RANK 2 LOCAL SYSTEMS AND ABELIAN VARIETIES II

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Abstract. Let $X/\mathbb{F}_q$ be a smooth, geometrically connected, quasiprojective scheme. Let $\mathcal{E}$ be a semisimple overconvergent F-isocrystal on X. Suppose that irreducible summands $\mathcal{E}_i$ of $\mathcal{E}$ have rank 2, determinant $\bar{\mathbb{Q}}_p(-1)$, and infinite monodromy at ∞. Suppose further that for each closed point x of X, the characteristic polynomial of $\mathcal{E}$ at x is in $\mathbb{Q}[t] \subset \mathbb{Q}_p[t]$. Then there exists a non-trivial open set $U \subset X$ such that $\mathcal{E}|_U$ comes from a family of abelian varieties on U.

As an application, let L_1 be an irreducible lisse $\bar{\mathbb{Q}}_l$ sheaf on X that has rank 2, determinant $\bar{\mathbb{Q}}_l(-1)$, and infinite monodromy at ∞. Then all crystalline companions to L_1 exist (as predicted by Deligne's crystalline companions conjecture) if and only if there exists a non-trivial open set $U \subset X$ and an abelian scheme $\pi_U \colon A_U \to U$ such that $L_1|_U$ is a summand of $R^1(\pi_U)_*\bar{\mathbb{Q}}_l$.

1. Introduction

Throughout this article, p is a prime number and q is a power of p. If X/k is a smooth scheme over a perfect field of characteristic p, then $\mathbf{F\text{-}Isoc}^\dagger(X)$ denotes the category of overconvergent F-isocrystals on X and $\mathbf{F\text{-}Isoc}^\dagger(X)_{\overline{\mathbb{Q}}_p}$ denotes its $\overline{\mathbb{Q}}_p$-linearization. Overconvergent F-isocrystals are a p-adic analog of lisse l-adic sheaves.

Definition 1.1. Let X/k be a smooth, geometrically connected scheme over a perfect field k of characteristic p and let $\mathcal{E} \in \mathbf{F\text{-}Isoc}^\dagger(X)_{\overline{\mathbb{Q}}_p}$. We say that $\mathcal{E}$ has *infinite local monodromy at infinity* if for every triple $(X', \overline{X'}, f)$ where $\overline{X'}$ is smooth projective over k, $X' \subset \overline{X'}$ is a Zariski open subset, and $f \colon X' \to X$ is an alteration, the overconvergent F-isocrystal $f^*\mathcal{E}$ does not extend to an F-isocrystal on $\overline{X'}$.

This definition of infinite local monodromy at infinity applies equally well to lisse $\overline{\mathbb{Q}}_l$-sheaves and is compatible with the other notions of infinite local monodromy at infinity.

Theorem 1.2. *Let $X/\mathbb{F}_q$ be a smooth, geometrically connected, quasiprojective scheme. Let $\mathcal{E} \in$ $\boldsymbol{F\text{-}Isoc}^\dagger(X)$ be a semisimple overconvergent F-isocrystal. Suppose:*

- *for every closed point x of X, the polynomial $P_x(\mathcal{E}, t)$ has coefficients in $\mathbb{Q} \subset \mathbb{Q}_p$;*
- *every irreducible summand $\mathcal{E}_i \in \boldsymbol{F\text{-}Isoc}^\dagger(X)_{\overline{\mathbb{Q}}_p}$ of $\mathcal{E}$ has rank 2, determinant $\overline{\mathbb{Q}}_p(-1)$, and infinite local monodromy around infinity.*

Then $\mathcal{E}$ comes from a family of abelian varieties. More precisely, there exists a non-empty open subset $U \subset X$ and an abelian scheme $A_U \to U$, so that $\mathbb{D}(A_U[p^\infty]) \otimes \overline{\mathbb{Q}}_p \cong \mathcal{E}|_U$.

Here, if $G \to X$ is a p-divisible group, $\mathbb{D}(G)$ is the (contravariant) Dieudonné crystal attached to G. We have the following application. Deligne formulated what is now called the companions conjecture in [Del80, Conjecture 1.2.1]. For a guide to the crystalline companions conjecture, see [Ked16, Ked18].

Corollary 1.3. *Let $X/\mathbb{F}_q$ be a smooth, geometrically connected scheme. Let L_1 be an irreducible rank 2 lisse $\overline{\mathbb{Q}}_l$ sheaf with infinite monodromy around infinity and determinant $\overline{\mathbb{Q}}_l(-1)$. Then the following are equivalent:*

(1) there exists a non-empty open subset $U \subset X$ and an abelian scheme $\pi \colon A_U \to U$ such that L_1 is a summand of $R^1(\pi_U)_*\overline{\mathbb{Q}}_l$;
(2) all crystalline companions to L_1 exist (as predicted by Deligne's crystalline companions conjecture).

Corollary 1.4. *Let $X/\mathbb{F}_q$ be a smooth, geometrically connected scheme. Let $\mathcal{E}_1$ be an irreducible rank 2 object of $\textbf{F-Isoc}^{\dagger}(X)_{\overline{\mathbb{Q}}_p}$ with infinite monodromy around infinity and determinant $\overline{\mathbb{Q}}_p(-1)$. Suppose the (number) field $E_1 \subset \overline{\mathbb{Q}}_p$ generated by the coefficients of $P_x(\mathcal{E}_1, t)$ as x ranges through the closed points of X has a single prime over p. Then $\mathcal{E}_1$ comes from a family of abelian varieties: there exists a non-empty open subset $U \subset X$ and an abelian scheme $A_U \to U$ such that $\mathcal{E}_1|_U$ is a summand of $\mathbb{D}(A_U[p^\infty]) \otimes \overline{\mathbb{Q}}_p$.*

In particular, Corollaries 1.3 and 1.4 provide some evidence for a question of Drinfeld [Dri12, Question 1.4] and a conjecture of the authors [KP18, Conjecture 1.2]. Our motivation for formulating this conjecture was a celebrated theorem of Corlette-Simpson over $\mathbb{C}$ [CS08, Theorem 11.2], the proof of which uses non-abelian Hodge theory. In contrast to our earlier work [KP18], this article does not use Serre-Tate deformation theory nor does it use the algebraization/globalization techniques of [Har70].

We briefly sketch the proof. Drinfeld's first work on the Langlands correspondence for GL_2, together with Abe's work on the p-adic Langlands correspondence, implies Theorem 1.2 when $\dim(X) = 1$. To do the higher-dimensional case, we first assume that X admits a simple normal crossings compactification $\bar{X}$ and $\mathcal{E}$ is a logarithmic F-isocrystal with nilpotent residues. (We recall the notion of logarithmic F-isocrystals in Appendix A.) A technique of Katz, combined with slope bounds originally due to Lafforgue, allow one to construct a logarithmic Dieudonné crystal whose associated logarithmic F-isocrystal is isomorphic to $\mathcal{E}$. This logarithmic Dieudonné crystal yields a natural line bundle, which we call the *Hodge bundle* ω of the logarithmic Dieudonné crystal, on $\bar{X}$.

For any odd prime $l \neq p$, it is well-known that the Hodge line bundle is ample on $\mathscr{A}_{h,1,l}$ over $\mathrm{Spec}(\mathbb{Z}[1/l])$. We use Poonen's Bertini theorem over finite fields together with Drinfeld's result and Zarhin's trick to find a well-adapted family of extremely ample space-filling curves $\bar{C}_n$ of $\bar{X}$ that each map to the minimal compactification $\mathscr{A}^*_{h,1,l} \subset \mathbb{P}^m$ via some fixed power of the Hodge bundle $\omega|^r_{\bar{C}_n}$. (This step uses foundational work of Kato, Kedlaya, le Stum, and Trihan that we explain in Appendix A.) The finitude of the set $H^0(\bar{X}, \omega^r)$ ensures that infinitely many of these maps can be pieced together to a map $\bar{X} \dashrightarrow \mathscr{A}_{h,1,l} \subset \mathbb{P}^m$; hence we obtain an abelian scheme $\psi_U \colon B_U \to U$ over some open $U \subset X$. The space-filling properties of the $\bar{C}_n$ and Zarhin's work on the Tate isogeny theorem for fields finitely generated over $\mathbb{F}_q$ then allow us to conclude.

