightarrow & E\mid_D & \xrightarrow{j^*\theta} & E\mid_D\otimes j^*\Omega^1(\log S) & \xrightarrow{j^*\theta} & E\mid_D\otimes j^*\Omega^2(\log S) & \xrightarrow{j^*\theta} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR((E,\theta)\mid_D) & = & 0 & \longrightarrow & E\mid_D & \xrightarrow{\theta\mid_D} & E\mid_D\otimes\Omega^1_D(\log S) & \xrightarrow{\theta\mid_D} & E\mid_D\otimes\Omega^2_D(\log S) & \xrightarrow{\theta\mid_D} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & & & 0 & & 0 & & 0 & &
\end{array}$$

[7] Following [LSZ19], we call a Higgs bundle semistable if it initiates a Higgs-de Rham flow where all terms are Higgs semistable and defined over a common finite field. Then the strongly semistability of t (E,θ) and $(E,\theta)'$ follows from Footnote 6.

induces

$$\mathbb{H}^i(DR(E,\theta)\mid_D) \xrightarrow{\sim} H^i_{\mathrm{Hig}}(D,(E,\theta)\mid_D).$$

for all $i \le d-2$ and an injection

$$\mathbb{H}^{d-1}(DR(E,\theta)\mid_D) \hookrightarrow H^{d-1}_{\mathrm{Hig}}(D,(E,\theta)\mid_D).$$

Then the lemma follows that

$$\mathbb{H}^i(DR(E,\theta)\otimes j_*\mathcal{O}_D) \xrightarrow{\sim} \mathbb{H}^i(DR(E,\theta)\mid_D) \qquad \square$$

8. The Higgs bundle $(E_{\mathcal{X}},\theta_{\mathcal{X}})$ is graded.

Proposition 8.1. *Setup as in Theorem 1.4. Let (E_{X_K},θ_{X_K}) be the logarithmic Higgs bundle attached to ρ_X in Section 4. Let $(E_{\mathcal{X}},\theta_{\mathcal{X}})$ be the Higgs bundle constructed in Theorem 5.1. Then the Higgs bundle $(E_{\mathcal{X}},\theta_{\mathcal{X}})$ is graded.*

To prove this, recall that a Higgs bundle (E,θ) is graded if and only if it is invariant under the $\mathbb{G}_m$ action. i.e.

$$(E,\theta)\simeq(E,t\theta) \quad \text{ for all } t\in\mathbb{G}_m.$$

Proof. Recall that $(E_{\mathcal{D}},\theta_{\mathcal{D}})$ is graded. For any $t\in\mathbb{G}_m$, one has

$$f_{\mathcal{D}}\colon (E_{\mathcal{D}},\theta_{\mathcal{D}})\simeq(E_{\mathcal{D}},t\theta_{\mathcal{D}}).$$

By Corollary 7.5, one gets $f_{X_1}\colon (E_{X_1},\theta_{X_1})\simeq(E_{X_1},t\theta_{X_1})$. Consider the obstruction c to lift f_{X_1} to W_2, which is located in

$$c\in H^1_{\mathrm{Hig}}\Big(\mathcal{H}\mathrm{om}\big((E_{X_1},\theta_{X_1}),(E_{X_1},t\theta_{X_1})\big)\Big).$$

By Corollary 7.5, one has an injective map

$$res\colon H^1_{\mathrm{Hig}}\Big(\mathcal{H}\mathrm{om}\big((E_{X_1},\theta_{X_1}),(E_{X_1},t\theta_{X_1})\big)\Big)\hookrightarrow H^1_{\mathrm{Hig}}\Big(\mathcal{H}\mathrm{om}\big((E_{D_1},\theta_{D_1}),(E_{D_1},t\theta_{D_1})\big)\Big).$$

