Gonality Growth of Galois Covers

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1 Introduction

In this note, we will study the growth of gonality in Galois, unramified maps of curves $\pi : X \to Y$. For us, a curve will mean a geometrically reduced, geometrically irreducible, smooth projective scheme of dimension 1 over a field k unless stated otherwise. Throughout, we assume that $g(Y) \geq 2$. The two main results are the following:

Theorem 1. *Let $\pi : X \to Y$ be a Galois, unramified map between two l-gonal curves over k. Suppose Y has a k-point. If Y does not have a gonal map factoring through a genus 1 curve, then* $\deg \pi \leq l^2$.

Theorem 2. *Let $\pi : X \to Y$ be a Galois, unramified map with $\gamma_k(X) = l$. Suppose Y does not map to a genus 1 curve. Then* $\deg \pi < 2l^2$.

Corollary 1.1. *If $\pi : X \to Y$ is a Galois, unramified map of degree n and Y does not map to a genus 1 curve, then $\gamma_k(X) \geq \left(\frac{n}{2}\right)^{\frac{1}{2}}$.*

The hypothesis about elliptic curves is necessary for both results. For instance, suppose Y had a gonal map factoring through an elliptic curve E. Given any unramified Galois cover $E' \to E$, we can build the following cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \downarrow & & \downarrow \\ E' & \longrightarrow & E \end{array}$$

This constructs arbitrarily large Galois, unramified covers of Y. Moreover Lemma 2.1 implies $\gamma_k(X) \geq \gamma_k(Y)$ and hence $\gamma_k(X) = \gamma_k(Y)$.

Definition 1.1. *Let X be a smooth projective curve over a field k. The gonality of X (over k), denoted $\gamma_k(X)$, is the minimal degree of a non-constant map $X \to \mathbb{P}^1$. If $f : X \to \mathbb{P}^1$ is a minimal degree map, we say that f is gonal.*

Definition 1.2. *We say X admits an essentially unique map to C of degree m if there is a degree m map $f : X \to C$ of degree m and if all other degree m maps $g : X \to C$ are obtained by post-composing with an automorphism of C.*

In other words, degree m maps $X \to C$ form a torsor for $Aut(C)$. Equivalently, X admits an essentially unique map to C of degree m iff there is a unique, index m subfield of $k(X)$ that is abstractly isomorphic to $k(C)$.

The main idea is roughly to show that a map $\pi : X \to Y$ with $\gamma_k(X) = l$ is a fiber product (up to normalization) of a Galois map between curves of bounded genus with bounded ramification, using Proposition 2.1 and Proposition 3.1. This will allow us to bound the degree of π through Lemma 2.4. Our methods and constructions are similar to those of A. Tamagawa in section 2 of [4].

2 Basic Observations

The first observation about gonality is that it doesn't decrease through a map of curves. I learned this from a paper of B. Poonen, [3], which says that the idea for this proof go back at least to [2]. For completeness, I reproduce the proof here.

Lemma 2.1. *Let X, Y be smooth projective curves defined over a field k. If $\pi : X \to Y$ is a morphism defined over k, then the gonality of Y is no bigger than the gonality of X.*

Proof. For motivation, we first prove the Galois case: suppose π is a Galois cover (not necessarily unramified) with group G and degree n. Let f be a minimal degree map from X to $\mathbb{P}^1$, say of degree l. Now, f is an element of $k(X)$ which satisfies the degree-n polynomial

$$\prod_{\sigma \in G} (t - \sigma f) \in k(Y)[t]$$

The coefficients of this polynomial have degree at most nl when considered as elements of $k(X)$ by the strong triangle inequality. Thus, each coefficient has degree at most l when considered an element of $k(Y)$. At least one of the coefficients is non-constant because f is non-constant, so $\gamma_k(Y) \leq l$.