To deduce the general case, we use Kedlaya's semistable reduction theorem for overconvergent F-isocrystals.

2. Preliminaries

Before proving Theorem 1.2, we need several preliminary results. A key ingredient in the proof is the following, which is a byproduct of Drinfeld's first work on the Langlands correspondence for GL_2.

Theorem 2.1. *(Drinfeld) Let $C/\mathbb{F}_q$ be a smooth affine curve and let L_1 be a rank 2 irreducible $\overline{\mathbb{Q}}_l$ sheaf with determinant $\overline{\mathbb{Q}}_l(-1)$. Suppose L_1 has infinite local monodromy around some point at $\infty \in \overline{C} \backslash C$. Then L_1 comes from a family of abelian varieties in the following sense: let E be the field generated by the Frobenius traces of L_1 and suppose $[E : \mathbb{Q}] = g$. Then there exists an abelian scheme*

$$\pi_C \colon A_C \to C$$

of dimension g and an isomorphism $E \cong End_C(A) \otimes \mathbb{Q}$, realizing A_C as a GL_2-type abelian scheme, such that L_1 occurs as a summand of $R^1(\pi_C)_* \overline{\mathbb{Q}}_l$. Moreover, $A_C \to C$ is totally degenerate around ∞.

See [ST18, Proof of Proposition 19, Remark 20] for how to recover this result from Drinfeld's work. This amounts to combining [Dri83, Main Theorem, Remark 5] with [Dri77, Theorem 1].

We will also need the following useful lemma to ensure that, given the hypotheses of Theorem 1.2, every p-adic companion of $\mathcal{E}_i$ is again a summand of $\mathcal{E}$; moreover, the companion relation preserves multiplicity in the isotypic decomposition of $\mathcal{E}$.

Lemma 2.2. *Let $X/\mathbb{F}_q$ be a smooth, geometrically connected scheme.*

(1) *Let $l \neq p$ be a prime and let L be a lisse, semi-simple $\overline{\mathbb{Q}}_l$-sheaf on X, all of whose irreducible summands L_i have algebraic determinant. Suppose for all closed points x of X, we have:*

$$P_x(L,t) \in \mathbb{Q}[t] \subset \overline{\mathbb{Q}}_l[t].$$

Let L_i be an irreducible summand of L that occurs with multiplicity m_i and $\iota \in Aut_{\mathbb{Q}}(\overline{\mathbb{Q}}_l)$ be a field automorphism. Then the ι-companion to L_i, denoted ${}^{\iota}L_i$, is isomorphic to an irreducible summand of L that occurs with multiplicity m_i.

(2) *Let $\mathcal{F}$ be a semi-simple object of $\boldsymbol{F\text{-}Isoc}^{\dagger}(X)_{\overline{\mathbb{Q}}_p}$, all of whose irreducible summands $\mathcal{F}_i$ have algebraic determinant. Suppose for all closed points x of X, we have:*

$$P_x(\mathcal{F},t) \in \mathbb{Q}[t] \subset \overline{\mathbb{Q}}_p[t].$$

Let $\mathcal{F}_i$ be an irreducible summand of $\mathcal{F}$ that occurs with multiplicity m_i and let $\iota \in Aut_{\mathbb{Q}}(\overline{\mathbb{Q}}_p)$. Then the ι-companion of $\mathcal{F}_i$, denoted ${}^{\iota}\mathcal{F}_i$,exists and is isomorphic to a direct summand of $\mathcal{F}$ that occurs with multiplicity m_i.

Before beginning the proof, we note that l-adic companions are known the exist for smooth varieties over finite fields [Dri12, Theorem 1.1]. In contrast, p-adic companions are not yet known to exist except when X is a curve [Abe18], though Kedlaya has recently proposed a promising strategy.

Proof. We reduce the crystalline case to the étale case. (Note that we could have equivalently proceeded by reduction to curves using [AE19].) As $\mathcal{F}$ is semisimple, write an isotypic decomposition:

$$\mathcal{F} \cong \bigoplus_{i=1}^{a} \mathcal{F}_i^{m_i}.$$

Note that each $\mathcal{F}_i$ is pure by [AE19, Theorem 2.7]. Fix an isomorphism $\sigma\colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_l$. By [AE19, Theorem 4.2] or [Ked18, Corollary 3.5.3], the σ-companion to each $\mathcal{F}_i$ exists as an irreducible lisse $\overline{\mathbb{Q}}_l$-sheaf L_i. Setting L to be the semi-simple σ-companion of $\mathcal{F}$, we have:

$$L \cong \bigoplus_{i=1}^{a} L_i^{m_i}.$$

Set $\iota \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}}_p)$. Then $\mathcal{F}_j$ is the ι-companion to $\mathcal{F}_i$ if and only if L_j is the $\sigma \circ \iota \circ \sigma^{-1}$-companion to L_i. Therefore it suffices to prove the result in the étale setting.

Let M be an irreducible lisse $\overline{\mathbb{Q}}_l$-sheaf on X. Then M is pure by [Del12, Théorème 1.6] and class field theory. Then the multiplicity of M in the semisimple sheaf L is: $\dim(H^0(X, M^* \otimes L))$. By assumption we have that for all closed points x of X, $P_x(L,t) \in \mathbb{Q}[t] \subset \overline{\mathbb{Q}}_l[t]$. Let $\iota \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}}_l)$, and note that the semi-simple ι-companion to L is again isomorphic to L. Then the ι-companion to $M^* \otimes L$ is isomorphic to $({}^{\iota}M^*) \otimes L$. On the other hand, the exact argument of [AE19, 3.2] for lisse l-adic sheaves implies that $\dim(H^0(X, M^* \otimes L)) = \dim(H^0(X, {}^{\iota}(M^* \otimes L))$. Therefore $\dim(H^0(X, M^* \otimes L)) = \dim(H^0(X, ({}^{\iota}M^*) \otimes L))$, and the result follows.

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Remark 2.3. The argument of [AE19, 3.2] cited in the proof of Lemma 2.2 is based on [Laf02, Cor VI.3] and uses L-functions. See also [Ked18, Lemma 3.1.5, Theorem 3.3.1].

Remark 2.4. It follows from Lemma 2.2 that, in the context of Theorem 2.1, there is a decomposition:

$$R^1(\pi_C)_*\overline{\mathbb{Q}}_l \cong \bigoplus_{i=1}^{g} (L_i)$$

where the L_i form a complete set of $\overline{\mathbb{Q}}_l$ companions. There are exactly g non-isomorphic companions because the field generated by Frobenius traces of L_1 is isomorphic to E and the l-adic companions are in bijective correspondence with the embeddings $E \hookrightarrow \overline{\mathbb{Q}}_l$. In particular, each companion occurs with multiplicity 1. In fact, as $E \cong \mathrm{End}_C(A_C) \otimes \mathbb{Q}$, it follows that $E \otimes \overline{\mathbb{Q}}_l$ acts on $R^1(\pi_C)_*\overline{\mathbb{Q}}_l$. On the other hand, $E \otimes \overline{\mathbb{Q}}_l \cong \prod_i \overline{\mathbb{Q}}_l$, where i runs over the embeddings $E \hookrightarrow \overline{\mathbb{Q}}_l$. For each i, pick a non-trivial idempotent $e_i \in E \otimes \overline{\mathbb{Q}}_l$ whose image is the i^{th} component of the direct product decomposition. The above direct sum decomposition is induced by these e_i.

To apply Drinfeld's Theorem 2.1, we will use the following lemma.

