Abstract. Let $X/F_q$ be a smooth, geometrically connected, quasiprojective variety. Let $E$ be a semisimple overconvergent $F$-isocrystal on $X$. Suppose that irreducible summands $E_i$ of $E$ have rank 2, determinant $\overline{Q}_p(-1)$, and infinite monodromy at $\infty$. Suppose further that for each closed point $x$ of $X$, the characteristic polynomial of $E$ at $x$ is in $\mathbb{Q}[t] \subset \mathbb{Q}_p[t]$. Then there exists a non-trivial open set $U \subset X$ such that $E|_U$ comes from a family of abelian varieties on $U$.

As an application, let $L_1$ be an irreducible lisse $\mathbb{Q}_l$ sheaf on $X$ that has rank 2, determinant $\overline{Q}_l(-1)$, and infinite monodromy at $\infty$. Then all crystalline companions to $L_1$ exist (as predicted by Deligne's crystalline companions conjecture) if and only if there exists a non-trivial open set $U \subset X$ and an abelian scheme $\pi_U : A_U \to U$ such that $L_1|_U$ is a summand of $R^1(\pi_U)_*\overline{Q}_l$.

1. Introduction

Throughout this article, $p$ is a prime number and $q$ is a power of $p$. If $X/k$ is a smooth variety over a perfect field of characteristic $p$, $\textbf{F-Isoc}^1(X)$ denotes the category of overconvergent $F$-isocrystals on $X$ and $\textbf{F-Isoc}^1(X)_{\mathbb{Q}_p}$ denotes its $\mathbb{Q}_p$-linearization. Overconvergent $F$-isocrystals are a $p$-adic analog of lisse $l$-adic sheaves.

Definition 1.1. Let $X/k$ be a smooth, geometrically connected variety over a perfect field $k$ of characteristic $p$. Let $E \in \textbf{F-Isoc}^1(X)_{\mathbb{Q}_p}$. We say that $E$ has infinite local monodromy at infinity if for every triple $(X', \overline{X}', f)$ where $\overline{X}'$ is smooth projective over $k$, $X' \subset \overline{X}'$ is a Zariski open subset, and $f : X' \to X$ is an alteration, the overconvergent $F$-isocrystal $f^*E$ does not extend to an $F$-isocrystal on $\overline{X}$.

This definition of infinite local monodromy at infinity applies equally well to lisse $\overline{Q}_l$-sheaves and is compatible with the other notions of infinite local monodromy at infinity.

Theorem 1.2. Let $X/F_q$ be a smooth, geometrically connected, quasiprojective variety. Let $E \in \textbf{F-Isoc}^1(X)$ be a semisimple overconvergent $F$-isocrystal. Suppose:

- for every closed point $x$ of $X$, the polynomial $P_x(E, t)$ has coefficients in $\mathbb{Q} \subset \mathbb{Q}_p$;
- every irreducible summand $E_i \in \textbf{F-Isoc}^1(X)_{\mathbb{Q}_p}$ of $E$ has rank 2, determinant $\overline{Q}_p(-1)$, and infinite local monodromy around infinity.

Then $E$ comes from a family of abelian varieties. More precisely, there exists a non-empty open subset $U \subset X$ and an abelian scheme $A_U \to U$, so that $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p \cong E|_U$.

Here, if $G \to X$ is a $p$-divisible group, $\mathbb{D}(G)$ is the (contravariant) Dieudonné crystal attached to $G$. We have the following application. Deligne formulated what is now called the companions conjecture in [Del80, Conjecture 1.2.1]. For a guide to the crystalline companions conjecture, see [Ked16, Ked18].

Corollary 1.3. Let $X/F_q$ be a smooth, geometrically connected variety. Let $L_1$ be an irreducible rank 2 lisse $\overline{Q}_l$ sheaf with infinite monodromy around infinity and determinant $\overline{Q}_l(-1)$. Then the following are equivalent:

1. there exists a non-empty open subset $U \subset X$ and an abelian scheme $\pi : A_U \to U$ such that $L_1$ is a summand of $R^1(\pi_U)_*\overline{Q}_l$;
2. all crystalline companions to $L_1$ exist (as predicted by Deligne's crystalline companions conjecture).
Corollary 1.4. Let $X/\mathbb{F}_q$ be a smooth, geometrically connected variety. Let $\mathcal{E}_1$ be an irreducible rank 2 object of $\text{F-Isoc}^1(X)_{\overline{\mathbb{Q}}_p}$ with infinite monodromy around infinity and determinant $\mathbb{C}_p(-1)$. Suppose the (number) field $E_1 \subset \overline{\mathbb{Q}}_p$ generated by the coefficients of $P_x(\mathcal{E}_1, t)$ as $x$ ranges through the closed points of $X$ has a single prime over $p$. Then $\mathcal{E}_1$ comes from a family of abelian varieties: there exists a non-empty open subset $U \subset X$ and an abelian scheme $A_U \to U$ such that $\mathcal{E}_1|_U$ is a summand of $D(A_U[p^\infty]) \otimes \overline{\mathbb{Q}}_p$

In particular, Corollaries 1.3 and 1.4 provide some evidence for [Dri12, Question 1.4] and [KP18, Conjecture 1.2]. We emphasize that, in contrast to our earlier work [KP18], this article does not use Serre-Tate deformation theory nor does it use the algebraization/globalization techniques of [Har70].

We briefly sketch the proof. Drinfeld’s first work on the Langlands correspondence for $GL_2$, together with Abe’s work on the $p$-adic Langlands correspondence, implies Theorem 1.2 when $\dim(X) = 1$. To do the higher-dimensional case, we first assume that $X$ admits a simple normal crossings compactification $\overline{X}$ and $E$ is a logarithmic $F$-isocrystal with nilpotent residues. (We recall the notion of logarithmic $F$-isocrystals in Appendix A.) A technique of Katz, combined with slope bounds originally due to Laforeigne, allow one to construct a logarithmic Dieudonné crystal whose associated logarithmic $F$-isocrystal is isomorphic to $E$. This logarithmic Dieudonné crystal yields a natural line bundle, which we call the Hodge bundle $\omega$ of the logarithmic Dieudonné crystal, on $X$.

For any odd prime $l \neq p$, it is well-known that the Hodge line bundle is ample on $\mathfrak{A}_{h,1,l}$ over $\text{Spec}(\mathbb{Z}[1/l])$. We use Poonen’s Bertini theorem over finite fields together with Drinfeld’s result and Zarhin’s trick to find a well-adapted family of rather ample space-filling curves $\mathcal{C}_n$ of $\overline{X}$ that each mapping to the minimal compactification $\mathfrak{A}_{h,1,l} \subset \mathbb{P}^m$ via some fixed power of the Hodge bundle $\omega_{\mathcal{C}_n}^{[m]}$. The finitude of the set $H^0(\overline{X}, \omega^{[1]})$ ensures that infinitely many of these maps can be pieced together to a map $\overline{X} \to \mathfrak{A}_{h,1,l} \subset \mathbb{P}^m$; hence we obtain an abelian scheme $\psi_U : B_U \to U$ over some open $U \subset X$. The space-filling properties of the $\mathcal{C}_n$ and Zarhin’s work on the Tate isogeny theorem for fields finitely generated over $\mathbb{F}_q$ then allow us to conclude.