Since f_{D_1} is liftable, the image of c under res vanishes. Thus $c=0$ and there is lifting of f_{X_1}

$$f'_{X_2}\colon (E_{X_2},\theta_{X_2})\longrightarrow(E_{X_2},t\theta_{X_2}).$$

In general $f'_{X_2}\mid_{D_2}\neq f_{\mathcal{D}}\mid_{D_2}$. We consider the difference $c'\in H^0_{Hig}(D_1,\mathrm{End}(E_{D_1},\theta_{D_1}))$ between $f'_{X_2}\mid_{D_2}$ and $f_{\mathcal{D}}\mid_{D_2}$. Since the restriction map in 1) of proposition 7.5 is bijection, we has a unique preimage $res^{-1}(c')$ of c'. Then we get a new isomorphism

$$f_{X_2}\colon (E_{X_2},\theta_{X_2})\longrightarrow(E_{X_2},t\theta_{X_2}).$$

by modifying the lifting f'_{X_2} via $res^{-1}(c')$. Inductively on the truncated level, we can lift f_{X_1} to an unique isomorphism $f_{\mathcal{X}}\colon (E_{\mathcal{X}},\theta_{\mathcal{X}})\simeq(E_{\mathcal{X}},t\theta_{\mathcal{X}})$ such that $f_{\mathcal{X}}\mid_D \mathcal{D}=f_{\mathcal{D}}$. Thus $(E_{\mathcal{X}},\theta_{\mathcal{X}})$ is graded. $\square$

9. Higgs-de Rham flow HDF_{X_1} with initial term $(E_{X_1},\theta_{X_1})_0$

Setup as in Question 1.3, and assume that

(1) $N^2<p-\dim(X)$ and
(2) the representation ρ_X is residually irreducible.

By the work in Section 8, there is a graded logarithmic Higgs bundle $(E_{\mathcal{X}}, \theta_{\mathcal{X}})$ such that (E_{X_1}, θ_{X_1}) and (E_{X_K}, θ_{X_K}) are semistable. By the setup of Question 1.3, we assume that the p-adic Higgs bundle $(E_{\mathcal{D}}, \theta_{\mathcal{D}})$ is periodic. As explained above, our goal is to prove that $(E_{\mathcal{X}}, \theta_{\mathcal{X}})$ is p-adically periodic and that the nilpotence of each term in the attached Higgs-de Rham flow is $\leq p-2$. We first will address the periodicity of (E_{X_1}, θ_{X_1}).

Since $(E_{X_1}, \theta_{X_1})_0 = (E_{\mathcal{X}}, \theta_{\mathcal{X}})_0 \mid_{X_1}$ is a graded semistable (logarithmic) Higgs bundle with trivial Chern classes and $N \leq p$, it is preperiodic by Lan-Sheng-Zuo [LSZ19, Theorem 1.5] (and Footnote 6), i.e., there exists a preperiodic Higgs-de Rham flow HDF_{X_1} with initial term $(E_{X_1}, \theta_{X_1})_0$. Restrict onto D_1, then one gets a preperiodic Higgs-de Rham flow $\mathrm{HDF}_{X_1} \mid_{D_1}$ with initial term $(E_{D_1}, \theta_{D_1})'_0 := (E_{X_1}, \theta_{X_1}) \mid_{D_1}$

(8)
$$\begin{tikzcd}[column sep=small]
& (V_{D_1}, \nabla_{D_1}, \mathrm{Fil}_{D_1})'_0 \arrow[dr] & & (V_{D_1}, \nabla_{D_1}, \mathrm{Fil}_{D_1})'_1 \arrow[dr] & & \cdots \\
(E_{D_1}, \theta_{D_1})'_0 \arrow[ur] & & (E_{D_1}, \theta_{D_1})'_1 \arrow[ur] & & (E_{D_1}, \theta_{D_1})'_2 \arrow[ur] &
\end{tikzcd}$$

We will prove this is isomorphic to the original Higgs-de Rham flow over D_1. By Lan-Sheng-Zuo [LSZ19, Theorem 1.5], $(E_{D_1}, \theta_{D_1})'_0$ is semistable.[8] There are two extensions $(E_{\mathcal{D}}, \theta_{\mathcal{D}})_0$ and $(E_{\mathcal{X}}, \theta_{\mathcal{X}})_0 \mid_D$ of $(E_{\mathcal{D}_K}, \theta_{\mathcal{D}_K})_0$. Since one extension has stable reduction over X_1 and the other one has semistable reduction over X_1, it follows that

$$(E_{\mathcal{D}}, \theta_{\mathcal{D}})_0 \simeq (E_{\mathcal{X}}, \theta_{\mathcal{X}})_0 \mid_D$$

by the following Langton-style lemma.