Lemma 2.5. *Let $Y/\mathbb{F}_q$ be a smooth, geometrically connected, projective scheme and let α be a line bundle on Y. Let $M \subset \mathbb{P}^m_{\mathbb{F}_q}$ be a closed subset. Suppose there exists an infinite collection $(C_n)_{n\in\mathbb{N}}$ of smooth, projective, geometrically connected curves $C_n \subset Y$ such that*

1. *for each $n \in \mathbb{N}$, the natural map $H^0(Y, \alpha) \to H^0(C_n, \alpha|_{C_n})$ is an isomorphism;*
2. *for any infinite subset $S \subset \mathbb{N}$, the union:*
$$\bigcup_{n\in S} C_n$$
is Zariski dense in Y;
3. *for each curve C_n, there exists an $m+1$ globally generating sections*
$$t_{n,0}, \ldots, t_{n,m} \in H^0(C_n, \alpha|_{C_n})$$
such that the induced map to $\mathbb{P}^m$ factors through M:
$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[dr] & \mathbb{P}^m \\ & M \arrow[u] \end{tikzcd}$$

Then there exist global sections $\tilde{t}_0, \ldots, \tilde{t}_m \in H^0(Y, \alpha)$ such that the induced rational map $\tilde{f} \colon Y \dashrightarrow \mathbb{P}^m$ has image in M. Moreover, $\tilde{f}$ can be chosen to be compatible with infinitely many of the maps f_n.

Proof. There are finitely many ordered $m+1$-tuples of sections $H^0(Y, \alpha) \cong H^0(C_n, \alpha|_{C_n})$ because $H^0(Y, \alpha)$ is a finite dimensional vector space over $\mathbb{F}_q$. By the pigeonhole principle, in our infinite collection we may an find an $m+1$-tuple of sections $\tilde{t}_0, \ldots, \tilde{t}_m \in H^0(Y, \alpha)$ such that there exists an infinite set $S \subset \mathbb{N}$ with
$$(\tilde{t}_0, \ldots, \tilde{t}_m)|_{C_n} = (t_{n,0}, \ldots, t_{n,m})$$
for every $n \in S$. There is therefore an induced rational map $\tilde{f} \colon Y \dashrightarrow \mathbb{P}^m$ with $\tilde{f}|_{C_n} = f_n$ for each $n \in S$. On the other hand, the collection $(C_n)_{n\in S}$ is Zariski dense in Y by assumption and $\tilde{f}(C_n) \subset M$; therefore the image of $\tilde{f}$ lands inside of M, as desired. □

To use Lemma 2.5, the following definition will be useful.

Definition 2.6. Let $\bar{X}/k$ be a smooth, geometrically connected, projective scheme of dimension at least 2, let $Z \subset \bar{X}$ be a reduced simple normal crossings divisor, and set $X := X \backslash Z$. Let $\bar{U} \subset \bar{X}$ be an open subset whose complement has codimension at least 2. Let $(x_j)_{j=1}^s$ be a finite collection of closed points of $U := \bar{U} \cap X$. Let α be a line bundle on $\bar{X}$. We say that $\bar{C} \subset \bar{U}$ is a *good curve* for the quintuple $(\bar{X}, X, \bar{U}, \alpha, (x_j)_{j=1}^s)$ if

- $\bar{C}$ is the smooth complete intersection of smooth ample divisors of $\bar{X}$ that intersect Z in good position;
- $\bar{C}$ contains each of the closed points x_j, for $j = 1 \ldots s$;
- the natural map $H^0(\bar{X}, \alpha) \to H^0(\bar{C}, \alpha|_{\bar{C}})$ is an isomorphism.

In the proof of Theorem 1.2, we will need to know that good curves exist. This is guaranteed by the following two results.

Proposition 2.7. *Let Y/k be a smooth, geometrically connected, projective scheme of dimension $d \geq 2$ and let α be a line bundle on Y. Let $D \subset Y$ be an ample divisor. Then there exists an $s_0 > 0$ such that for any $s \geq s_0$, and for any integral divisor $E \in |sD|$ in the linear series, the natural map:*
$$H^0(Y, \alpha) \to H^0(E, \alpha|_E)$$
is an isomorphism.

Proof. For any $s > 0$, let $E \in |sD|$ be an integral divisor in the linear series. Then there is an exact sequence:

$$0 \to \alpha(-E) \to \alpha \to \alpha|_E \to 0.$$

If $h^0(Y, \alpha(-E)) = h^1(Y, \alpha(-E)) = 0$, then by the long exact sequence in cohomology, the restriction map $H^0(Y, \alpha) \to H^0(E, \alpha|_E)$ is an isomorphism. Our task is therefore to show that for all sufficiently large s, $h^0(Y, \alpha(-sD)) = h^1(Y, \alpha(-sD)) = 0$.

Let $\mathfrak{L}$ be the canonical bundle of Y. Then by Serre duality, $h^i(Y, \alpha(-sD)) = h^{d-i}(Y, \alpha^\vee(sD) \otimes \mathfrak{L})$. It follows from Serre vanishing that there exists an $s_0 > 0$ such that for any $s \geq s_0$ and for any $i < d$, $h^{d-i}(Y, \alpha^\vee(sD) \otimes \mathfrak{L}) = 0$. Therefore for any $s \geq s_0$ and for any $i < d$, $h^i(Y, \alpha(-sD)) = 0$ and the result follows. □

Lemma 2.8. *Let $\bar{X}/k$ be a smooth, geometrically connected, projective scheme of dimension at least 2, let $Z \subset \bar{X}$ be a reduced simple normal crossings divisor, and set $X := X \backslash Z$. Let $\bar{U} \subset \bar{X}$ be an open subset whose complement has codimension at least 2. Let $(x_j)_{j=1}^s$ be a finite collection of closed points of $U := \bar{U} \cap X$. Let α be a line bundle on $\bar{X}$. Then there is a good curve $\bar{C} \subset \bar{U}$ for the quintuple $(\bar{X}, X, \bar{U}, \alpha, (x_j)_{j=1}^s)$*

Proof. By induction, it suffices to construct a smooth ample divisor $\bar{D} \subset \bar{X}$ such that

- $\bar{D} \cap \bar{U}$ has complementary codimension at least 2 in $\bar{D}$;
- $\bar{D}$ intersections Z transversely;
- $\bar{D}$ contains x_j, for $j = 1 \ldots s$; and
- the natural map $H^0(\bar{X}, \alpha) \to H^0(\bar{D}, \alpha|_{\bar{D}})$ is an isomorphism.

First suppose k is infinite. Then the existence of such $\bar{D}$ follows from Bertini's theorem [Sta20, Tag 0FD4] by Proposition 2.7. Now suppose k is finite. Then, the existence of such $\bar{D}$ follows from Poonen's Bertini theorem over finite fields [Poo04, Theorem 1.3], together with Proposition 2.7. □

3. Proofs of Theorem 1.2 and Corollaries 1.3, 1.4

Proof of Theorem 1.2. We proceed in several steps.

Step 1: *organizing the summands of $\mathcal{E}$.* As $\mathcal{E}_i$ is irreducible, has determinant $\overline{\mathbb{Q}}_p(-1)$, and has rank 2, the slopes of $(\mathcal{E}_i)_x$ are in the interval $[0, 1]$ for every closed point x of X, see e.g. [DK17, Theorem 1.1.5].