To deduce the general case, we use Kedlaya’s semistable reduction theorem for overconvergent $F$-isocrystals.

2. Preliminaries

Before proving Theorem 1.2, we need several preliminary results. A key ingredient in the proof is the following, which is a hyproduct of Drinfeld’s first work on the Langlands correspondence for $GL_2$.

Theorem 2.1. (Drinfeld) Let $C/\mathbb{F}_q$ be a smooth affine curve and let $L_1$ be a rank 2 irreducible $\mathcal{U}_l$ sheaf with determinant $\mathcal{U}_l(-1)$. Suppose $L_1$ has infinite local monodromy around some point at $\infty \in \mathcal{U}_l \setminus C$. Then $L_1$ comes from a family of abelian varieties in the following sense: let $E$ be the field generated by the Frobenius traces of $L_1$ and suppose $[E : \mathbb{Q}] = g$. Then there exists an abelian scheme $\pi_C : A_C \to C$

of dimension $g$ and an isomorphism $E \cong \text{End}_{C}(A) \otimes \mathbb{Q}$, realizing $A_C$ as a $GL_2$-type abelian scheme, such that $L_1$ occurs as a summand of $R^1(\pi_C)_*\mathcal{U}_l$. Moreover, $A_C \to C$ is totally degenerate around $\infty$.

See [ST18, Proof of Proposition 19, Remark 20] for how to recover this result from Drinfeld’s work.³

We will also need the following useful lemma to ensure that, given the hypotheses of Theorem 1.2, every $p$-adic companion of $E_1$ is again a summand of $E$; moreover, the companion relation preserves multiplicity in the isotypic decomposition of $E$.

Lemma 2.2. Let $X/\mathbb{F}_q$ be a smooth, geometrically connected variety and let $l \neq p$ be a prime.

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1Our motivation for formulating this conjecture was a celebrated theorem of Corlette-Simpson over $\mathbb{C}$, see [CS08, Theorem 11.2]. The result of Corlette-Simpson uses non-abelian Hodge theory.

2This step uses foundational work of Kato, Kedlaya, le Stum, and Tríhan that we explicate in Appendix A.

3This amounts to combining [D683, Main Theorem, Remark 5] with [D677, Theorem 1].
(1) let L be a lisse, semi-simple \( \mathbb{Q}_l \)-sheaf on X, all of whose irreducible summands \( L_i \) have algebraic determinant. Suppose for all closed points \( x \) of X, we have:

\[
P_x(L, t) \in \mathbb{Q}[t] \subset \mathbb{Q}_l[t].
\]

Let \( L_i \) be an irreducible summand of L that occurs with multiplicity \( m_i \) and \( \iota \in \text{Aut}_{\mathbb{Q}}(\mathcal{O}_X) \) be a field automorphism. Then the \( \iota \)-companion to \( L_i \), denoted \( ^{\iota}L_i \), is isomorphic to an irreducible summand of L that occurs with multiplicity \( m_i \).

(2) Let \( F \) be a semi-simple object of \( \textbf{F}\mathit{-Isoc}^!(X)_{\mathbb{Q}_p} \), all of whose irreducible summands \( F_i \) have algebraic determinant. Suppose for all closed points \( x \) of X, we have:

\[
P_x(F, t) \in \mathbb{Q}[t] \subset \mathbb{Q}_p[t].
\]

Let \( F_i \) be an irreducible summand of \( F \) that occurs with multiplicity \( m_i \) and let \( \iota \in \text{Aut}_{\mathbb{Q}}(\mathcal{O}_X) \). Then the \( \iota \)-companion of \( F_i \), denoted \( ^{\iota}F_i \), exists and is isomorphic to a direct summand of \( F \) that occurs with multiplicity \( m_i \).

Before beginning the proof, we note that \( l \)-adic companions are known to exist for smooth varieties over finite fields [Dri12, Theorem 1.1]. In contrast, \( p \)-adic companions are not yet known to exist except when \( X \) is a curve [Abe18].

**Proof.** We reduce the crystalline case to the étale case. (Note that we could have equivalently proceeded by reduction to curves using [AE19].) As \( F \) is semisimple, write an isotypic decomposition:

\[
F \cong \bigoplus_{i=1}^a F_i^{m_i}.
\]

Note that each \( F_i \) is pure by [AE19, Theorem 2.7]. Fix an isomorphism \( \sigma : \mathbb{Q}_p \to \mathcal{O}_X^\times \). By [AE19, Theorem 4.2] or [Ked18, Corollary 3.5.3], the \( \sigma \)-companion to each \( F_i \) exists as an irreducible lisse \( \mathbb{Q}_l \)-sheaf \( L_i \). Setting \( L \) to be the semi-simple \( \sigma \)-companion of \( F \), we have:

\[
L \cong \bigoplus_{i=1}^a L_i^{m_i}.
\]

Set \( \iota \in \text{Aut}_{\mathbb{Q}}(\mathcal{O}_X) \). Then \( F_j \) is the \( \iota \)-companion to \( F_i \) if and only if \( L_j \) is the \( \sigma \circ \iota \circ \sigma^{-1} \)-companion to \( L_i \). Therefore it suffices to prove the result in the étale setting.

Let \( M \) be an irreducible lisse \( \mathbb{Q}_l \)-sheaf on X. Then \( M \) is pure by [Dell2, Théorème 1.6] and class field theory. Then the multiplicity of \( M \) in the semisimple sheaf \( L \) is: \( \text{dim}(H^0(X, M^* \otimes L)) \). By assumption we have that for all closed points \( x \) of X, \( P_x(L, t) \in \mathbb{Q}[t] \subset \mathbb{Q}_l[t] \). Let \( \iota \in \text{Aut}_{\mathbb{Q}}(\mathcal{O}_X) \), and note that the semi-simple \( \iota \)-companion to \( L \) is again isomorphic to \( L \). Then the \( \iota \)-companion to \( M^* \otimes L \) is isomorphic to \( \langle ^{\iota}M \rangle^* \otimes L \). On the other hand, the exact argument of [AE19, 3.2] for lisse \( l \)-adic sheaves implies that \( \text{dim}(H^0(X, M^* \otimes L)) = \text{dim}(H^0(X, \langle ^{\iota}M \rangle \otimes L)) \). This implies the result. \( \square \)

**Remark 2.3.** It follows from Lemma 2.2 that, in the context of Theorem 2.1, there is a decomposition:

\[
R^1(\pi_C)_* \mathbb{Q}_l \cong \bigoplus_{i=1}^g \langle L_i \rangle
\]

where the \( L_i \) form a complete set of \( \mathbb{Q}_l \) companions.\(^5\) In particular, each companion occurs with multiplicity 1. In fact, as \( E \cong \text{End}_{\mathbb{C}}(A_C) \otimes \mathcal{Q}_l \), it follows that \( E \otimes \mathbb{Q}_l \) acts on \( R^1(\pi_C)_* \mathbb{Q}_l \). On the other hand, \( E \otimes \mathbb{Q}_l \cong \prod_i \mathbb{Q}_l \), where \( i \) runs over the embeddings \( E \hookrightarrow \mathbb{Q}_l \). For each \( i \), pick a non-trivial idempotent \( e_i \in E \otimes \mathbb{Q}_l \) whose image is the \( i \)-th component of the direct product decomposition. The above direct sum decomposition is induced by these \( e_i \).