Lemma 9.1. *(Lan-Sheng-Zuo) Let R be a discrete valuation ring with fraction field K and residue field k. Let X/R be a smooth projective scheme and let $S \subset R$ be a relative (strict) normal crossings divisor. Let $(E_1, \theta_1)_X$ and $(E_2, \theta_2)_X$ be logarithmic Higgs bundle on X that are isomorphic over K. Suppose that $(E_1, \theta_1)_{X_k}$ is semistable and $(E_2, \theta_2)_{X_k}$ is stable. Then $(E_1, \theta_1)_X$ and $(E_2, \theta_2)_X$ are isomorphic.*

Proof. The non-logarithmic version may be found as Lemma 6.4 in 1311.6424v1. The argument in the logarithmic setting requires no changes. ☐

Now, we identify $(E_{\mathcal{D}}, \theta_{\mathcal{D}})_0$ with $(E_{\mathcal{X}}, \theta_{\mathcal{X}})_0 \mid_D$ with this isomorphism. In particular, one has

$$(E_{D_1}, \theta_{D_1})'_0 = (E_{\mathcal{X}}, \theta_{\mathcal{X}})_0 \mid_{D_1} = (E_{\mathcal{D}}, \theta_{\mathcal{D}})_0 \mid_{D_1} =: (E_{D_1}, \theta_{D_1})_0.$$

On the other hand, one has periodic Higgs-de Rham flow $\mathrm{HDF}_{\mathcal{D}} \mid_{D_1}$ with initial term $(E_{D_1}, \theta_{D_1})_0$

(9)
$$\begin{tikzcd}[column sep=small]
& (V_{D_1}, \nabla_{D_1}, \mathrm{Fil}_{D_1})_0 \arrow[dr] & & (V_{D_1}, \nabla_{D_1}, \mathrm{Fil}_{D_1})_1 \arrow[dr] & & \cdots \\
(E_{D_1}, \theta_{D_1})_0 \arrow[ur] & & (E_{D_1}, \theta_{D_1})_1 \arrow[ur] & & (E_{D_1}, \theta_{D_1})_2 \arrow[ur] &
\end{tikzcd}$$

Lemma 9.2. $\mathrm{HDF}_{X_1} \mid_{D_1} = \mathrm{HDF}_{\mathcal{D}} \mid_{D_1}$.

Proof. Since $(E_{D_1}, \theta_{D_1})_0$ is stable and $\mathrm{HDF}_{\mathcal{D}} \mid_{D_1}$ is periodic, all terms appeared in $\mathrm{HDF}_{\mathcal{D}} \mid_{D_1}$ are stable. This is because

- $(V_{D_1}, \nabla_{D_1})_i$ is stable if and only if $(E_{D_1}, \theta_{D_1})_i$ is stable;

[8] Strictly speaking, [LSZ19, Theorem 1.5] only deals with the non-logarithmic case. However the argument immediately carries to the logarithmic case, as follows. As $(E_{X_1}, \theta_{X_1})'_0$ is preperiodic, so is $(E_{D_1}, \theta_{D_1})'_0$. Preperiodic logarithmic Higgs bundles with trivial Chern classes are automatically semistable, see also Footnote 6.

- If $(E_{D_1},\theta_{D_1})_{i+1}$ is stable, then $(V_{D_1},\nabla_{D_1})_i$ is stable.

Since Cartier inverse functor is compatible with restriction, one has $(V_{D_1},\nabla_{D_1})_0' = (V_{D_1},\nabla_{D_1})_0$. By [LSZ19, Lemma 7.1][9], $\mathrm{Fil}'_{D_1,0} = \mathrm{Fil}_{D_1,0}$. Taking the associated graded, one gets $(E_{D_1},\theta_{D_1})_1' = (E_{D_1},\theta_{D_1})_1$. Inductively, one shows that $\mathrm{HDF}_{X_1} \mid_{D_1} = \mathrm{HDF}_{\mathcal{D}} \mid_{D_1}$. □

By the fully faithfulness of the restriction functor as in Lemma 7.4, the map $\varphi_{D_1}\colon (E_{D_1},\theta_{D_1})_f \xrightarrow{\sim} (E_{D_1},\theta_{D_1})_0$ that witnesses the periodicity of HDF_{D_1} can be lifted canonically to a map $\varphi_{X_1}\colon (E_{X_1},\theta_{X_1})_f \xrightarrow{\sim} (E_{X_1},\theta_{X_1})_0$. This implies that the Higgs-de Rham flow HDF_{X_1} is also f-periodic.