Write the isotypic decomposition of $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)_{\overline{\mathbb{Q}}_p}$:

$$\mathcal{E} \cong \bigoplus_{i=1}^{a} (\mathcal{E}_i)^{m_i}.$$

The field generated by the coefficients of $P_x(\mathcal{E}, t)$ as x ranges through closed points of X is $\mathbb{Q}$. Therefore, by [Dri18, E.10] and either [AE19, Theorem 4.2] or [Ked18, Corollary 3.5.3], we can pick an l and a field isomorphism $\sigma\colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_l$ such that the semi-simple σ companion L to $\mathcal{E}$ exists and in fact may be defined over $\mathbb{Q}_l$, i.e. corresponds to a representation:

$$\pi_1(X) \to \mathrm{GL}_N(\mathbb{Q}_l).$$

(We emphasize that L is independent of the choice of σ because because the field generated by coefficients of characteristic polynomials $P_x(\mathcal{E}, t)$ as x ranges over closed points of X is $\mathbb{Q}$.) By compactness of $\pi_1(X)$, we may conjugate the representation into $\mathrm{GL}_N(\mathbb{Z}_l)$. We refer to the attached lisse $\mathbb{Z}_l$-sheaf as $\tilde{L}$. Similarly, for each i we denote by L_i the σ-companion to L_i (the L_i indeed do depend on the choice of σ). The companion relation implies that:

$$L \cong \bigoplus_{i=1}^{a} L_i^{m_i}$$

Let $E_i \subset \overline{\mathbb{Q}}_p$ denote the (number) field generated by the coefficients of $P_x(\mathcal{E}_i, t)$ as x ranges through the closed points of X. Note that for each $\mathcal{E}_i$, all p-adic companions exist and are summands of $\mathcal{E}$ by Lemma 2.2. For each $\mathcal{E}_i$, set $\mathcal{F}_i$ to be the sum of all distinct p-adic companions of $\mathcal{E}_i$. Note that there are $[E_i \colon \mathbb{Q}]$ distinct p-adic companions of $\mathcal{E}_i$, parametrized by the embeddings $E_i \hookrightarrow \overline{\mathbb{Q}}_p$. By reordering the indices, we write the decomposition of $\mathcal{E}$ as follows:

$$\mathcal{E} \cong \bigoplus_{i=1}^{b} \mathcal{F}_i^{m_i} \tag{3.1}$$

for some integer $1 \leq b \leq a$. (Under this reordering, the collection of $(\mathcal{E}_i)_{i=1}^b$ are all mutually not companions and for each $b+1 \leq j \leq a$, there exists a unique $1 \leq i \leq b$ such that $\mathcal{E}_j$ is a companion of $\mathcal{E}_i$.) Set

$$g = \sum_{j=1}^{b} m_i [E_i \colon \mathbb{Q}]. \tag{3.2}$$

Step 2: ***the proof in a simplified situation.*** We first assume that X admits a simple normal crossings compactification $\bar{X}$ such that $\mathcal{E}$ extends to a logarithmic F-isocrystal with nilpotent residues on $\bar{X}$ and moreover that $\tilde{L}$ has trivial residual representation. Write $Z := \bar{X} \backslash X$ for the boundary.

By Lemma A.6, there exists a Zariski open $\bar{U} \subset \bar{X}$ with complementary codimension at least 2, and a logarithmic Dieudonné crystal $(M_{\bar{U}}, F, V)$ on $\bar{U}$ (with the logarithmic structure coming from $Z \cap U$). Let

$$(N_{\bar{U}}, F, V) := (M_{\bar{U}}, F, V)^4 \oplus ((M_{\bar{U}}, F, V)^t)^4,$$

where the t denotes the *dual logarithmic Dieudonné crystal*. We also consider this logarithmic Dieudonné crystal as we will need to use Zarhin's trick. We set $U := \bar{U} \backslash (\bar{U} \cap Z)$.

After Remark A.7, it follows that we may define Hodge line bundles ω_M and ω_N on $\bar{U}$ attached to the two logarithmic Dieudonné crystals. As $\bar{U} \subset \bar{X}$ has complementary codimension at least 2 and $\bar{X}$ is smooth, it follows that ω_M and ω_N extend canonically to line bundles on all of $\bar{X}$.

The Hodge line bundle α on the fine moduli scheme $\mathscr{A}_{8g,1,l} \otimes \mathbb{F}_q$ is ample by [Mor85, Ch. IX, Théorème 3.1, p. 210] or [FC90, Ch. V, Theorem 2.5(i), p. 152]. Let g be as in Equation 3.2 and choose an r so that the α^r is very ample on $\mathscr{A}_{8g,1,l}$. As $8g > 1$, it follows from the Koecher principle that $H^0(\mathscr{A}_{8g,1,l} \otimes \mathbb{F}_q, \alpha^r)$ is a finite dimensional $\mathbb{F}_q$-vector space for all $r \in \mathbb{Z}$ [FC90, Ch. V, Theorem 1.5 (ii)]. Fix a basis $s_0, \dots, s_m$ of the vector space:

$$s_0, \dots, s_m \in H^0(\mathscr{A}_{8g,1,l} \otimes \mathbb{F}_q, \alpha^r) \tag{3.3}$$

once and for all. There is an induced embedding $\mathscr{A}_{8g,1,l} \subset \mathbb{P}^m$. As is customary, denote by $\mathscr{A}^*_{8g,1,l}$ the Zariski closure of $\mathscr{A}_{8g,1,l}$ in $\mathbb{P}^m$; we call this the *minimal compactification*. Abusing notation, we also denote by α the Hodge line bundle on $\mathscr{A}^*_{8g,1,l}$. The Koecher principle implies that $H^0(\mathscr{A}_{8g,1,l} \otimes \mathbb{F}_q, \alpha^r) = H^0(\mathscr{A}^*_{8g,1,l} \otimes \mathbb{F}_q, \alpha^r)$: this follows from [FC90, Ch. V, Theorem 1.5 (ii), Theorem 2.5 (iii)].

It follows from [Del12] there exists a *finite* number of closed points $(x_j)_{j=1}^s$ of U such that for each $\mathcal{E}_i$, the field generated by the coefficients of $P_{x_j}(\mathcal{E}_i, t) \in \overline{\mathbb{Q}}_p[t]$ as $j = 1 \dots s$ is $E_i \subset \overline{\mathbb{Q}}_p$.

If $\bar{C} \subset \bar{U}$ is a good curve for the quintuple $(\bar{X}, X, \bar{U}, \omega_N^r, (x_j)_{j=1}^s)$ as in Definition 2.6, set $C := \bar{C} \cap X$. Then the following three properties hold.

- Each $\mathcal{E}_i|_C$ is irreducible by [AE19, Theorem 2.6].
- The field generated by Frobenius traces of $\mathcal{E}_i|_C$ is E_i.
- Each $\mathcal{E}_i|_C$ has infinite monodromy around ∞. Indeed, from the positivity of $\bar{C}$, and the good position assumption, it follows that $\bar{C}$ intersects each component of Z is a non-empty and transverse way. Then the assumption that $\mathcal{E}_i$ has infinite monodromy around Z implies $\bar{C}$ has infinite monodromy around $\bar{C} \cap Z$.

It follows from the construction of p-to-l companions for curves, Theorem 2.1, and Remark 2.4 together with Equation 3.1 that if $\bar{C} \subset \bar{X}$ is a good curve, then there exists an abelian scheme $\pi_C \colon A_C \to C$ of relative dimension g such that

$$R^1(\pi_C)_*\overline{\mathbb{Q}}_l \cong L|_C.$$

As we assumed that the $\mathbb{Z}_l$-lattice $\tilde{L}$ has trivial residual representation, it follows that we may replace A_C by an l-primarily-isogenous abelian scheme to ensure that the étale group scheme $A_C[l] \to C$ is split.

Similarly, we have that $\mathbb{D}(A_C[p^\infty]) \otimes \overline{\mathbb{Q}}_p \cong \mathcal{E}|_C$. Therefore $\mathbb{D}(A_C[p^\infty])$ is isogenous to $(M, F, V)_C$ as Dieudonné crystals on C. We claim that we may replace A_C by an (p-primarily) isogenous abelian scheme in order to ensure that:

$$\mathbb{D}(A_C[p^\infty]) \cong (M, F, V)_C$$

as Dieudonné crystals on C. To see this, use [dJ95] to construct a p-divisible group G_C on C where $\mathbb{D}(G_C) \cong (M_C, F, V)$. It follows that $A_C[p^\infty]$ and G_C are isogenous. Pick an isogeny $A_C[p^\infty] \to G_C$; the kernel $K \subset A$ is a p-primary finite flat group scheme. There is a *p-primary* isogeny of abelian schemes $A_C \to A_C/K$. As the group of l-torsion points of an abelian scheme is l-primary, it follows that A_C/K also has trivial l-torsion. Replace A_C by A_C/K.