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\(^4\)This argument is based on [La02, Cor VI.3] and uses L-functions. See also [Ked18, Lemma 3.1.5, Theorem 3.3.1].

\(^5\)There are exactly \( g \) non-isomorphic companions because the field generated by Frobenius traces of \( L_1 \) is isomorphic to \( E \) and the \( l \)-adic companions are in bijective correspondence with the embeddings \( E \hookrightarrow \mathbb{Q}_l \).
To apply Drinfeld’s Theorem 2.1, we need the following geometric lemma. We will later use Lemma 2.4 in conjunction with Poonen’s Bertini theorem to construct a sequence of space-filling curves.

**Lemma 2.4.** Let $Y/\mathbb{F}_q$ be a smooth, geometrically connected, projective variety and let $\alpha$ be a line bundle on $Y$. Let $M \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a closed subset. Suppose there exists an infinite collection $(C_n)_{n \in \mathbb{N}}$ of smooth, projective, geometrically connected curves $C_n \subset Y$ such that

1. for each $n \in \mathbb{N}$, the natural map $H^0(Y, \alpha) \to H^0(C_n, \alpha|_{C_n})$ is an isomorphism;
2. for any infinite subset $S \subset \mathbb{N}$, the union

$$
\bigcup_{n \in S} C_n
$$

is Zariski dense in $Y$;
3. for each curve $C_n$, there exists an $m + 1$ globally generating sections

$$
t_{n,0}, \ldots, t_{n,m} \in H^0(C_n, \alpha|_{C_n})
$$

such that the induced map to $\mathbb{P}^m$ factors through $M$:

$$
\begin{array}{ccc}
C_n & \xrightarrow{f_n} & \mathbb{P}^m \\
\downarrow & & \downarrow \\
M & & 
\end{array}
$$

Then there exist global sections $\tilde{t}_0, \ldots, \tilde{t}_m \in H^0(Y, \alpha)$ such that the induced rational map $\tilde{f} : Y \dashrightarrow \mathbb{P}^m$ has image in $M$. Moreover, $\tilde{f}$ can be chosen to be compatible with infinitely many of the maps $f_n$.

**Proof.** There are finitely many ordered $m + 1$-tuples of sections $H^0(Y, \alpha) \cong H^0(C_n, \alpha|_{C_n})$ because $H^0(Y, \alpha)$ is a finite dimensional vector space over $\mathbb{F}_q$. By the pigeonhole principle, in our infinite collection we may find an $m + 1$-tuple of sections $t_{0,0}, \ldots, t_{m} \in H^0(Y, \alpha)$ such that there exists an infinite set $S \subset \mathbb{N}$ with

$$(t_{0,0}, \ldots, t_{m})|_{C_n} = (t_{n,0}, \ldots, t_{n,m})$$

for every $n \in S$. There is therefore an induced rational map $\tilde{f} : Y \dashrightarrow \mathbb{P}^m$ with $\tilde{f}|_{C_n} = f_n$ for each $n \in S$. On the other hand, the collection $(C_n)_{n \in S}$ is Zariski dense in $Y$ by assumption and $\tilde{f}(C_n) \subset M$; therefore the image of $\tilde{f}$ lands inside of $M$, as desired. \hfill $\square$

### 3. Proofs of Theorem 1.2 and Corollaries 1.3, 1.4

**Proof of Theorem 1.2.** As $\mathcal{E}_i$ is irreducible, has determinant $\overline{\mathbb{Q}}_p(-1)$, and has rank 2, the slopes of $(\mathcal{E}_i)_x$ are in the interval $[0, 1]$ for every closed point $x$ of $X$, see e.g. [DK17, Theorem 1.1.5].

Write the isotypic decomposition of $\mathcal{E}$ in $\mathbf{F-Isoc}^\dagger(\mathcal{X})_{\overline{\mathbb{F}}_p}$:

$$
\mathcal{E} \cong \bigoplus_{i=1}^a (\mathcal{E}_i)^{m_i}.
$$

The field generated by the coefficients of $P_x(\mathcal{E}, t)$ as $x$ ranges through closed points of $X$ is $\mathbb{Q}$. Therefore, by [Dri18, E.10] and either [AE19, Theorem 4.2] or [Ked18, Corollary 3.5.3], we can pick an $l$ and a field isomorphism $\sigma : \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_l$ such that the semi-simple $\sigma$ companion $L$ to $\mathcal{E}$ exists and in fact may be defined over $\mathbb{Q}_l$, i.e. corresponds to a representation:

$$
\pi_1(X) \to \text{GL}_N(\mathbb{Q}_l).
$$

(We emphasize that $L$ is independent of the choice of $\sigma$ because the field generated by coefficients of characteristic polynomials $P_x(\mathcal{E}, t)$ as $x$ ranges over closed points of $X$ is $\mathbb{Q}$.) By compactness of $\pi_1(X)$, we may conjugate the representation into $\text{GL}_N(\mathbb{Z}_l)$. We refer to the attached lisse $\mathbb{Z}_l$-sheaf...
as \( \hat{L} \). Similarly, for each \( i \) we denote by \( L_i \) the \( \sigma \)-companion to \( L_i \) (the \( L_i \) indeed do depend on the choice of \( \sigma \)). The companion relation implies that:

\[
L \cong \bigoplus_i L_i^{m_i}
\]

We first assume that \( X \) admits a simple normal crossings compactification \( \hat{X} \) such that \( \mathcal{E} \) extends to a logarithmic \( F \)-isocrystal with nilpotent residues on \( X \) and moreover that \( \hat{L} \) has trivial residual representation. Write \( Z := \hat{X} \setminus X \) for the boundary.

By Lemma A.5, there exists a Zariski open \( \bar{U} \subset \hat{X} \) with complementary codimension at least 2, and a logarithmic Dieudonné crystal \((M_\sigma, F, V)\) on \( \bar{U} \) (with the logarithmic structure coming from \( Z \cap U \)). Let

\[
(N_\sigma, F, V) := (M_\sigma, F, V)^4 \oplus ((M_\sigma, F, V)^t)^4,
\]

where the \( t \) denotes the dual logarithmic Dieudonné crystal. We also consider this logarithmic Dieudonné crystal as we will need to use Zarhin’s trick. We set \( U := \bar{U} \setminus (\bar{U} \cap Z) \).

After Remark A.6, it follows that we may define Hodge line bundles \( \omega_M \) and \( \omega_N \) on \( \bar{U} \) attached to the two logarithmic Dieudonné crystals. As \( \bar{U} \subset \hat{X} \) has complementary codimension at least 2 and \( \hat{X} \) is smooth, it follows that \( \omega_M \) and \( \omega_N \) extend canonically to line bundles on all of \( \hat{X} \).