The general (not-necessarily Galois or even separable) case is only slightly more complicated; let $P(t)$ be the characteristic polynomial of f as an element of $k(X)/k(Y)$. We can find some field M containing $k(X)$ such that in M the polynomial P splits as

$$P(t) = \prod_{i=1}^{n} (t - f_i)$$

Let $s = [M : k(X)]$. Then the functions f_i, considered as elements of M, have degree sl. Thus the coefficients of the characteristic polynomial, considered as functions in M, have degree at most sln, by the strong triangle inequality. Hence, as elements of $k(Y)$, they have degree at most l. □

Lemma 2.2. *If $\pi : X \to Y$ is a map where $\gamma_k(X) = \gamma_k(Y) = l$, then any minimal-degree map $f : X \to \mathbb{P}^1$ is a primitive element of the field extension $k(X) over k(Y)$.*

Proof.

$$\begin{array}{ccc} k(X) \supseteq & k(Y)[f] & \supset k(Y) \\ & \cup & \\ & k(f) & \end{array}$$

$[k(X) : k(f)] = l$ so $[k(Y)[f] : k(f)] \leq l$. Let D be the smooth projective model of $k(Y)[f]$. We have a factorization: $\pi : X \to D \to Y$. Lemma 2.1 implies that $\gamma_k(D) = l$, hence $k(Y)[f] = k(X)$ as desired. □

Proposition 2.1. *Same situation as Lemma 2.2. If X admits an essentially unique map to C that continues to a gonal map, then we have the following square which is cartesian up to normalization. The bottom arrow is Galois.*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \;= X/G \\ {\scriptstyle m:1}\downarrow & & \downarrow{\scriptstyle m:1} \\ C & \xrightarrow{\rho} & C/G \end{array}$$

Proof. As the map $X \to C$ is unique, G acts on C. Let $n = |G|$. We need to show G acts faithfully of C, or equivalently that $\deg \rho = n$. Let $f : X \to C \to \mathbb{P}^1$ be a gonal map. We know that f is a primitive element for the extension $k(X)/k(Y)$ and hence has degree n over $k(Y)$. Thus, the degree of f over any subfield of $k(Y)$, i.e. $k(C)^G$, is at least n. By definiton, $f \in k(C)$. The bottom map arises from a group quotient, so it has degree at most n, with equality iff G acts faithfully. □

Remark 2.1. *G does not necessarily act faithfully on C if $\gamma_k(X) \neq \gamma_k(Y)$!*

Lemma 2.3. *Suppose we have a diagram*

$$\begin{array}{ccc} X & \xrightarrow[\pi]{} & Y \;= X/G \\ {\scriptstyle m:1}\downarrow & & \downarrow{\scriptstyle m:1} \\ C & \xrightarrow{\rho} & C/G \end{array}$$

which is cartesian up to normalization and with π unramified. Then all of the ramification indices of ρ divide m.

Proof. We will show below that because π is unramified, the ramification index e_c of a point $c \in C$ over $s \in C/G$ must divide *each* of the ramification indices

e_y over s. Thus e_c divides their sum, which is m.

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ c & \longrightarrow & s \end{array}$$

Pick uniformizers t and u at s and c respectively. Then the order of vanishing of t at x is $v_x(t) = v_y(t) = e_y$ because π is unramified. On the other hand, $v_x(t) = v_c(t)v_x(u)$. Thus $v_c(t)|e_y$, as desired. □

Lemma 2.4. *Let P be a genus 0 curve and $\tau : C \to P$ a Galois morphism of curves, branched over a finite set $S \subset P$, with ramification numbers e_c. Denote the genus of C by g and suppose $g \neq 1$. Then $\deg \tau \leq |(2g-2)| lcm(e_c)$.*