By construction, the l-torsion is trivial; hence A_C has semistable reduction along $\bar{C} \cap Z$. Let $A_{\bar{C}} \to \bar{C}$ be the Néron model. It follows from Remark A.8 that the logarithmic Dieudonné crystal of $A_{\bar{C}} \to \bar{C}$ is isomorphic to $(M, F, V)_{\bar{C}}$. Again by Remark A.8, the Hodge bundle of the $A_{\bar{C}} \to \bar{C}$ is isomorphic to $\omega_M|_{\bar{C}}$.

Set $B_C := (A_C \times_C A_C^t)^4$. By Zarhin's trick [Mor85, Chapitre IX, Lemme 1.1, p. 205], B_C admits a principal polarization. By construction, we have that

- B_C has trivial l-torsion, and
- $\mathbb{D}(B_C[p^\infty]) \cong (N_C, F, V)$

Once more, by Remark A.8 it follows that there is an isomorphism of logarithmic Dieudonné crystals:

$$\mathbb{D}(B_{\bar{C}}[p^\infty]) \cong (N, F, V)_{\bar{C}}.$$

The Hodge line bundle of $B_{\bar{C}}$ is hence isomorphic to $\omega_N|_{\bar{C}}$. However, we emphasize that the choice $B_C \to C$ is not canonical!

We have an induced morphism to a fine moduli scheme $C \to \mathscr{A}_{8g,1,l}$. This extends to a morphism

$$\bar{C} \to \mathscr{A}^*_{8g,1,l} \tag{3.4}$$

to the minimal compactification. We now claim the pullback of α, the Hodge line bundle on $\mathscr{A}^*_{8g,1,l}$, is isomorphic to $\omega_N|_{\bar{C}}$. Here is the reason: choose a toroidal compactification $\bar{\mathscr{A}}_{8g,1,l}$. We then have a commutative diagram:

$$\begin{tikzcd} \bar{C} \arrow[r, "h"] \arrow[dr] & \bar{\mathscr{A}}_{8g,1,l} \arrow[d, "\varphi"] \\ & \mathscr{A}^*_{8g,1,l}. \end{tikzcd} \tag{3.5}$$

By [FC90, Ch. V, Theorem 2.5, p. 152], there is a semi-abelian scheme $G \to \bar{\mathscr{A}}_{8g,1,l}$ such that $\varphi^*\alpha$ is isomorphic to the Hodge line bundle of $G \to S$. Now, [FC90, Ch. I, Proposition 2.7, p.9] implies that h^*G is isomorphic to the semi-abelian scheme given by the open subset of $A_{\bar{C}} \to \bar{C}$ obtained by removing the non-identity components along $\bar{C} \backslash C$. In particular, it follows that the Hodge line bundle of h^*G is compatible with the Hodge line bundle constructed in Remark A.8.

In Equation 3.3, we have already fixed a basis of sections

$$s_0, \ldots, s_m \in H^0(\mathscr{A}_{8g,1,l} \otimes \mathbb{F}_q, \alpha^r) = H^0(\mathscr{A}^*_{8g,1,l} \otimes \mathbb{F}_q, \alpha^r);$$

after pulling back the sections to $\bar{C}$ via Equation 3.4, we obtain an $m+1$-tuple of sections $t_0, \ldots, t_m$ in $H^0(\bar{C}, \omega_N^r|_{\bar{C}})$ that define the morphism $\bar{C} \to \mathscr{A}^*_{8g,1,l} \subset \mathbb{P}^m$.

In conclusion, for every good curve $\bar{C} \subset \bar{U}$ for the quintuple $(\bar{X}, X, \bar{U}, \omega_N^r, (x_j)_{j=1}^s)$, we have constructed an $m+1$-tuple of globally generating sections $t_0, \ldots, t_m \in H^0(\bar{C}, \omega_N^r|_{\bar{C}})$ such that the induced map lands in $\mathscr{A}^*_{8g,1,l} \subset \mathbb{P}^m$, such that C maps into $\mathscr{A}_{8g,1,l}$, and such that the induced abelian variety on $B_C \to C$ is isomorphic to $(A_C \times_C A_C^t)^4$ where $A_C \to C$ is an abelian scheme with $\mathbb{D}(A_C[p^\infty]) \cong (M, F, V)|_C$ as Dieudonné crystals on C. Therefore we also have that $\mathbb{D}(B_C[p^\infty]) \cong (N, F, V)|_C$.

For each $n > 0$, let P_n denote the union of the set of closed points of U whose residue field is contained in $\mathbb{F}_{q^{n!}}$ with $(x_j)_{j=1}^s$. By Lemma 2.8, it follows that for each $n > 0$, there exists a good curve $\bar{C}_n \subset \bar{U}$ for the quintuple $(\bar{X}, X, \bar{U}, \omega_N^r, P_n)$.

For each $n \in \mathbb{N}$, by the above remarks we obtain an $m+1$-tuple of globally generating sections

$$t_{n,0}, \ldots, t_{n,m} \in H^0(\bar{C}_n, \omega_N^r|_{\bar{C}_n})$$

such that the induced map factors $f_n\colon \bar{C}_n \to \mathscr{A}^*_{8g,1,l} \subset \mathbb{P}^m$. Moreover, any infinite subcollection of the $\bar{C}_n$ are Zariski dense because they are space-filling. By Lemma 2.5, it follows that there exists an infinite set $S \subset \mathbb{N}$ and sections $\tilde{t}_0, \ldots, \tilde{t}_m \in H^0(\bar{X}, \omega_N^r)$ such that the induced rational map $\tilde{f}\colon X \dashrightarrow \mathbb{P}^m$ lands in $\mathscr{A}^*_{8g,1,l}$ and moreover, for each $n \in S$, we have an equality of morphisms $\tilde{f}|_{\bar{C}_n} = f_n$.

By shrinking U, we therefore obtain a map $\tilde{f}\colon U \to \mathscr{A}_{8g,1,l}$ and hence an abelian scheme $B_U \to U$ such that $B_U[l]$ is a trivial étale cover of U. The maps $f_n\colon \bar{C}_n \to \mathscr{A}_{8g,1,l}$ were all constructed such that the induced abelian scheme $B_{C_n} \to C_n$ is compatible with $(N_{C_n}, F, V) \otimes \mathbb{Q}_p \cong (\mathcal{E} \oplus \mathcal{E}^*(-1))^4|_C$. On the other hand, if u is a closed point of U, then u lies on C_n for all $n \gg 0$: indeed, if the residue field of u is $\mathbb{F}_{q^e}$, then C_n contains u for all $n \geq e$. Therefore $B_U \to U$ is compatible with $(L \oplus L^*(-1))^4|_U$.