Let \( E_i \subset \overline{\mathbb{P}}_p \) denote the (number) field generated by the coefficients of \( P_x(\mathcal{E}_i, t) \) as \( x \) ranges through the closed points of \( X \). For each \( \mathcal{E}_i \), set \( \mathcal{F}_i \) to be the sum of all distinct \( p \)-adic companions of \( \mathcal{E}_i \). \(^6\) (Note that there are \([E_i; \mathbb{Q}]\) distinct \( p \)-adic companions of \( \mathcal{E}_i \), parametrized by the embeddings \( E_i \hookrightarrow \overline{\mathbb{P}}_p \).) By reordering the indices, we write the decomposition of \( \mathcal{E} \) as follows:

\[
\mathcal{E} \cong \bigoplus_{i=1}^b \mathcal{F}_i^{m_i}
\]

for some integer \( 1 \leq b \leq a \). (Under this reordering, the collection of \((\mathcal{E}_i)^b_{i=1}\) are all mutually non companions and for each \( b+1 \leq j \leq a \), there exists a unique \( 1 \leq i \leq b \) such that \( \mathcal{E}_j \) is a companion of \( \mathcal{E}_i \).) Set \( g = \sum_{i=1}^b m_i [E_i; \mathbb{Q}] \).

The Hodge line bundle \( \alpha \) on the fine moduli scheme \( \mathcal{A}_{g,1,1} \otimes \mathbb{Q}_p \) is ample by [Mor85, Ch. IX, Théorême 3.1, p. 210] or [FC90, Ch. V, Theorem 2.5(1), p. 152]. Pick an \( r \) so that the \( \alpha^r \) is very ample on \( \mathcal{A}_{g,1,1} \). Fix a basis \( s_0, \ldots, s_m \) of the vector space:

\[
H^0(\mathcal{A}_{g,1,1} \otimes \mathbb{Q}_p, \alpha^r)
\]

once and for all. There is an induced embedding \( \mathcal{A}_{g,1,1} \subset \mathbb{P}^m \). As is customary, denote by \( \mathcal{A}_{g,1,1}^* \) the Zariski closure of \( \mathcal{A}_{g,1,1} \) in \( \mathbb{P}^m \); after [FC90, Ch. V], we call this the minimal compactification.

It follows from [Del12] there exists a finite number of closed points \( \{x_j\}_{j=1}^s \) of \( \bar{U} \) such that for each \( \mathcal{E}_i \), the field generated by the coefficients of \( P_{x_j}(\mathcal{E}_i, t) \in \mathbb{Q}_p[t] \) as \( j = 1 \ldots s \) is \( E_i \subset \overline{\mathbb{P}}_p \).

**Definition 3.1.** In the context of the above, we say that \( \tilde{C} \subset \bar{U} \) is a good curve if

- \( \tilde{C} \) is the smooth complete intersection of smooth ample divisors of \( \bar{X} \) that intersect \( Z \) in good position;
- \( \tilde{C} \) contains each of the closed points \( x_j \), for \( j = 1 \ldots s \);
- the natural map \( H^0(\bar{X}, \omega_N^r) \to H^0(\tilde{C}, \omega_N^r|_{\tilde{C}}) \) is an isomorphism.

If \( \tilde{C} \subset \bar{X} \) is a good curve, set \( C := \tilde{C} \setminus (\tilde{C} \cap Z) \). Then:

- each \( \mathcal{E}_i|_{\tilde{C}} \) is irreducible by [AE19, Theorem 2.6];
- the field generated by Frobenius traces of \( \mathcal{E}_i|_{\tilde{C}} \) is \( E_i \); and
- each \( \mathcal{E}_i|_{\tilde{C}} \) has infinite monodromy around \( \infty \).\(^7\)

\(^6\) All \( p \)-adic companions exist and are summands of \( \mathcal{E} \) by Lemma 2.2.

\(^7\) From the positivity of \( \tilde{C} \), and the good position assumption, it follows that \( \tilde{C} \) intersects each component of \( Z \) is a non-empty and transverse way. Then the assumption that \( \mathcal{E}_i \) has infinite monodromy around \( Z \) implies \( \tilde{C} \) has infinite monodromy around \( \tilde{C} \cap Z \).
It follows from the construction of $p$-to-$l$ companions for curves, Theorem 2.1, and Remark 2.3 together with Equation 3.1 that if $C \subset X$ is a good curve, then there exists an abelian scheme $\pi_C : A_C \to C$ of relative dimension $g$ such that

$$R^1(\pi_C)_* \mathcal{O}_C \cong L|_C.$$  

As we assumed that the $\mathbb{Z}$-lattice $\hat{L}$ has trivial residual representation, it follows that we may replace $A_C$ by an $l$-primarily-isogenous abelian scheme so that $R^1(\pi_C)_* \mathcal{O}_C \cong \hat{L}$ and hence that the étale group scheme $A_C[l] \to C$ is split.

Similarly, we have that $\mathcal{D}(A_C[p^\infty]) \otimes \mathcal{O}_p \cong \mathcal{E}$. Therefore $\mathcal{D}(A_C[p^\infty])$ is isogenous to $(M,F,V)_C$ as Dieudonné crystals on $C$. Replacing $A_C$ by an $(p$-primarily) isogenous abelian scheme, we may ensure that:

$$\mathcal{D}(A_C[p^\infty]) \cong (M,F,V)_C$$

as Dieudonné crystals on $C$.\footnote{To see this, use [LJ95] to construct a $p$-divisible group $G_C$ on $C$ where $\mathcal{D}(G_C) \cong (M,F,V)$. It follows that $A_C[p^\infty]$ and $G_C$ are isogenous. Pick an isogeny $A_C[p^\infty] \to G_C$; the kernel $K \subset A$ is a $p$-primary finite flat group scheme. Replace $A_C$ by $A_C/K$.}

By our assumptions, the $l$-torsion is trivial; hence $A_C$ has semistable reduction along $C \cap Z$. Let $A_C \to \tilde{C}$ be the Néron model. It follows from Remark A.7 that the logarithmic Dieudonné crystal of $A_C \to \tilde{C}$ is isomorphic to $(M,F,V)_C$. Again by Remark A.7, the Hodge bundle of the $A_C \to \tilde{C}$ is isomorphic to $\omega_{M,\tilde{C}}$.

Set $\mathcal{B}_C := (A_C \times_C A_C^1)^g$. By Zarhin’s trick [Mor85, Chapitre IX, Lemme 1.1, p. 205], $\mathcal{B}_C$ admits a principal polarization. By construction, we have that

- $\mathcal{B}_C$ has trivial $l$-torsion, and
- $\mathcal{D}(\mathcal{B}_C[p^\infty]) \cong (N_C,F,V)$

Once more, by Remark A.7 it follows that there is an isomorphism of logarithmic Dieudonné crystals:

$$\mathcal{D}(\mathcal{B}_C[p^\infty]) \cong (N,F,V)_C.$$  

The Hodge line bundle of $\mathcal{B}_C$ is hence isomorphic to $\omega_N|_{\tilde{C}}$. However, we emphasize that the choice $\mathcal{B}_C \to \tilde{C}$ is not canonical!