10. Higgs-de Rham flow $\mathrm{HDF}_{\mathcal{X}}$ with initial term $(E_{\mathcal{X}},\theta_{\mathcal{X}})_0$

In this section, we use two results of Krishnamoorthy-Yang-Zuo [KYZ20] to lift HDF_{X_1} onto $\mathcal{X}$.

Proposition 10.1. *There is an unique f-periodic Higgs-de Rham flow $HDF_{\mathcal{X}}$ over $\mathcal{X}$, which*

- *lifts HDF_{X_1},*
- *with initial term $(E_{\mathcal{X}},\theta_{\mathcal{X}})$ and satisfying*
- *$HDF_{\mathcal{X}} \mid_{\mathcal{D}} \simeq HDF_{\mathcal{D}}$.*

We prove this result inductively on the truncated level; in particular, we may assume we have already lifted HDF_{X_1} to an f-periodic Higgs-de Rham flow HDF_{X_n} over X_n, where $n \geq$ is a positive integer:

(10)
$$\begin{tikzcd}
 & (V_{X_n},\nabla_{X_n},\mathrm{Fil}_{X_n})_0 \arrow[dr] & & (V_{X_n},\nabla_{X_n},\mathrm{Fil}_{X_n})_{f-1} \arrow[dr] & \\
(E_{X_n},\theta_{X_n})_0 \arrow[ur] & & \cdots \arrow[ur] & & (E_{X_n},\theta_{X_n})_f \arrow[llll, bend left, "\varphi_n"]
\end{tikzcd}$$

satisfying $HDF_{X_n} \mid_{D_n} \simeq HDF_{\mathcal{D}} \mid_{D_n}$ and with initial term (E_{X_n},θ_{X_n}). We only need to lift HDF_{X_n} to an f-periodic Higgs-de Rham flow $HDF_{X_{n+1}}$ over X_{n+1} satisfying $HDF_{X_{n+1}} \mid_{D_{n+1}} \simeq HDF_{\mathcal{D}} \mid_{D_{n+1}}$ and with initial term (E_{X_n},θ_{X_n}) as following.

First, taking the $(n+1)$-truncated level Cartier inverse functor, one gets

$$(V_{X_{n+1}},\nabla_{X_{n+1}})_0 := C_{n+1}^{-1}((E_{X_{n+1}},\theta_{X_{n+1}})_0,(V_{X_n},\nabla_{X_n},\mathrm{Fil}_{X_n})_{f-1},\varphi_n)$$

which is a vector bundle over X_n together with an integrable connection. This filtration lifts $(V_{X_n},\nabla_{X_n})_0$ and satisfies $(V_{X_{n+1}},\nabla_{X_{n+1}})_0 \mid_{D_{n+1}} = (V_{\mathcal{D}},\nabla_{\mathcal{D}})_0 \mid_{D_{n+1}}$.

Proposition 10.2. *Notation as above.*

1. *There is a unique Hodge (Griffith transversal) filtration* $\mathrm{Fil}_{X_{n+1},0}$ *on* $(V_{X_{n+1}},\nabla_{X_{n+1}})$, *which lifts* Fil_{X_n} *and satisfying* $\mathrm{Fil}_{X_{n+1}} \mid_{D_{n+1}} = \mathrm{Fil}_{\mathcal{D}} \mid_{D_{n+1}}$.
2. *Taking the associated graded, we obtain a Higgs bundle*

$$(E_{X_{n+1}},\theta_{X_{n+1}}) = \mathrm{Gr}(V_{X_{n+1}},\nabla_{X_{n+1}},\mathrm{Fil}_{X_{n+1}}),$$

[9] The argument in *loc. cit* is written for flat connections, but the proof extends more generally to logarithmic flat connections.

which lifts (E_{X_n}, θ_{X_n}) and satisfies

$$(E_{X_{n+1}}, \theta_{X_{n+1}}) \mid_{D_{n+1}} = (E_{D_{n+1}}, \theta_{D_{n+1}}).$$

To prove this result, we need a result of Krishnamoorthy-Yang-Zuo [KYZ20] about the obstruction to lifting the Hodge filtration. We recall this result.