For each $n \in S$ we have that $\tilde{f}|_{\bar{C}_n} = f_n$ by Lemma 2.5. Then, by construction there exists an abelian scheme $A_{C_n} \to C_n$ of dimension g with

$$B_U|_{C_n} \cong (A_{C_n} \times_{C_n} A^t_{C_n})^4.$$

Consider the map of representations induced by the first $\mathbb{Z}_l$-cohomology of the abelian schemes $B_U \to U$ and $B_U|_{C_n} \to C_n$:

$$\begin{tikzcd}
\pi_1(C_n) \arrow[rr] \arrow[dr] & & \pi_1(U) \arrow[dl] \\
& \mathrm{GL}_{16g}(\mathbb{Z}_l). &
\end{tikzcd} \tag{3.6}$$

Then [Kat01, Lemma 6(b)] implies that for $n \gg 0$, the two representations have the same image (which lands in $\mathrm{GL}_{2g}(\mathbb{Z}_l)^8$). By the fundamental work of Tate-Zarhin on Tate's isogeny theorem for abelian varieties over finitely generated fields of positive characteristic [Mor85, Ch. XII, Théorème 2.5(i), p. 244], it follows that the natural injective map $\mathrm{End}_U(B_U) \hookrightarrow \mathrm{End}_{C_n}(B_U|_{C_n})$ is an isomorphism when tensored with $\mathbb{Z}_l$ and hence also when tensored with $\mathbb{Q}_l$. It follows that the map

$$\mathrm{End}_U(B_U) \otimes \mathbb{Q} \to \mathrm{End}_{C_n}(B_U|_{C_n}) \otimes \mathbb{Q}$$

is an isomorphism as both sides are finite dimensional semi-simple $\mathbb{Q}$-algebras of the same rank.

We know that $\mathrm{End}_{C_n}(B_U|_{C_n})$ has a *nontrivial idempotent* e_{C_n} that projects onto a copy of A_{C_n}. After replacing e_{C_n} by a high integer multiple, we may lift e_{C_n} to $e_U \in \mathrm{End}_U(B_U)$. Set the image of e_U to be the abelian scheme $\pi_U\colon A_U \to U$. It follows from Equation 3.6 that A_U is compatible with $\mathcal{E}$ (equivalently: L), as desired.

Step 3: ***the proof in the general case via reduction to Step 2.*** There exists a projective divisorial compactification $\bar{X}$ of X. (This means that $\bar{X}$ is normal and the boundary is an effective Cartier divisor.) By Kedlaya's semistable reduction theorem (see [Ked16, Theorem 7.6] for a meta-reference), there is a generically étale alteration $\varphi\colon X' \to X$ together with a simple normal crossings compactification $\bar{X}'$ such that the overconvergent pullback $\mathcal{E}'$ extends to a logarithmic F-isocrystal with nilpotent residues. After replacing X' with a further finite étale cover, we may guarantee that the residual representation of L' is trivial.

We have proven the theorem for $\mathcal{E}'$ on X': there exists an open subset $W' \subset X'$ and an abelian scheme $A_{W'} \to W'$ with $\mathbb{D}(A_{W'}[p^\infty]) \cong \mathcal{E}'|_{W'}$. After shrinking W' and W, we may assume that $\varphi|_{W'} \colon W' \to W$ is finite étale, of degree d.

Set $B_W := \mathfrak{Res}_W^{W'}(A_{W'})$ to be the Weil restriction of scalars. This is an abelian scheme over W that is compatible with L^d. Recall that we wrote an isotypic decomposition:

$$L \cong \bigoplus_{i=1}^{a} (L_i)^{m_i}$$

where each L_i is irreducible on X (and hence on W). Let $E_i \subset \overline{\mathbb{Q}}_l$ denote the field generated by the traces of Frobenius on L_i as x ranges through the closed points of W. We claim that we may find a smooth curve $C \subset W$ with the following properties:

(1) each $L_i|_C$ is irreducible;
(2) the field generated by Frobenius traces of $L_i|_C$ is $E_i \subset \overline{\mathbb{Q}}_l$;
(3) each $L_i|_C$ has infinite monodromy around ∞; and
(4) the induced monodromy representations coming from $B_W \to W$ and $B_W|_C \to C$

$$\begin{tikzcd} \pi_1(C) \arrow[rr] \arrow[dr] & & \pi_1(W) \arrow[dl] \\ & \mathrm{GL}_{2gd}(\mathbb{Z}_l) & \end{tikzcd}$$

have the same image.

We have a projective normal compactification $\bar{X}$ of X, which is smooth away from a closed subset of codimension at least 2. Let $F = \bar{X} \backslash X$ and let $F' \subset F$ be the singular locus of $\bar{X}$. For each L_i, there is an irreducible component F_j of F that witnesses the fact that L_i has infinite monodromy at ∞: having infinite monodromy at ∞ means that a certain inertia group has infinite image in the representation.

Pick a closed point $y_j \in F_j \backslash (F_j \cap F')$ for each j. Then, by using [Dri12, C.2], we may construct an infinite set of curves $(C_n)_{n\in\mathbb{N}}$ where each $C_n \subset W$ is a smooth, geometrically connected curve that contains all closed points of W whose residue fields are contained in $\mathbb{F}_{q^{n!}}$ and that pass through the y_j transversally (i.e., with a tangent direction that is not contained in F_j). (We remark that this is a consequence of [Poo04, Theorem 1.3].)

Each $L_i|_{C_n}$ has infinite monodromy around ∞. By [Kat01, Lemma 6(b)], it follows that for all $n \gg 0$, C_n satisfies (4). For $n \gg 0$, [Kat01, Lemma 6(b)] and [Del12] guarantees that setting $C := C'_n$ satisfies the above four conditions.

Again, by using Drinfeld's Theorem 2.1, Remark 2.4, and Equation 3.1, there exists an abelian scheme $A_C \to C$ that is compatible with $L|_C$. On the one hand, using the Tate isogeny theorem [Mor85, Ch. XII, Théorème 2.5] it follows that A_C^d is isogenous to $B_W|_C$. On the other hand, another application the Tate isogeny theorem together with property (4) of C implies that the natural map:

$$\mathrm{End}_W(B_W) \to \mathrm{End}_C(B_W|_C)$$

is an isomorphism after tensoring with $\mathbb{Q}$. As $B_W|_C$ is isogenous to A_C^d, it follows that $\mathrm{End}_C(B_W|_C) \otimes \mathbb{Q}$ has an element e_C projecting onto a factor of A_C. After replacing e_C with a high integer multiple, we may lift to to $e_W \in \mathrm{End}_W(B_W)$. Set the image of e_W to be the abelian scheme $A_W \to W$; this is compatible with $L|_W$, as desired. □

Proof of Corollary 1.3. Suppose there exists $\pi_U \colon A_U \to U$ such that $R^1(\pi_U)_*\overline{\mathbb{Q}}_l$ has L_1 as a summand. A theorem of Zarhin implies that $R^1(\pi_U)_*\mathbb{Q}_l$ is semi-simple [Mor85, Chapitre XII, Theorem 2.5, p. 244-245]. The field generated by the characteristic polynomials of $R^1(\pi_U)_*\mathbb{Q}_l$ is clearly $\mathbb{Q}$.

Similarly, $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$ is a semi-simple object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(U)$ by [Pál15]. As $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$ is isomorphic to the rational crystalline cohomology of $A_U \to U$, it follows from [KM74] that that $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$ and $R^1(\pi_U)_*\mathbb{Q}_l$ are companions. It follows from Lemma 2.2 that all crystalline companions of $L_1|_U$ exist and moreover are summands of $\mathbb{D}(A_U[p^\infty])\otimes\overline{\mathbb{Q}}_p$. Then by [Ked18, Corollary 3.3.3], all crystalline companions to L_1 exist.