We have an induced morphism to a fine moduli scheme $C \to \mathcal{A}_{g,1,l}$. This extends to a morphism

$$\tilde{C} \to \mathcal{A}_{g,1,l}^*$$

to the minimal compactification. We now claim the pullback of $\alpha$, the Hodge line bundle on $\mathcal{A}_{g,1,l}^*$, is isomorphic to $\omega_N|_{\tilde{C}}$. Here is the reason: choose a toroidal compactification $\mathcal{A}_{g,1,l}^\tau$. We then have a commutative diagram:

\begin{equation}
\begin{array}{ccc}
\tilde{C} & \xrightarrow{h} & \mathcal{A}_{g,1,l}^\tau \\
\downarrow \varphi & & \downarrow \\
\mathcal{A}_{g,1,l}^*
\end{array}
\end{equation}

By [FC90, Ch. V, Theorem 2.5, p. 152], there is a semi-abelian scheme $G \to \mathcal{A}_{g,1,l}^\tau$ such that $\varphi^*\alpha$ is isomorphic to the Hodge line bundle of $G \to S$. Now, [FC90, Ch. I, Proposition 2.7, p.9] implies that $h^*G$ is isomorphic to the semi-abelian scheme given by the open subset of $A_C \to \tilde{C}$ obtained by removing the non-identity components along $\tilde{C}\setminus C$. In particular, it follows that the Hodge line bundle of $h^*G$ is compatible with the Hodge line bundle constructed in Remark A.7.

We have already picked a basis of sections

$$s_0, \ldots, s_m \in H^0(\mathcal{A}_{g,1,l} \otimes \mathcal{O}_\tilde{C}, \alpha^\tau),$$

extending the above arrow to the following:

$$\tilde{C} \to \mathcal{A}_{g,1,l}^\tau \subseteq \mathbb{P}^m.$$
After pulling back to $C$, we obtain an $m + 1$-tuple of sections $t_0, \ldots, t_m$ in $H^0(C, \omega^*_N|_C)$ that define the morphism $\tilde{C} \to \mathbb{P}^m$. To repeat, for every good curve $\tilde{C} \subset \tilde{X}$, we have constructed an $m + 1$-tuple of globally generating sections $t_0, \ldots, t_m \in H^0(\tilde{C}, \omega^*_N|_C)$ such that the induced map lands in $\mathcal{A}_{8g,1,1} \subset \mathbb{P}^m$, such that $C$ maps into $\mathcal{A}_{8g,1,1}$, and such that the induced abelian variety on $C$ has Dieudonné crystal isomorphic to $(\overline{N}, F, V)_{\mathbb{C}}$.

We claim there exists an infinite collection of good curves $(\tilde{C}_n)_{n \in \mathbb{N}}$ such that for each $n$, the curve $C_n \subset U$ contains all closed points of $U$ whose residue field is contained in $\mathbb{F}_{q^n}$. Indeed, this is an immediate consequence of [Poo04, Theorem 1.3] (note that the condition that $H^0(\tilde{X}, \omega^*_N) \to H^0(\tilde{C}, \omega^*_N|_C)$ is an isomorphism is guaranteed by taking $C$ to be the complete intersection of sufficiently ample smooth divisors of $\tilde{X}$: use induction, Serre duality, and Serre vanishing.)

For each $n \in \mathbb{N}$, by the above remarks we obtain an $m + 1$-tuple of globally generating sections

$$t_{n,0}, \ldots, t_{n,m} \in H^0(\tilde{C}_n, \omega^*_N|_{\tilde{C}_n})$$

such that the induced map factors $f_n: \tilde{C}_n \to \mathcal{A}_{8g,1,1} \subset \mathbb{P}^m$. Moreover, any infinite subcollection of the $C_n$ are Zariski dense because they are space-filling. By Lemma 2.4, it follows that there exists an infinite set $S \subset \mathbb{N}$ and sections $t_{0,n}, \ldots, t_{m,n} \in H^0(X, \omega^*_N)$ such that the induced rational map $f: X \dashrightarrow \mathbb{P}^m$ lands in $\mathcal{A}_{8g,1,1}$ and moreover, for each $n \in S$, we have an equality of morphisms $f|_{\tilde{C}_n} = f_n$.

By shrinking $U$, we therefore obtain a map $\tilde{f}: U \to \mathcal{A}_{8g,1,1}$ and hence an abelian scheme $B_U \to U$ such that $B_U|_{\{l\}}$ is a trivial étale cover of $U$. The $C_n$ are space filling and the maps $f_n: \tilde{C}_n \to \mathcal{A}_{8g,1,1}$ were all constructed such that the induced abelian scheme $B_{C_n} \to C_n$ is compatible with $(N_{C_n}, F, V) \otimes \mathbb{Q}_p \cong (E \oplus E^*(-1))^4|_C$. Therefore $B_U \to U$ is compatible with $(L \oplus L^*(-1))^4|_U$.

For each $n \in S$ we have that $f|_{\tilde{C}_n} = f_n$ by Lemma 2.4. Then, by construction there exists an abelian scheme $A_{C_n} \to C_n$ of dimension $g$ with

$$B_U|_{C_n} \cong (A_{C_n} \times C_n A^*_n)^4.$$  

Consider the map of representations induced by the first $\mathbb{Z}_l$-cohomology of the abelian schemes $B_U \to U$ and $B_U|_{C_n} \to C_n$:

$$\pi_1(C_n) \to \pi_1(U) \\
\text{GL}_{16g}(\mathbb{Z}_l).$$

Then [Kat01, Lemma 6(b)] implies that for $n \gg 0$, the two representations have the same image. By the fundamental work of Tate-Zarhin on Tate’s isogeny theorem for abelian varieties over finitely generated fields of positive characteristic [Mor85, Ch. XII, Théorème 2.5(i), p. 244], it follows that the natural injective map $\text{End}_{\mathbb{C}}(B_U) \hookrightarrow \text{End}_{\mathbb{C}}(B_U|_{C_n})$ is an isomorphism when tensored with $\mathbb{Z}_l$ and hence also when tensored with $\mathbb{Q}_l$. It follows that the map

$$\text{End}_{\mathbb{C}}(B_U) \otimes \mathbb{Q} \to \text{End}_{\mathbb{C}}(B_U|_{C_n}) \otimes \mathbb{Q}$$

is an isomorphism as both sides are finite dimensional semi-simple $\mathbb{Q}$-algebras of the same rank.

We know that $\text{End}_{\mathbb{C}}(B_U|_{C_n})$ has a nontrivial idempotent $e_{C_n}$ that projects onto a copy of $A_{C_n}$. After replacing $e_{C_n}$ by a high integer multiple, we may lift $e_{C_n}$ to $e_U \in \text{End}_{\mathbb{C}}(B_U)$. Set the image of $e_U$ to be the abelian scheme $\pi_U: A_U \to U$. It follows from Equation 3.3 that $A_U$ is compatible with $E$ (equivalently: $L$).

We have proven the theorem when $X$ has an simple normal crossings compaction $\tilde{X}$ such that $E$ extends to a logarithmic $F$-isocrystal with nilpotent residues and the residual representation of $L$ is trivial. We now prove the theorem in general, using our above work.

There exists a projective divisorial compactification $\tilde{X}$ of $X$. By Kedlaya’s semistable reduction theorem (see [Ked16, Theorem 7.6] for a meta-reference), there is a generically étale alteration $\varphi: X' \to X$ together with a simple normal crossings compactification $\tilde{X}'$ such that the overconvergent pullback

\[^9\text{This means that } \tilde{X} \text{ is normal and the boundary is an effective Cartier divisor.}\]
$\mathcal{E}'$ extends to a logarithmic $F$-isocrystal with nilpotent residues. After replacing $X'$ with a further étale cover, we may guarantee that the residual representation of $L'$ is trivial.