Let $(V, \nabla, \mathrm{F}^*)$ be a filtered de Rham bundle over X_n, i.e. a vector bundle bundle together with a flat connection with logarithmic singularities and a filtration by subbundles that satisfies Griffiths transversality over X_n . Denote its modulo p reduction by $(\overline{V}, \overline{\nabla}, \overline{\mathrm{F}}^*)$. Let $(\widetilde{V}, \widetilde{\nabla})$ be a lifting of the flat bundle (V, ∇) on X_{n+1}. The $\overline{\mathrm{F}}^*$ induces a Hodge filtration $\overline{\mathrm{Fil}}^*$ on $(\mathrm{End}(\overline{V}), \overline{\nabla}^{\mathrm{End}})$ defined by

$$\overline{\mathrm{Fil}}^\ell \mathrm{End}(\overline{V}) = \sum_{\ell_1} (\overline{V}/\overline{\mathrm{F}}^{\ell_1})^\vee \otimes \overline{\mathrm{F}}^{\ell_1+\ell-1}.$$

As $(\mathrm{End}(\overline{V}), \overline{\nabla}^{\mathrm{End}}, \overline{\mathrm{Fil}}^*)$ satisfies Griffith transversality, the de Rham complex $(\mathrm{End}(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\nabla}^{\mathrm{End}})$ induces the following complex denoted by $\mathscr{C}$

$$0 \to \mathrm{End}(\overline{V})/\overline{\mathrm{Fil}}^0\mathrm{End}(\overline{V}) \xrightarrow{\overline{\nabla}^{\mathrm{End}}} \mathrm{End}(\overline{V})/\overline{\mathrm{Fil}}^{-1}\mathrm{End}(\overline{V}) \otimes \Omega^1_{X_1} \xrightarrow{\overline{\nabla}^{\mathrm{End}}} \mathrm{End}(\overline{V})/\overline{\mathrm{Fil}}^{-2}\mathrm{End}(\overline{V}) \otimes \Omega^2_{X_1} \to \cdots$$

which is the quotient of $(\mathrm{End}(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\nabla}^{\mathrm{End}})$ by the sub complex $(\overline{\mathrm{Fil}}^{-*}\mathrm{End}(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\nabla}^{\mathrm{End}})$, i.e., we have the following exact sequence of complexes of sheaves over X_1:

$$0 \to (\overline{\mathrm{Fil}}^{-*}\mathrm{End}(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\nabla}^{\mathrm{End}}) \to (\mathrm{End}(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\nabla}^{\mathrm{End}}) \to \mathscr{C} \to 0.$$

Denote $(\overline{E}, \overline{\theta}) = \mathrm{Gr}(\overline{V}, \overline{\nabla}, \overline{\mathrm{F}}^*)$. Then $(\mathrm{End}(\overline{E}), \overline{\theta}^{\mathrm{End}}) = \mathrm{End}((\overline{E}, \overline{\theta}))$ is also a graded Higgs bundle. Here is the key observation about the complex $\mathscr{C}$: it is a successive extension of direct summands of the Higgs complex of the graded Higgs bundle $(\mathrm{End}(\overline{E}), \overline{\theta}^{\mathrm{End}})$

$$\left\{ (\mathrm{End}(\overline{E})^{p-*,-p+*} \otimes \Omega^*_{X_1}, \overline{\theta}^{\mathrm{End}}) \middle| p = 0, 1, 2, \cdots \right\}.$$

Theorem 10.3. [KYZ20, Theorem 3.9] *Notation as above.*

(1) *The obstruction of lifting the filtration* F^* *onto* $(\widetilde{V}, \widetilde{\nabla})$ *with Griffiths transversality is located in* $\mathbb{H}^1(\mathscr{C})$.

(2) *If the above the obstruction vanishes, then the lifting space is an* $\mathbb{H}^0(\mathscr{C})$*-torsor.*