Proof. We may suppose k is algebraically closed. Note that because C is geometrically irreducible, the map cannot be unramified. Let n be the degree of τ. Let δ_c be such that the ramification divisor R at c is $e_c + \delta_c - 1$. Here, $\delta_c = 0$ iff ramification is tame at c. Applying the Riemann-Hurwitz formula, we see that

$$2g - 2 = -2n + \sum_{s \in S} \sum_{c \in \tau^{-1}(s)} (e_c - 1 + \delta_c)$$

The map τ is Galois so e_c and δ_c are constant in fibers. Similarly, the size of the fiber at s is thus $\frac{n}{e_s}$. Expanding, we get

$$2g - 2 = -2n + \sum_{s \in S} (e_s - 1 + \delta_s) \sum_{c \in \tau^{-1}(s)} 1$$

$$2g - 2 = (|S| - 2)n + \sum_{s \in S} (\delta_s - 1) \frac{n}{e_s}$$

$$\frac{2g-2}{n} = (|S| - 2) + \sum_{s \in S} \frac{\delta_s - 1}{e_s}$$

Now, the denominator of the RHS is bounded by $\mathrm{lcm}(e_c)$ and hence n is bounded as desired. □

Remark 2.2. *The Galois assumption is crucial! Lemma 2.4 is not true otherwise.*

3 Unique Curves

Proposition 3.1. *Suppose $f : X \to \mathbb{P}^1$ is a degree l map. Then there exists a curve C of genus g, an integer m, and an essentially unique map $\rho : X \to C$ of degree m such that there is factorization of f through ρ:*

$$X \to C \to \mathbb{P}^1$$

such that $gm < l^2$.

Proof. If X had a unique g_l^1, we could set $C = \mathbb{P}^1$. Otherwise, X has another g_l^1, say f'. Taking the product, we get a map

$$(f.f') : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

Call the image of this map D_1, its normalization C_1, and set $a_1 = \deg(C_1 \to \mathbb{P}^1)$. Then

$$g(C_1) = p_g(D_1) \leq p_a(D_1) \leq (a_1 - 1)^2$$

If the induced map $X \to C_1$ is the unique degree $\frac{l}{a_1}$ map between X and C_1, we can set $C = C_1$ and we are done. Otherwise, there are at least 2, and we get a map

$$X \to C_1 \times C_1$$

of type $(\frac{l}{a_1}, \frac{l}{a_1})$. Call the image curve D_2, its normalization C_2, and let $a_2 = \deg(C_2 \to C_1)$. Then, the adjunction formula tells us

$$2g(D_2) - 2 \leq 2p_a(D_2) - 2 = (D_2)^2 + D_2.K$$

The Hodge Index Theorem implies that $(D_2)^2 \leq 2a_2^2$ ([1] Exercise V.1.9). Moreover, we know that $D_2.K = 2(2g(C_1) - 2)a_2$. Putting all of this together, we get that

$$g(C_2) \leq a_2^2 + 2a_2(a_1 - 1)^2 - 2a_2 + 1 = (a_2 - 1)^2 + 2a_2(a_1 - 1)^2$$

If $X \to C_2$ is an isomorphism, then we can set $C = X$. If $X \to C_2$ is the unique, degree $\frac{l}{a_1 a_2}$ map between X and C_2, we can set $C = C_2$. Otherwise, we continue the procedure until it terminates. At the end of the day, we will get a map $X \to C$, where

$$g(C) \leq (a_n - 1)^2 + 2a_n(a_{n-1} - 1)^2 + \ldots + 2^{n-1}a_n \ldots a_2(a_1 - 1)^2$$

where $a_1 a_2 \ldots a_n | l$ and $m = \deg(X \to C) = \frac{l}{a_1 \ldots a_n}$. Moreover, this will be the unique degree m map between X and C by construction. Now, the proposition will follow from the following lemma. □

Lemma 3.1. *If $a_i \geq 2$ are integers with $a_1 \ldots a_n = d$, then*

$$(a_1 - 1)^2 + 2a_1(a_2 - 1)^2 + \ldots + 2^{n-1}a_1 \ldots a_{n-1}(a_n - 1)^2 < d^2$$