We have proven the theorem for $\mathcal{E}'$ on $X'$: there exists an open subset $W' \subset X'$ and an abelian scheme $A_{W'} \to W'$ with $D(A_{W'}[p^\infty]) \cong \mathcal{E}'|_{W'}$. After shrinking $W'$ and $W$, we may assume that $\varphi|_{W'} : W' \to W$ is finite étale, of degree $d$.

Set $B_W := \text{Res}^W_{W'}(A_{W'})$ to be the Weil restriction of scalars. This is an abelian scheme over $W$ that is compatible with $L^d$. Recall that we wrote an isotypic decomposition:

$$L \cong \bigoplus_{i=1}^a (L_i)^{m_i},$$

where each $L_i$ is irreducible on $X$ (and hence on $W$). Let $E_i \subset \overline{\mathbb{Q}}_l$ denote the field generated by the traces of Frobenius on $L_i$ as $x$ ranges through the closed points of $W$. We claim that we may find a smooth curve $C \subset W$ with the following properties:

1. Each $L_i|_C$ is irreducible;
2. The field generated by Frobenius traces of $L_i|_C$ is $E_i \subset \overline{\mathbb{Q}}_l$;
3. Each $L_i|_C$ has infinite monodromy around $\infty$; and
4. The induced monodromy representations coming from $B_W \to W$ and $B_W|_C \to C$

have the same image.

We have a projective normal compactification $\overline{X}$ of $X$, which is smooth away from a closed subset of codimension at least 2. Let $F = \overline{X} \setminus X$ and let $F' \subset F$ be the singular locus of $\overline{X}$. For each $L_i$, there is an irreducible component $F_j$ of $F$ that witnesses the fact that $L_i$ has infinite monodromy at $\infty$: having infinite monodromy at $\infty$ means that a certain inertia group has infinite image in the representation.

Pick a closed point $y_j \in F_j \setminus (F_j \cap F')$ for each $j$. Then, by using [Dri12, C.2], we may construct an infinite set of curves $(C_n)_{n \in \mathbb{N}}$ where each $C_n \subset W$ is a smooth, geometrically connected curve that contains all closed points of $W$ whose residue fields are contained in $\mathbb{F}_{q^n}$ and that pass through the $y_j$ transversally (i.e., with a tangent direction that is not contained in $F_j$). (We remark that this is a consequence of [Poc04, Theorem 1.3].)

Each $L_i|_{C_n}$ has infinite monodromy around $\infty$. By [Kat01, Lemma 6(b)], it follows that for all $n \gg 0$, $C_n$ satisfies (4). For $n \gg 0$, [Kat01, Lemma 6(b)] and [Del12] guarantees that setting $C := C_n$ satisfies the above four conditions.

Again, by using Drinfeld’s Theorem 2.1 and Remark 2.3, there exists $A_C \to C$ that is compatible with $L|_C$. On the one hand, using the Tate isogeny theorem [Mor85, Ch. XII, Théorème 2.5] it follows that $A_C^d$ is isogenous to $B_W|_C$. On the other hand, another application the Tate isogeny theorem together with property (4) of $C$ implies that the natural map

$$\text{End}_W(B_W) \to \text{End}_C(B_W|_C)$$

is an isomorphism after tensoring with $\mathbb{Q}$. As $B_W|_C$ is isogenous to $A_C^d$, it follows that $\text{End}_C(B_W|_C) \otimes \mathbb{Q}$ has an element $e_C$ projecting onto a factor of $A_C$. After replacing $e_C$ with a high integer multiple, we may lift to $e_W \in \text{End}_W(B_W)$. Set the image of $e_W$ to be the abelian scheme $A_W \to W$; this is compatible with $L|_W$, as desired. \[\square\]

Proof of Corollary 1.3. Suppose there exists $\pi_U : A_U \to U$ such that $R^1(\pi_U)_* \mathbb{Q}$ has $L_1$ as a summand. A theorem of Zarhin implies that $R^1(\pi_U)_* \mathbb{Q}$ is semi-simple [Mor85, Chapitre XII, Théorème 2.5 on p. 244-245]. The field generated by the characteristic polynomials of $R^1(\pi_U)_* \mathbb{Q}$ is clearly $\mathbb{Q}$.\[\square\]
Similarly, $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$ is a semi-simple object of $\mathbf{F}$-$\mathbf{Isoc}^1(U)$ by [Pal15]. As $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$ is isomorphic to the rational crystalline cohomology of $A_U \to U$, it follows from [KM74] that $\mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$ and $R^1(\pi_U)_*\mathbb{Q}_l$ are companions. It follows from Lemma 2.2 that all crystalline companions of $L_1|U$ exist and moreover are summands of $\mathbb{D}(A_U[p^\infty]) \otimes \overline{\mathbb{Q}}_p$. Then by [Ked18, Corollary 3.3.3], all crystalline companions to $L_1$ exist.

Conversely, suppose all crystalline companions $(E_i)_{i=1}^n$ to $L_1$ exist. They all have infinite monodromy at $\infty$ be the companion relation. There exists a $p$-adic local field $K$ with each $E_i$ an object of $\mathbf{F}$-$\mathbf{Isoc}^1(X)_K$. Set $E := \bigoplus_{i=1}^n E_i$, considered as an object of $\mathbf{F}$-$\mathbf{Isoc}^1(X)$ (by restricting scalars from $K$ to $\mathbb{Q}_p$, so the rank of $E$ is $2b|K : \mathbb{Q}|$). Then $E$ satisfies the hypotheses of Theorem 1.2, and moreover $L_1$ is a companion of a summand of $E$. It follows that there is an open set $U \subset X$ together with an abelian scheme $\pi_U : A_U \to U$ such that $E \cong \mathbb{D}(A_U[p^\infty]) \otimes \mathbb{Q}_p$. Again using Zarhin’s semi-simplicity, $L_1|U$ is a summand of $R^1(\pi_U)_*\mathbb{Q}_l$, as desired. \hfill $\Box$

**Proof of Corollary 1.4.** Under the assumption on $E_1$, all $p$-adic companions to $E_1$ exist by [KP18, Corollary 4.16]. (This result is straightforward; they are all Galois twists of each other.) Fix $\sigma : \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_l$. Then the $\sigma$-companion to $E_1$ exists by [AE19, Theorem 4.2] or [Ked18, Corollary 3.5.3]. Apply Corollary 1.3. \hfill $\Box$

**Appendix A. Logarithmic F-crystals**

We first recall the notion of a logarithmic $F$-crystal/isocrystal. While this notion is due to Kato [Kat89, Section 6], our treatment is copied from recent work of Kedlaya.

**Definition A.1.** A smooth pair over a perfect field $k$ is a pair $(Y, Z)$ where $Y/k$ is a smooth variety and $Z \subset Y$ is a strict normal crossings divisor.

**Definition A.2.** Let $(Y, Z)$ be a smooth pair over a perfect field $k$ of characteristic $p > 0$. A smooth chart for $(Y, Z)$ is a sequence of elements $t_1, \ldots, t_n$ of elements of $O_Y(Y)$ such that the

- induced map $\tilde{f} : Y \to \mathbb{A}^n$ is étale, and
- there exists an $m \in [0, n]$ such that the zero-loci of $\tilde{t}_i$, for $i = 0 \ldots m$, are exactly the irreducible components of $Z$.