Associated to the filtered de Rham bundle $(V_{X_1}, \nabla_{X_1}, \mathrm{Fil}_{X_1})$ and $(V_{D_1}, \nabla_{D_1}, \mathrm{Fil}_{D_1}) = (V_{X_1}, \nabla_{X_1}, \mathrm{Fil}_{X_1}) \mid_{D_1}$, one has complex $\mathscr{C}_{X_1}$ of sheaves over X_1

$$0 \to \mathrm{End}(V_{X_1})/\overline{\mathrm{Fil}}^0\mathrm{End}(V_{X_1}) \xrightarrow{\nabla^{\mathrm{End}}_{X_1}} \mathrm{End}(V_{X_1})/\overline{\mathrm{Fil}}^{-1}\mathrm{End}(V_{X_1}) \otimes \Omega^1_{X_1} \xrightarrow{\nabla^{\mathrm{End}}_{X_1}} \cdots$$

and complex $\mathscr{C}_{D_1}$ of sheaves over D_1

$$0 \to \mathrm{End}(V_{D_1})/\overline{\mathrm{Fil}}^0\mathrm{End}(V_{D_1}) \xrightarrow{\nabla^{\mathrm{End}}_{D_1}} \mathrm{End}(V_{D_1})/\overline{\mathrm{Fil}}^{-1}\mathrm{End}(V_{D_1}) \otimes \Omega^1_{D_1} \xrightarrow{\nabla^{\mathrm{End}}_{D_1}} \cdots$$

satisfying

$$\mathscr{C}_{D_1} = \mathscr{C}_{X_1} \mid_{D_1} .$$

Proposition 10.4. *The restriction induces*

1). *an isomorphism* $\mathbb{H}^0(\mathscr{C}_{X_1}) \xrightarrow{\sim} \mathbb{H}^0(\mathscr{C}_{D_1})$*, and*
2). *an injection* $\mathbb{H}^1(\mathscr{C}_{X_1}) \hookrightarrow \mathbb{H}^1(\mathscr{C}_{D_1})$.

Proof. Since the complex $\mathscr{C}$ is a successive extension of direct summands of the Higgs complex, one gets the results by Lemma 7.4 and the five lemma. $\square$

Proof of Proposition 10.2. The uniqueness of the Hodge filtration follows Theorem 10.3 and Proposition 10.4. For the existence, we denote by $c \in \mathbb{H}^1(\mathscr{C}_{X_1})$ the obstruction to lift Fil_{X_n} onto X_{n+1}. Since Fil_{D_n} is liftable, the image $res(c) \in \mathbb{H}^1(\mathscr{C}_{D_1})$ of c under res vanishes. Since res is an injection, $c = 0$ and $\mathrm{Fil}_{\mathcal{D}} \mid_{D_{n+1}}$ is also liftable. We choose a lifting $\mathrm{Fil}'_{X_{n+1}}$ and denote by $c' \in \mathbb{H}^0(\mathscr{C}_{D_1})$ the difference between $\mathrm{Fil}'_{X_{n+1}} \mid_{D_{n+1}}$ and $\mathrm{Fil}_{\mathcal{D}} \mid_{D_{n+1}}$. Since res is an isomorphism, one uses $res^{-1}(c')$ to modify the original filtration $\mathrm{Fil}'_{X_{n+1}}$. Then one gets a new filtration $\mathrm{Fil}_{X_{n+1}}$ such that $\mathrm{Fil}_{X_{n+1}} \mid_{D_{n+1}} = \mathrm{Fil}_{\mathcal{D}} \mid_{D_{n+1}}$.

2) follows 1) directly. □

Run the Higgs-de Rham flow with initial term $((E_{X_{n+1}}, \theta_{X_{n+1}})_0, (V_{X_n}, \nabla_{X_n}, \mathrm{Fil}_{X_n})_{f-1}, \varphi_n)$ together with the Hodge filtrations constructed as in Proposition 10.2. Then one constructs a Higgs-de Rham flow $HDF_{X_{n+1}}$ over X_{n+1}

$$\begin{tikzcd}[column sep=small] & (V_{X_{n+1}}, \nabla_{X_{n+1}}, \mathrm{Fil}_{X_{n+1}})_0 \arrow[dr] & & (V_{X_{n+1}}, \nabla_{X_{n+1}}, \mathrm{Fil}_{X_{n+1}})_{f-1} \arrow[dr] & & \cdots \\ (E_{X_{n+1}}, \theta_{X_{n+1}})_0 \arrow[ur] & & \cdots \arrow[ur] & & (E_{X_{n+1}}, \theta_{X_{n+1}})_f \arrow[ur] & \end{tikzcd} \tag{11}$$

satisfying

- with initial term $((E_{X_{n+1}}, \theta_{X_{n+1}})_0, (V_{X_n}, \nabla_{X_n}, \mathrm{Fil}_{X_n})_{f-1}, \varphi_n)$;
- $HDF_{X_{n+1}} \mid_{D_{n+1}} \simeq HDF_{\mathcal{D}} \mid_{D_{n+1}}$.