Proof. Induction on n. The base case is trivial, so suppose it is true for k. Say $a_1 \ldots a_k = p$. We must prove that

$$2^k a_1 \ldots a_k (a_{k+1} - 1)^2 < p^2(a_{k+1}^2 - 1)$$

This follows from the fact that $2^k \leq p$ and $(a_{k+1} - 1)^2 < (a_{k+1}^2 - 1)$. □

Remark 3.1. *Phil Engel remarked that Lemma 3.1 can be improved to $(d-1)^2$.*

4 Proofs

Proof of Theorem 1. We have the following diagram, cartesian up to normalization, where $g(C) < l^2$ by Proposition 2.1 and Proposition 3.1.

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \ = X/G \\ {\scriptstyle m:1}\downarrow & & \downarrow{\scriptstyle m:1} \\ C & \xrightarrow{\rho} & C/G \end{array}$$

Note that because $\gamma_k(X) = \gamma_k(Y) = l$ and $\gamma_k(C) \geq \gamma_k(C/G)$, $\gamma_k(C) = \gamma_k(C/G) = \frac{l}{m}$ and hence the map $Y \to C/G$ can be continued to a gonal map.

There are three cases: $g(C/G) \geq 2$, $g(C/G) = 1$, or $g(C/G) = 0$. In the first case, the Riemann-Hurwitz formula implies that $\deg(\pi) < l^2$. The second case cannot happen by assumption.

We are left with the case that $g(C/G) = 0$. We assumed Y had a k-point, so C/G has a k-point and is hence isomorphic to $\mathbb{P}^1$. Then $l = m$, so $C \cong \mathbb{P}^1$, because $\gamma_k(C) = \gamma_k(C/G)$. In this case, Lemma 2.3 and Lemma 2.4 imply $\deg \rho \leq 2l$. Putting all of the pieces together, we see that in all cases $\deg \rho \leq l^2$, as desired. ☐

Remark 4.1. *If $l > 2$, we in fact get $\deg \pi < l^2$.*

Proof of Theorem 2. We again have the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \ = X/G \\ {\scriptstyle m:1}\downarrow & & \downarrow \\ C & \xrightarrow{\rho} & C/G \end{array}$$

where there is essentially one degree m map $X \to C$. Here, the argument in Proposition 2.1 fails and G need not act faithfully on C. Let the stabilizer of G acting on C be $H \trianglelefteq G$. The induced map $X/H \to Y$ is Galois and unramified with group G/H. Moreover, $\gamma_k(X/H) = \frac{\gamma_k(X)}{|H|}$.

$$\begin{array}{ccccc} X & \longrightarrow & X/H & \longrightarrow & Y \\ {\scriptstyle m:1}\downarrow & & \downarrow{\scriptstyle \frac{m}{|H|}:1} & & \downarrow{\scriptstyle \frac{m}{|H|}:1} \\ C & \xrightarrow{\text{id}} & C & \longrightarrow & C/(G/H) \end{array}$$

As before, there are three cases: $g(C/(G/H)) \geq 2$, $g(C/(G/H)) = 1$, and $g(C/(G/H)) = 0$. In the first case, note that $|H| \leq m$. The Riemann-Hurwitz formula applied to the map $C \to C/(G/H)$ gives that $|G/H| \leq g(C)$. Proposition 3.1 therefore implies

$$deg(\pi) = |H||G/H| \leq mg(C) < l^2$$

The second case cannot happen by assumption; we assumed that Y did not map to an elliptic curve.

We are left with the case that $C/(G/H)$ is a genus 0 curve. We know that $g(C)m < l^2$. Lemma 2.4 and Lemma 2.3 applied to the right square implies that

$$|G/H| \leq (2g(C) - 2)(\frac{m}{|H|}) < \frac{2l^2}{|H|}$$

Thus $|G| < 2l^2$, as desired. □

References

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