Let $(Y, Z)$ be a smooth pair over a perfect field $k$ of characteristic $p > 0$. Let $\tilde{t}_1, \ldots, \tilde{t}_n$ be a smooth chart of $(Y, Z)$. Let $P_0$ be the formal scheme given by the formal completion of $W(k)[t_1, \ldots, t_n]$ along $(p)$. By topological invariance of the étale site, there exists a unique smooth formal scheme $P$ together with an étale morphism $f : P \to P_0$ lifting $\tilde{f}$. We call the pair $(P, t_1, \ldots, t_n)$ the lifted smooth chart of $(Y, Z)$ associated to the original chart.

Let $\sigma_0 : P_0 \to P_0$ be the Frobenius lift with $\sigma^*(t_i) = t_i^p$ for $i \in [0, \ldots, n]$. Then there exists an associated Frobenius lift $\sigma : P \to P$.

**Definition A.3.** Let $(Y, Z)$ be a smooth pair over a perfect field $k$ and let $\tilde{t}_1, \ldots, \tilde{t}_n$ be a smooth chart of $(Y, Z)$. Keep notations as above. A logarithmic crystal with nilpotent residues on $(Y, Z)$ is a pair $(M, \nabla)$ where

- $M$ is a $p$-torsion free coherent module over $P^{10}$; and
- $\nabla$ is an integrable, topologically quasi-nilpotent connection on $M$ (with respect to $W(k)$) with logarithmic poles and nilpotent residues along the zero-loci of $f^*(\tilde{t}_i)$.

A logarithmic $F$-crystal with nilpotent residues is a triple $(M, \nabla, F)$ where $(M, \nabla)$ is a logarithmic crystal with nilpotent residues and $F$ is an injective, horizontal morphism

$$F : \sigma^*(M) \to M$$

of coherent $P$-modules. A logarithmic Dieudonné crystal with nilpotent residues is a quadruple $(M, \nabla, F, V)$ where $(M, \nabla, F)$ is a logarithmic $F$-crystal in finite, locally free modules with nilpotent residues.

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10Note that we don’t demand that $M$ is locally free!
residues and $V$ is an injective, horizontal map

$$V: (M) \to \sigma^* M$$

such that $FV = VF = p$. 

This definition extends to general smooth pairs by Zariski gluing; every smooth pair admits a finite open covering on which the restriction admits a smooth chart. We often drop the connection $\nabla$ from the notation.

There is a natural category of logarithmic crystals with nilpotent residues on $(Y, Z)$ (where morphisms are $P$-linear and horizontal), and the category of logarithmic isocrystals with nilpotent residues is defined to be the induced isogeny category. One similarly defines the category of logarithmic $F$-isocrystals with nilpotent residues.

**Remark A.4.** Let $(Y, Z)$ be a smooth pair over $k$ and let $U = Y \setminus Z$. We denote by $Y$ the (fine, saturated) logarithmic scheme given by $(Y, \alpha: \mathcal{O}_U \to \mathcal{O}_Y)$. Then our definition of a logarithmic crystal is compatible with the definition of Katz (see [Kat79, Theorem 6.2]), our definition of a logarithmic $F$-crystal in finite, locally free modules is compatible with the definition of Kato-Trihan [KT03, 4.1] and our definition of a logarithmic $F$-isocrystal is compatible with the definition given by Shiho (see [Shi00, Definition 4.1.3]).

The mathematical content of the following lemma is essentially [Kat79, Theorem 2.6.1] (and relatedly [Cre87, Lemma 2.5.1]); we have simply rewritten Katz’s argument to the logarithmic setting. The key is that Katz’s slope bounds holding on the open subset where the logarithmic structure is trivial guarantee that they hold everywhere. We use Kato’s definition of logarithmic $F$-crystals only for convenience to discuss global objects; all of the computations use the local definitions given above.

**Lemma A.5.** Let $(Y, Z)$ be a smooth pair over a perfect field $k$ of positive characteristic and let $U := Y \setminus Z$. Let $\mathcal{E}$ be a logarithmic $F$-isocrystal on $(Y, Z)$.

1. Suppose the Newton slopes of $\mathcal{E}_U$ are all non-negative. Then there exists an open subset $W \subset Y$, whose complementary codimension is at least 2, and a logarithmic $F$-crystal in finite, locally free modules $(M^n, F)$ on the smooth pair $(W, W \cap Z)$ such that $(M^n, F) \otimes \mathbb{Q} \cong \mathcal{E}_W$.

2. Suppose the Newton slopes of $\mathcal{E}_U$ are in the interval $[0,1]$. Then there exists an open subset $W \subset Y$, whose complementary codimension is at least 2, and a logarithmic Dieudonné crystal in finite, locally free modules $(M^n, F, V)$ on the smooth pair $(W, W \cap Z)$ such that $(M^n, F) \otimes \mathbb{Q} \cong \mathcal{E}_W$.

**Proof.** By definition of a logarithmic $F$-isocrystal, there exists a logarithmic crystal in coherent (not necessarily locally free!) modules $M$ and a map $F: \text{Frob}_Y^* M \to M \otimes \mathbb{Q}$ that is isomorphic to $\mathcal{E}$ when thought of as a logarithmic $F$-isocrystal. Here, $\text{Frob}_Y$ refers to the absolute Frobenius (on the f.s. log scheme $Y$ induced from the smooth pair $(Y, Z)$) and the $*$ refers to pullback on the logarithmic crystalline topos. This is compatible with our above definitions.

As $M$ is a logarithmic crystal in coherent modules, there exists a non-negative integer $\nu$ so that

$$F: (\text{Frob}_Y)^* M \to p^{-\nu} M.$$

We have assumed that the Newton slopes of $\mathcal{E}$ are all non-negative. Slope bounds of Katz (see the proof on [Kat79, p. 151-152]) imply then that there exists a non-negative $\nu$ such that all $n \geq 0$,

$$F^n: (\text{Frob}_Y)^* M \to p^{-\nu} M.$$

We explicate this in local coordinates. Take an affine open neighborhood $T \subset Y$ such that $(T, T \cap Z)$ has a smooth chart $(\bar{t}_1, \ldots, \bar{t}_n)$. Let $(P, t_1, \ldots, t_n)$ be the associated lifted smooth chart; note that $P = \text{Spf}(A)$ where $A$ is a noetherian $W(k)$ algebra equipped with the $p$-adic topology. Then the logarithmic crystal yields a finitely generated $A$ module $M_A$ and the Frobenius structure induces a $M_A$-linear (continuous) homomorphism $F: \sigma^* M_A \to p^{-\nu} M_A$.

As $U \cap T \subset T$ is open dense, it follows from Equation A.1 that

$$F^n: (\sigma^n)^* M_A \to p^{-\nu} M_A.$$
By varying $T$, one deduces that $F^n : (\text{Frob}_T^n)^* M \to p^{-n} M$.

Consider the module

$$M'_A := \sum_{n \geq 0} F^n ((\sigma_A^n)^* M_A) \subset p^{-n} M_A.$$  

As $A$ is noetherian, $M'_A$ is finitely generated, being a submodule of a finitely generated module. Moreover, $M'_A$ is stable under $F$. Finally, $M'_A$ is the finite sum of (logarithmic) horizontal submodules. Therefore the pair $(M'_A, F)$ is in fact a logarithmic $F$-crystal in coherent modules. We have an isomorphism $(M'_A, F) \otimes \mathbb{Q} \cong \mathcal{E}_T$ in the category of logarithmic $F$-isocrystals with nilpotent residues on $(T, Z \cap T)$.