Conversely, suppose all crystalline companions $(\mathcal{E}_i)_{i=1}^b$ to L_1 exist. They all have infinite monodromy at ∞ be the companion relation. There exists a p-adic local field K with each $\mathcal{E}_i$ an object of $\mathbf{F\text{-}Isoc}^\dagger(X)_K$. Set $\mathcal{E} := \bigoplus_{i=1}^b \mathcal{E}_i$, considered as an object of $\mathbf{F\text{-}Isoc}^\dagger(X)$ (by restricting scalars from K to $\mathbb{Q}_p$, so the rank of $\mathcal{E}$ is $2b[K:\mathbb{Q}]$). Then $\mathcal{E}$ satisfies the hypotheses of Theorem 1.2, and moreover L_1 is a companion of a summand of $\mathcal{E}$. It follows that there is an open set $U \subset X$ together with an abelian scheme $\pi_U\colon A_U \to U$ such that $\mathcal{E} \cong \mathbb{D}(A_U[p^\infty)\otimes\mathbb{Q}_p$. Again using Zarhin's semi-simplicity, $L_1|_U$ is a summand of $R^1(\pi_U)_*\overline{\mathbb{Q}}_l$, as desired. □

Proof of Corollary 1.4. Under the assumption on E_1, all p-adic companions to $\mathcal{E}_1$ exist by [KP18, Corollary 4.16]. (This result is straightforward; they are all Galois twists of each other.) Fix $\sigma\colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_l$. Then the σ-companion to $\mathcal{E}_1$ exists by [AE19, Theorem 4.2] or [Ked18, Corollary 3.5.3]. Apply Corollary 1.3. □

Appendix A. Logarithmic F-crystals

We first recall the notion of a logarithmic F-crystal/isocrystal. While this notion is due to Kato [Kat89, Section 6], our treatment is copied from recent work of Kedlaya.

Definition A.1. A *smooth pair* over a perfect field k is a pair (Y,Z) where Y/k is a smooth variety and $Z\subset Y$ is a strict normal crossings divisor.

Definition A.2. Let (Y,Z) be a smooth pair over a perfect field k of characteristic $p>0$. A *smooth chart* for (Y,Z) is a sequence of elements $\bar{t}_1,\dots,\bar{t}_n$ of elements of $\mathcal{O}_Y(Y)$ such that the

- induced map $\bar{f}\colon Y\to\mathbb{A}^n$ is étale, and
- there exists an $m\in[0,n]$ such that the zero-loci of $\bar{t}_i$, for $i=0\dots m$, are exactly the irreducible components of Z.

Let (Y,Z) be a smooth pair over a perfect field k of characteristic $p>0$. Let $\bar{t}_1,\dots,\bar{t}_n$ be a smooth chart of (Y,Z). Let P_0 be the formal scheme given by the formal completion of $W(k)[t_1,\dots,t_n]$ along (p). By topological invariance of the étale site, there exists a unique smooth formal scheme P together with an étale morphism $f\colon P\to P_0$ lifting $\bar{f}$. We call the pair $(P,t_1,\dots,t_n)$ the *lifted smooth chart* of (Y,Z) associated to the original chart.

Let $\sigma_0\colon P_0\to P_0$ be the Frobenius lift with $\sigma^*(t_i)=t_i^p$ for $i\in[0,\dots,n]$. Then there exists an associated Frobenius lift $\sigma\colon P\to P$.

Definition A.3. Let (Y,Z) be a smooth pair over a perfect field k and let $\bar{t}_1,\dots,\bar{t}_n$ be a smooth chart of (Y,Z). Keep notations as above. A *logarithmic crystal with nilpotent residues* on (Y,Z) is a pair (M,∇) where

- M is a p-torsion free coherent module over P; and
- ∇ is a integrable, topologically quasi-nilpotent connection on M (with respect to $W(k)$) with logarithmic poles and nilpotent residues along the zero-loci of $f^*(t_i)$.

A *logarithmic F-crystal with nilpotent residues* is a triple (M,∇,F) where (M,∇) is a logarithmic crystal with nilpotent residues and F is an injective, horizontal morphism

$$F\colon\sigma^*(M)\to M$$

of coherent P-modules. A *logarithmic Dieudonné crystal with nilpotent residues* is a quadruple (M,∇,F,V) where (M,∇,F) is a logarithmic F-crystal in finite, locally free modules with nilpotent residues and V is an injective, horizontal map

$$V\colon(M)\to\sigma^*M$$

such that $FV=VF=p$.

Remark A.4. In the definition of a logarithmic F-crystal with nilpotent residues, we do not demand that M is locally free.

This definition extends to general smooth pairs by Zariski gluing; every smooth pair admits a finite open covering on which the restriction admits a smooth chart. We often drop the connection ∇ from the notation.

There is a natural category of logarithmic crystals with nilpotent residues on (Y, Z) (where morphisms are P-linear and horizontal), and the category of logarithmic isocrystals with nilpotent residues is defined to be the induced isogeny category. One similarly defines the category of logarithmic F-isocrystals with nilpotent residues.

Remark A.5. Let (Y, Z) be a smooth pair over k and let $U = Y\backslash Z$. We denote by $\underline{Y}$ the (fine, saturated) logarithmic scheme given by $(Y, \alpha\colon \mathcal{O}_U^* \hookrightarrow \mathcal{O}_Y)$. Then our definition of a logarithmic crystal is compatible with the definition of Kato (see [Kat89, Theorem 6.2]), our definition of a logarithmic F-crystal in finite, locally free modules is compatible with the definition of Kato-Trihan [KT03, 4.1] and our definition of a logarithmic F-isocrystal is compatible with the definition given by Shiho (see [Shi00, Definition 4.1.3]).

The mathematical content of the following lemma is essentially [Kat79, Theorem 2.6.1] (and relatedly [Cre87, Lemma 2.5.1]); we have simply rewritten Katz's argument to the logarithmic setting. The key is that Katz's slope bounds holding on the open subset where the logarithmic structure is trivial guarantee that they hold everywhere. We use Kato's definition of logarithmic F-crystals only for convenience to discuss global objects; all of the computations use the local definitions given above.

Lemma A.6. *Let (Y, Z) be a smooth pair over a perfect field k of positive characteristic and let $U := Y\backslash Z$. Let $\mathcal{E}$ be a logarithmic F-isocrystal on (Y, Z).*

1. *Suppose the Newton slopes of $\mathcal{E}_U$ are all non-negative. Then there exists an open subset $W \subset Y$, whose complementary codimension is at least 2, and a logarithmic F-crystal in finite, locally free modules (M'', F) on the smooth pair $(W, W \cap Z)$ such that $(M'', F) \otimes \mathbb{Q} \cong \mathcal{E}_W$.*
2. *Suppose the Newton slopes of $\mathcal{E}_U$ are in the interval $[0, 1]$. Then there exists an open subset $W \subset Y$, whose complementary codimension is at least 2, and a logarithmic Dieudonné crystal in finite, locally free modules (M'', F, V) on the smooth pair $(W, W\cap Z)$ such that $(M'', F)\otimes\mathbb{Q} \cong \mathcal{E}_W$.*

Proof. By definition of a logarithmic F-isocrystal, there exists a logarithmic crystal in coherent (not necessarily locally free!) modules M and a map $F\colon Frob_{\underline{Y}}^* M \to M \otimes \mathbb{Q}$ that is isomorphic to $\mathcal{E}$ when thought of as a logarithmic F-isocrystal. Here, $Frob_{\underline{Y}}$ refers to the absolute Frobenius (on the f.s. log scheme $\underline{Y}$ induced from the smooth pair (Y, Z)) and the $*$ refers to pullback on the logarithmic crystalline topos. This is compatible with our above definitions.

As M is a logarithmic crystal in *coherent* modules, there exists a non-negative integer ν so that

$$F\colon (Frob_Y)^* M \to p^{-\mu} M.$$

We have assumed that the Newton slopes of $\mathcal{E}$ are all non-negative. Slope bounds of Katz (see the proof on [Kat79, p. 151-152]) imply then that there exists a non-negative ν such that all $n \geq 0$,

$$F^n\colon (Frob_U^n)^* M_U \to p^{-\nu} M_U. \tag{A.1}$$

We explicate this in local coordinates. Take an affine open neighborhood $T \subset Y$ such that $(T, T\cap Z)$ has a smooth chart $(\bar{t}_1, \dots, \bar{t}_n)$. Let $(P, t_1, \dots, t_n)$ be the associated lifted smooth chart; note that $P = \mathrm{Spf}(A)$ where A is a noetherian $W(k)$ algebra equipped with the p-adic topology. Then the logarithmic crystal yields a finitely generated A module M_A and the Frobenius structure induces a M_A-linear (continuous) homomorphism $F\colon \sigma^* M_A \to p^{-\mu} M_A$.