For the last step, we need to show this lifting flow is f-periodic. In other words, we need the following result.

Lemma 10.5. *There exists an isomorphism*

$$\varphi_{X_{n+1}} \colon (E_{X_{n+1}}, \theta_{X_{n+1}})_f \xrightarrow{\sim} (E_{X_{n+1}}, \theta_{X_{n+1}})_0$$

which lifts φ_{X_n} and satisfying $\varphi_{X_{n+1}} \mid_{D_{n+1}} = \varphi_{D_{n+1}}$.

To prove this result, we need another result on the obstruction class to lifting a Higgs bundle.

Theorem 10.6. [KYZ20, Proposition 4.2] *Let (E, θ) be a logarithmic Higgs bundle over X_n. Denote $(\overline{E}, \overline{\theta}) = (E, \theta) \mid_{X_1}$. Then*

1). *if (E, θ) has lifting $(\widetilde{E}, \widetilde{\theta})$, then the lifting set is an $H^1_{Hig}(X_1, \mathrm{End}((\overline{E}, \overline{\theta})))$-torsor;*
2). *the infinitesimal automorphism group of $(\widetilde{E}, \widetilde{\theta})$ over (E, θ) is $H^0_{Hig}(X_1, \mathrm{End}((\overline{E}, \overline{\theta})))$.*

Directly from Corollary 7.5, one gets results.

Proposition 10.7. *The restriction induces*

1). *an isomorphism $res \colon H^0_{Hig}(X_1, \mathrm{End}(E_{X_1}, \theta_{X_1})) \xrightarrow{\sim} H^0_{Hig}(D_1, \mathrm{End}(E_{D_1}, \theta_{D_1}))$, and*
2). *an injection $res \colon H^1_{Hig}(X_1, \mathrm{End}(E_{X_1}, \theta_{X_1})) \hookrightarrow H^1_{Hig}(D_1, \mathrm{End}(E_{D_1}, \theta_{D_1}))$.*

Proof of Lemma 10.5. We identify $(E_{X_n}, \theta_{X_n})_0$ and $(E_{X_n}, \theta_{X_n})_f$ via φ_{X_n}. Since both $(E_{X_{n+1}}, \theta_{X_{n+1}})_0$ and $(E_{X_{n+1}}, \theta_{X_{n+1}})_f$ lift $(E_{X_n}, \theta_{X_n})_0$, they differ by an element

$$c \in H^1_{Hig}(X_1, \mathrm{End}(E_{X_1}, \theta_{X_1})).$$

Since $HDF_{X_{n+1}} \mid_{D_{n+1}}$ is f-periodic, one has $res(c) = 0 \in H^1_{Hig}(D_1, \mathrm{End}(E_{D_1}, \theta_{D_1}))$. By the injection of the restriction map in 2) of proposition 10.7, $c = 0$ and there is an isomorphism

$$\varphi'_{X_{n+1}} \colon (E_{X_{n+1}}, \theta_{X_{n+1}})_f \longrightarrow (E_{X_{n+1}}, \theta_{X_{n+1}})_0.$$

In general $\varphi'_{X_{n+1}}\mid_{D_{n+1}}\neq\varphi_{D_{n+1}}$. We consider the difference

$$c'\in H^0_{Hig}(D_1,\mathrm{End}(E_{D_1},\theta_{D_1}))$$

between $\varphi'_{X_{n+1}}\mid_{D_{n+1}}$ and $\varphi_{D_{n+1}}$. Since the restriction map in 1) of proposition 10.7 is bijection, we have a unique preimage $res^{-1}(c')$ of c'. We obtain a new isomorphism

$$\varphi_{X_{n+1}}\colon (E_{X_{n+1}},\theta_{X_{n+1}})_f\longrightarrow (E_{X_{n+1}},\theta_{X_{n+1}})_0.$$

by modifying the lifting $\varphi'_{X_{n+1}}$ via $res^{-1}(c')$. Then $\varphi_{X_{n+1}}$ satisfies our required property, i.e. $\varphi_{X_{n+1}}\mid_{D_{n+1}}=\varphi_{D_{n+1}}$. □

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