Now, set $M''_A := (M_A)'^*$. This is a coherent reflexive sheaf on the ring $A$, and hence is locally free away on an open set of $\text{Spec}(A)$ whose complement has codimension at least 3. $M''_A$ is manifestly stable under the connection and $F$. In particular, we can find an open subset $T'' \subset T$ with complementary codimension at least 2 such that the logarithmic $F$-crystal $(M''_A, F)_{T''}$ is a crystal in finite, locally free modules.

After initially choosing a pair $(M, F : \text{Frob}_T M \to p^{-n} M)$ representing $\mathcal{E}$, the constructions we have made are canonical. Therefore, ranging over $T$, we may glue the $(M''_A, F)_{T''}$; that is, there is an open subset $W \subset T$ with complementary codimension at least 2 and a logarithmic $F$-crystal $(M'', F)_W$ in finite, locally free modules on the smooth pair $(W, Z \cap W)$ that is a lattice inside of $\mathcal{E}_W$.

We now indicate how to complete the result if the Newton polygons on $U$ are in the interval $[0, 1]$. Let $(M, F)$ be a logarithmic $F$-crystal in finite, locally free modules on a smooth pair $(Y, Z)$ over a perfect field $k$ and suppose the Newton slopes on $U$ are no greater than 1. Set $V := F^{-1} \circ p$. Then $V$ does not necessarily stabilize $M$; however, the pair $(M, V)_U$ is a logarithmic $\sigma^{-1}$-$F$-isocrystal in the language of [Kat79].

By the coherence argument as above, we may find $\eta$ such that:

$$V : (\text{Frob}_Y^{-1})^* M \to p^{-n} M$$

on all of $Y$. Again, using Katz’s slope bounds on $U$ (which hold equally well for $\sigma^{-1}$-$F$-crystals) and the same coherence argument, one shows that after possibly increasing $\eta$, we in fact have

$$V^n : (\text{Frob}_Y^{-n})^* M \to p^{-n} M$$

for all $n \geq 0$. Now run exactly the above argument with $V$ instead of $F$: then

$$M' := \sum_{n \geq 0} V^n (\text{Frob}_Y^{-1})^* M \subset p^{-n} M$$

will be coherent, horizontal, and stabilized by $V$. Recall that $FV = VF = p$; therefore $M'$ is also stabilized by $F$! Then $M'' := (M')''$ is a reflexive logarithmic crystal on $(Y, Z)$ that is stabilized by both $F$ and $V$. Exactly as above, there exists an open subset $W \subset Y$ of complementary codimension at least 2 such that $(M'', F, V)_W$ is a logarithmic Dieudonné crystal in finite, locally free modules, as desired. 

\[ \square \]

**Remark A.6.** Let $(Y, Z)$ be a smooth pair over $k$ and let $(M, F)$ be a logarithmic $F$-crystal in finite, locally free modules on $(Y, Z)$. We construct a natural line bundle $\omega$, which we call the *Hodge line bundle*, attached to $(M, F)$.

Evaluating $M$ on the trivial thickening of $(Y, Z)$, we obtain a vector bundle $M_{(Y, Z)}$ on $Y$ together with an integrable connection with logarithmic poles on $Z$ and a horizontal map

$$F_{(Y, Z)} : \text{Frob}_Y^* M_{(Y, Z)} \to M_{(Y, Z)}.$$  

As $F_{(Y, Z)}$ is a homomorphism of vector bundles, it follows that the kernel is a vector bundle on $Y$. Set $\omega := \det(\ker(F_{(Y, Z)}))$. We call this kernel the *Hodge line bundle* associated to $(P, F)$.

\[11\] Fortunately, Katz’s entire paper is written in the context of $\sigma^a$-$F$-crystals for any $a \neq 0$, not just the positive $a$. In particular, all of Katz’s results also hold for $\sigma^{-1}$-$F$-crystals. Katz does not deal with logarithmic crystals, but we only use the slope bounds on the open set $U$. 


As a reference for this remark: in the case when $Z$ is empty, one finds this construction in [dJ98, 2.5.2 and 2.5.5]. In the setting of logarithmic Dieudonné crystals, Kato-Trihan construct the dual object: $\text{Lie}(M,F,V)$, see [KT03, 5.1] and especially Lemma 5.3 of loc. cit.

**Remark A.7.** We have the following relationships.

1. Let $Y/k$ be a smooth scheme over a perfect field $k$. Let $A_Y \to Y$ be an abelian scheme. Then there is an associated Dieudonné crystal $(M,F,V) = D(A_Y[p^\infty])$ on $Y$ [BBM82]. The Hodge bundle of $(M,F)$ is isomorphic to the Hodge line bundle of the abelian scheme $A_Y \to Y$ by [BBM82, 3.3.5 and 4.3.10].

2. Let $C/k$ be a smooth curve over a perfect field $k$, let $U \subset C$ be a dense open subset, and let $Z \subset C$ be the reduced complement. Let $A_U \to U$ be an abelian scheme with semi-stable reduction along $Z$. Call the Néron model $A_C \to C$. Then there is an attached logarithmic Dieudonné crystal on $(C,Z)$, which we call $D(A_C[p^\infty])$ [KT03, 4.4-4.8]. By construction, $D(A_C[\infty])|_U$ is isomorphic to the crystalline Dieudonné functor of the $p$-divisible group $A_U[p^\infty]$. Moreover, the Hodge line bundle of $A_C \to C$, the open subset of $A_C \to C$ obtained by discarding the non-identity components over $Z$, is isomorphic to the Hodge bundle of $D(A_C[p^\infty])$: this follows from [KT03, Example 5.4(b)], with the caveat that they work with the covariant Dieudonné functor.

Finally, we argue that $D(A_C[p^\infty])$ is the unique logarithmic Dieudonné crystal with nilpotent residues on $(C,Z)$ that extends $D(A_U[p^\infty])$. First of all, note that we only need to check that there is at most one extension logarithmic $F$-crystal: in our setting, $V$ is determined by $F$ under the relation $FV = VF = p$. By [LT01], it follows that $D(A_U[p^\infty]) \otimes \mathbb{Q}_p$ is isocrystalline. By the hypothesis that $A_U \to U$ has semistable reduction along $Z$, Kato-Trihan have constructed $\log F$-crystal with nilpotent residues on $(C,Z)$ as above. Finally, by a full-faithfulness result of Kedlaya [Ked07, Proposition 6.3.2], it follows that the extension of $D(A_U[p^\infty])$ to a logarithmic $F$-crystal with nilpotent residues on $(C,Z)$ is unique.

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12Kato-Trihan construct a covariant Dieudonné functor. We assume ours is contravariant, which may be accomplished by taking a dual as in [KT03, 4.1].

13See also the description of gluing as in [KT03, Lemma 4.4.1].

14When $k \cong \mathbb{F}_p$, which is all we need, the fact that the residues are nilpotent may be derived as follows: semistable reduction guarantees tameness of the $p$-adic companion, and then one may compare local $\sigma$-factors via compatibility in the Langlands correspondence. See e.g. the proof of [AE19, Proposition 2.8].