As $U \cap T \subset T$ is open dense, it follows from Equation A.1 that

$$F^n\colon (\sigma^n)^* M_A \to p^{-\nu} M_A.$$

. By varying T, one deduces that $F^n\colon (Frob_Y^n)^* M \to p^{-\nu} M$.

Consider the module

$$M'_A := \sum_{n\geq 0} F^n\left((\sigma_X^n)^* M_A\right) \subset p^{-\nu} M_A.$$

As A is noetherian, M'_A is finitely generated, being a submodule of a finitely generated module. Moreover, M'_A is stable under F. Finally, M'_A is the finite sum of (logarithmic) horizontal submodules. Therefore the pair (M'_A, F) is in fact a logarithmic F-crystal in coherent modules. We have an isomorphism $(M'_A, F) \otimes \mathbb{Q} \cong \mathcal{E}_T$ in the category of logarithmic F-isocrystals with nilpotent residues on $(T, Z \cap T)$.

Now, set $M''_A := (M'_A)^{**}$. This is a coherent reflexive sheaf on the ring A, and hence is locally free away on an open set of $\mathrm{Spec}(A)$ whose complement has codimension at least 3. M''_A is manifestly stable under the connection and F. In particular, we can find an open subset $T'' \subset T$ with complementary codimension at least 2 such that the logarithmic F-crystal $(M''_A, F)_{T''}$ is a crystal in finite, locally free modules.

After initially choosing a pair $(M, F\colon Frob^*_{\underline{Y}} M \to p^{-\mu} M)$ representing $\mathcal{E}$, the constructions we have made are canonical. Therefore, ranging over $\overline{T}$, we may glue the $(M'', F)_{T''}$; that is, there is an open subset $W \subset T$ with complementary codimension at least 2 and a logarithmic F-crystal $(M'', F)_W$ in finite, locally free modules on the smooth pair $(W, Z \cap W)$ that is a lattice inside of $\mathcal{E}_W$.

We now indicate how to complete the result if the Newton polygons on U are in the interval $[0, 1]$. Let (M, F) be a logarithmic F-crystal in finite, locally free modules on a smooth pair (Y, Z) over a perfect field k and suppose the Newton slopes on U are no greater than 1. Set $V := F^{-1} \circ p$. Then V *does not* necessarily stabilize M; however, the pair $(M, V)_U$ is a logarithmic σ^{-1}-F-isocrystal in the language of [Kat79]. (Fortunately, Katz's entire paper is written in the context of σ^a-F-crystals for *any* $a \neq 0$, not just the positive a. In particular, all of Katz's results also hold for σ^{-1}-F-crystals. Katz does not deal with logarithmic crystals, but we only use the slope bounds on the open set U.) By the coherence argument as above, we may find η such that:

$$V\colon (Frob^{-1}_{\underline{Y}})^* M \to p^{-\eta} M$$

on all of Y. Again, using Katz's slope bounds on U (which hold equally well for σ^{-1}-F-crystals) and the same coherence argument, one shows that after possibly increasing η, we in fact have

$$V^n\colon (Frob^{-n}_{\underline{Y}})^* M \to p^{-\eta} M$$

for all $n \geq 0$. Now run exactly the above argument with V instead of F: then

$$M' := \sum_{n \geq 0} V^n (Frob^{-1}_{\underline{Y}})^* M) \subset p^{-\eta} M$$

will be coherent, horizontal, and stabilized by V. Recall that $FV = VF = p$; therefore M' is also stabilized by F! Then $M'' := (M')^{**}$ is a reflexive logarithmic crystal on (Y, Z) that is stabilized by both F and V. Exactly as above, there exists an open subset $W \subset Y$ of complementary codimension at least 2 such that $(M'', F, V)_W$ is a logarithmic Dieudonné crystal in finite, locally free modules, as desired.

□

Remark A.7. Let (Y, Z) be a smooth pair over k and let (M, F, V) be a logarithmic Dieudonné crystal (in finite, locally free modules) on (Y, Z). We construct a natural line bundle ω, which we call the *Hodge line bundle*, attached to (M, F, V).

Evaluating M on the trivial thickening of (Y, Z), we obtain a vector bundle $M_{(Y,Z)}$ on Y together with an integrable connection with logarithmic poles on Z and a horizontal map:

$$F_{(Y,Z)}\colon Frob^*_{\underline{Y}} M_{(Y,Z)} \to M_{(Y,Z)}.$$

The kernel is a vector bundle on Y. Set $\omega := \det(\ker(F_{(Y,Z)}))$. We call this kernel the *Hodge line bundle* associated to (P, F).

As a reference for this remark: in the case when Z is empty, one finds this construction in [dJ98, 2.5.2 and 2.5.5]. In the setting of logarithmic Dieudonné crystals, Kato-Trihan construct the dual object: $Lie(M, F, V)$, see [KT03, 5.1] and especially Lemma 5.3 of *loc. cit.* Our hypothesis that (Y, Z) is a smooth pair over a perfect field k imply that the conditions of 5.1 on p. 563 of *loc. cit.* hold: étale locally, there is a p-basis of Y such that each (regular) component of Z is cut out by some member of the p-basis.

Remark A.8. We have the following relationships.

(1) Let Y/k be a smooth scheme over a perfect field k. Let $A_Y \to Y$ be an abelian scheme. Then there is an associated Dieudonné crystal $(M, F, V) = \mathbb{D}(A_Y[p^\infty])$ on Y [BBM82]. The Hodge bundle of (M, F) is isomorphic to the Hodge line bundle of the abelian scheme $A_Y \to Y$ by [BBM82, 3.3.5 and 4.3.10].

(2) Let C/k be a smooth curve over a perfect field k, let $U \subset C$ be a dense open subset, and let $Z \subset C$ be the reduced complement. let $A_U \to U$ be an abelian scheme with semi-stable reduction along Z. Call the Néron model $A_C \to C$. Then there is an attached logarithmic Dieudonné crystal on (C, Z), which we call $\mathbb{D}(A_C[p^\infty])$ [KT03, 4.4-4.8]. Kato-Trihan construct a covariant Dieudonné functor. We assume ours is contravariant, which may be accomplished by taking a dual as in [KT03, 4.1].

By construction, $\mathbb{D}(A_C[\infty])|_U$ is isomorphic to the crystalline Dieudonné functor of the p-divisible group $A_U[p^\infty]$. (See also the description of gluing as in [KT03, Lemma 4.4.1].) Moreover, the Hodge line bundle of $A^o_C \to C$, the open subset of $A_C \to C$ obtained by discarding the non-identity components over Z, is isomorphic to the Hodge bundle of $\mathbb{D}(A_C[p^\infty])$: this follows from [KT03, Example 5.4(b)], with the caveat that they work with the covariant Dieudonné functor.

Finally, we argue that when $k \cong \mathbb{F}_q$, $\mathbb{D}(A_C[p^\infty])$ is the unique logarithmic Dieudonné crystal with nilpotent residues on (C, Z) that extends $\mathbb{D}(A_U[p^\infty])$. First of all, note that we only need to check that there is at most one extension as a logarithmic F-crystal: in our setting, V is determined by F under the relation $FV = VF = p$. By [LT01], it follows that $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$ is overconvergent. By the hypothesis that $A_U \to U$ has semistable reduction along Z, Kato-Trihan have constructed an attached log F-crystal on (C, Z) as above. We claim the residues of this F-crystal are nilpotent. This may be seen as follows: semistable reduction guarantees tameness of the l-adic companion, and then one may compare local ε-factors via compatibility in the Langlands correspondence. (See the proof of [AE19, Proposition 2.8].) Finally, by a full-faithfulness result of Kedlaya [Ked07, Proposition 6.3.2], it follows that the extension of $\mathbb{D}(A_U[p^\infty])$ to a logarithmic F-crystal with nilpotent residues on (C, Z) is unique.

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